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Definition (of sequential continuity at a point). *Let (X, d) and (X', d') be two metric spaces and let $B : X \to X'$ be a correspondence of the set X into the set X'. The correspondence B is said sequentially continuous at a point $x_* \in X$ if whenever a sequence x in X d-converges to x_* (in symbols, $x \to_d x_*$) and whenever a sequence x' in X' d'-converges to a point $x'_* \in X'$ (in symbols, $x' \to_{d'} x'_*$) and verifies $x'_n \in B(x_n)$, for every natural n, then the (limit) relation $x'_* \in B(x_*)$ holds true.*

Remark. Obviously, a correspondence is sequentially continuous if and only if it is sequentially continuous at every point of its domain. Moreover, a correspondence is sequentially continuous if and only if its graph is closed.

0.1 The Banach fixed point theorem

Definition (of orbit). *Let X be a non-empty set and let x_0 be a point of X. Let $f : X \to X$ be a mapping of X into itself. We call **orbit of** f **starting from** x_0 the sequence of iterated images $(f^n(x_0))_{n=0}^{\infty}$, where f^n is defined inductively by the composition $f^n = f \circ f^{n-1}$, if n is a positive integer, and the power f^0 is the identity id_X. If S is a subset of X, we call **orbit of** f **starting from** S the sequence of iterated images $(f^n(S))_{n=0}^{\infty}$.*

Lemma. *Let (X, d) be a metric space, let $f : X \to X$ be a Lipschitz continuous function with a constant L and let B be a bounded subset of the space. Then, for every non-negative integer n, the following inequality holds*

$$^{d}f^n(B) \leq L^n \, {}^{d}B.$$

Consequently, if the function f is a contraction, the sequence of diameters $({}^{d}f^{n}(B))_{n=0}^{\infty}$ is vanishing.

Proof. It's easy to see, by induction, that, for every non-negative integer n,

$$^{d}f^{n}(B) \leq L^{n\ d}(B)\,.$$

Indeed, let us prove the inequality for $n = 1$. Let y and y' lie in $f(B)$, then there are two points x and x' in B such that $y = f(x)$ and $y' = f(x')$. By Lipshitz continuity, we have

$$\begin{aligned} d(y,y') &= d(f(x),f(x')) \leq \\ &\leq Ld(x,x') \leq \\ &\leq L\ {}^{d}B. \end{aligned}$$

Since the preceding inequality holds for all y and y' in $f(B)$, we deduce

$$^{d}(f(B)) \leq L\ {}^{d}(B),$$

as we desired. If the result is true for $n-1$, we have

$$\begin{aligned} {}^{d}(f^{n}(B)) &= {}^{d}(f(f^{n-1}(B))) \leq \\ &\leq L\ {}^{d}(f^{n-1}(B)) \leq \\ &\leq LL^{n-1\ d}(B) \leq \\ &\leq L^{n\ d}(B)\,. \end{aligned}$$

If $L < 1$, from the above inequality, since $^{d}(B) < \infty$, the result follows immediately. ■

Remark (about Cauchy sequences). A sequence x in a metric space (X,d) is said a Cauchy sequence if, for every positive real r, there exists an index n_0 of the sequence such that, for every couple of integers $m,n > n_0$, is $d(x_m,x_n) < r$. It is clear that, a sequence x is a Cauchy sequence if and only if, setting $B_n = (x_i)_{i=n}^{\infty}$ for every natural n, the sequence of diameters $({}^{d}B_n)_{n=0}^{\infty}$ vanishes.

Theorem (Banach-Picard fixed point theorem). *Let (X,d) be a complete metric space and let $f : X \to X$ be a contraction with a constant $L \in\ ]0,1[$. Then, the contraction has one and only one fixed point x^*. Precisely, for every point x_0 of the space, the orbit x of f starting from the point x_0 converges to x^* and the fixed point is localized in the closed ball centered at x_0 and of radius*

$$\frac{d\,(x_1,x_0)}{1-L}L^{n},$$

for each natural n. In other words, the fastness of convergence of the sequence x towards the fixed point is given by the inequality

$$d\left(x_n, x^*\right) \leq \frac{d\left(x_1, x_0\right)}{1-L} L^n,$$

for every $n \in \mathbb{N}_0$.

Proof. Let x_0 be a point of the space. If the orbit x converges, then it converges to a fixed point of f, by continuity of f. Let us prove that x converges. Since the metric space is complete, it's enough to prove that it is a Cauchy sequence. Note that, setting $B_n = (x_i)_{i=n}^{\infty}$, for every natural n, is $B_n = f(B_{n-1})$, for every positive integer n. So, by the lemma, if B_0 is bounded then the sequence of diameters $({}^d B_n)_{n=0}^{\infty}$ is vanishing, and hence the sequence x is a Cauchy sequence. To prove that B_0 is bounded, we claim that it is contained in the closed ball $\overline{B}_d(x_0, R)$ centered at x_0 and of radius

$$R := \frac{d(x_0, f(x_0))}{1-L}.$$

Indeed, for every positive integer k, the following inequalities hold

$$\begin{aligned}
d(x_0, f^k(x_0)) &\leq \sum_{i=1}^{k} d(f^{i-1}(x_0), f^i(x_0)) \leq \\
&\leq \sum_{i=1}^{k} L^{i-1} d(x_0, f(x_0)) \leq \\
&\leq d(x_0, f(x_0)) \sum_{i=1}^{k} L^{i-1} \leq \\
&\leq d(x_0, f(x_0)) \sum_{i=1}^{\infty} L^{i-1} = \\
&= \frac{d(x_0, f(x_0))}{1-L},
\end{aligned}$$

so the claim is proved. Concerning the uniqueness, let x_0 and x_0' be two fixed points of f, the set $B := \{x_0, x_0'\}$ is bounded, so the sequence of diameters $({}^d f^n(B))_{n=0}^{\infty}$ vanishes, but $f^n(B) = B$, for every natural number n, so the diameter of B must be 0, i.e., $x_0 = x_0'$. ■

The Banach fixed point theorem can be generalized as follows.

Theorem. *Let (X, d) be a complete metric space and let $f : X \to X$ be a mapping. Assume there exists a non-empty subset S of the space such that* *the orbit of f starting from it has the corresponding sequence of diameters vanishing; assume, moreover, that there is a point x_0 of the space such that the corresponding orbit of f starting from it is eventually in S. Then, the mapping f has at least one fixed point.*

Chapter 1

Decision-form Games

1.1 Strategy spaces and strategy base of game

The context. We deal with *two-player games*. We shall consider two non-void sets E and F, viewed as the respective sets of strategies at disposal of two players. The aim is to form ordered pairs of strategies $(x, y) \in E \times F$, called *strategy profiles* or *bistrategies*, via the (individual or collective) selection of their components x and y, done by the two players in the sets E and F, respectively, in order that the strategy x of the first player is a good reaction to the strategic behavior y of the second player and vice versa.

Let us formalize our starting point.

Definition (strategy base and bistrategy space). *Let (E, F) be a pair of non-empty sets, we call it* ***strategy base of a two-player game****. The first set E is said* **the first player's** ***strategy set****; the second set F is said* **the second player's** ***strategy set****. Any element x of E is said a* ***first player's strategy*** *and any element y in F is said a* ***second player's strategy****. Every pair of strategies $(x, y) \in E \times F$ is said* ***a bistrategy of the strategy base*** *(E, F) and the cartesian product $E \times F$ is said* ***the bistrategy space of the base*** *(E, F).*

Interpretation and terminology. We call the two players of a game Emil and Frances: Emil, simply, stands for "first player"; Frances stands for "second player". Emil's aim is to choose a strategy x in the set E , Frances' aim is to choose a strategy y in F, in order to form a bistrategy (x, y) such that the strategy x is an Emil's good response to the Frances' strategy y and vice versa.

We can make a first distinction between bases of game.

Definition (finite and infinite bases). *A strategy base is said **finite** if it has finitely many bistrategies, **infinite** on the contrary.*

Example (of infinite strategy bases). Two producers offer the identical good on a same market. They can interact *a la Cournot* or *a la Bertrand*. In the first case they choose the quantities to produce; in the second one, they choose the unitary prices of the good. The strategy spaces E and F of the two players coincide with the interval $[0, +\infty[$, or they are infinite subsets of this semi-line. In both cases, the strategy base is the pair of strategy spaces (E, F), and it is infinite.

1.2 Decision rules

A standard way for Emil and Frances to choose their reactions to the strategies of the other player is the adoption of *decision rules*. Let us formalize this basic concept (see also [2] and [3]).

Definition (decision rule). *Let (E, F) be a strategy base of a two-player game. An **Emil's decision rule on the base** (E, F) is a correspondence from F to E, say $e : F \to E$. Symmetrically, a **Frances' decision rule on the base** (E, F) is a correspondence from E to F, say $f : E \to F$.*

1.3 Decision-form games

Let us formalize the basic concept of our discourse.

Definition (decision-form game). *Let (E, F) be a strategy base of a two-player game. A **two-player decision-form game on the base** (E, F) is a pair (e, f) of decision rules of the players Emil and Frances, respectively, on the strategy base (E, F).*

Example (of a game). Let $E = [-1, 2]$ and $F = [-1, 1]$ be the strategy sets of two players. The multifunctions $e : F \to E$ and $f : E \to F$, defined by

$$e(y) = \begin{cases} -1 & if \;\; y < 0 \\ E & if \;\; y = 0 \\ 2 & if \;\; y > 0 \end{cases}, \quad f(x) = \begin{cases} -1 & if \;\; x < 1 \\ F & if \;\; x = 1 \\ 1 & if \;\; x > 1 \end{cases},$$

for every strategy x in E and y in F, are decision rules, of Emil and Frances respectively, on the base (E, F). The pair (e, f) is a two-player decision-form game on the base (E, F).

Definition (finite and infinite games). *A game is said **finite** if it has a finite number of bistrategies, **infinite** on the contrary.*

Definition (symmetric games). *A decision-form game is said **symmetric** if the decision rules of the two players coincide (consequently, the two players have the same strategy space).*

Definition (of univocal game). *A decision-form game is said, with abuse of language, **univocal** if its decision rules are everywhere defined and univocal, that is if its decision rules are functions.*

1.4 Possible reactions

Definition (of possible reaction and of capability of reaction). *Let (e, f) be a decision-form game. Let y be a Frances' strategy, the elements of the image of y by the correspondence e (that is, the elements of the set $e(y)$), i.e., the direct corresponding strategies of y by the rule e, are called **Emil's possible responses**, or* **Emil's** ***possible reactions, to the Frances' strategy*** *y. Analogously, let x be an Emil's strategy, the elements of the image of x by the decision rule f (that is, the elements of the set $f(x)$), i.e. the direct corresponding strategies of x by the rule f, are said* **Frances'** ***possible responses,*** *or* **Frances'** ***possible reactions, to the Emil's strategy*** *x. The* *set of Emil's possible reactions (responses) to the Frances' strategy y is said the* ***Emil's reaction set to the Frances' strategy*** *y. Finally, we say that* ***Emil can react to the Frances' strategy*** *y if the corresponding reaction set $e(y)$ is non-void.*

Interpretation. In the conditions of the above definition, the decision rule e associates, with each strategy $y \in F$ (of Frances), all those strategies x of E among which Emil can choose his response, when Frances is playing y. Analogously, the decision rule f associates, with every strategy $x \in E$, played by Emil, all those strategies y in F among which Frances can choose her own response, to react to the Emil's action x.

Example (of reaction). Let $e : F \to E$ and $f : E \to F$ be two decision rules, with strategy spaces $E = [-1, 2]$ and $F = [-1, 1]$, defined by

$$e(y) = \begin{cases} -1 & if \;\; y < 0 \\ E & if \;\; y = 0 \\ 2 & if \;\; y > 0 \end{cases}, \quad f(x) = \begin{cases} -1 & if \;\; x < 1 \\ F & if \;\; x = 1 \\ 1 & if \;\; x > 1 \end{cases},$$

for every x in E and y in F. The only possible Emil's response, to a Frances' strategy $y < 0$ is the strategy -1. Emil can choose an arbitrary strategy in E, if Frances plays 0; Emil has only the reaction strategy 2, if Frances plays a strategy $y > 0$. The only possible Frances' response to an Emil's strategy $x < 1$ is the strategy -1; Frances can choose an arbitrary strategy in F if Emil plays 1; Frances has only the reaction strategy 1 if Emil uses a strategy $x > 1$.

Definition (of equilibrium). *We call* ***equilibrium*** *of a decision form game (e, f) each bistrategy (x, y) of the game such that the strategy x is a possible reaction to the strategy y, with respect to the decision rule e, and y is a possible reaction to x, with respect to f. In other terms, an equilibrium of (e, f) is any bistrategy of the game belonging to the intersection of the graph of f with the inverse (symmetric) graph of e.*

1.5 Some classic examples

In this section we present some elementary finite games. These games are the decision-form version of some classic example of normal-form game theory.

Example (Matching pennies). To win a prize, two players 1 and 2 must write a number, chosen among -1 and 1, hiding the choice to the other player. After this, the choices are revealed simultaneously. If the numbers coincide, player 1 wins, if they are different player 2 wins. The preceding scenario can be formalized as a decision-form game $G = (e_1, e_2)$, with both strategy spaces coincident with the finite set $E = \{-1, 1\}$ and decision rules $e_1, e_2 : E \to E$, defined by $e_1(s) = s$ and $e_2(s) = -s$, for every strategy s in E. It is a univocal non-symmetric game.

Example (Coordination game). To win a prize, two partners of a pair $(1, 2)$ must write a number, chosen among -1 and 1, hiding the choice to the other partner. After this, the choices are revealed simultaneously. If the numbers coincide, the pair wins, if they are different, the pair loses. The preceding scenario can be formalized by a decision-form game $G = (e_1, e_2)$, with strategy spaces coincident with the set $E = \{-1, 1\}$ and decision rules $e_1, e_2 : E \to E$, defined by $e_1(s) = s$ and $e_2(s) = s$, for every strategy s in E. It is a univocal and symmetric game.

An opposite version of the above game is the following one.

Example (Discoordination game). To win a prize, two partners of a pair $(1, 2)$ must write a number, chosen among -1 and 1, hiding the choice to the other partner. After this, the choices are revealed simultaneously. If the numbers coincide, the pair loses, if they are different, the pair wins. The preceding scenario can be formalized as a decision-form game $G = (e_1, e_2)$, with strategy spaces coincident with the set $E = \{-1, 1\}$ and decision rules $e_1, e_2 : E \to E$, defined by $e_1(s) = -s$ and $e_2(s) = -s$, for every strategy s in E. It is a univocal and symmetric game.

1.6 Disarming strategies

Our definition of game does not exclude the existence of Emil's strategies x such that the Frances' reaction set to x, that is the image $f(x)$, is empty. In other words, it may happen that Frances could not be able to react to a certain Emil's strategy x, as she does not consider any own strategy appropriate to face up to the Emil's action x. It makes harder and harder the comprehension of what we can define as a solvable game or the solution of a game. This consideration prompts us to give the following definition.

Definition (of a disarming strategy). *Let (e, f) be a game. The Emil's strategies x to which Frances cannot react, i.e. such that the image $f(x)$ is empty, are called* ***Emil's disarming strategies (for Frances)****. The Frances' strategies y to which Emil cannot react, namely such that the reaction set $e(y)$ is empty, are called* ***Frances' disarming strategies (for Emil)****.*

Example (of disarming strategies). Let $E = [-1, 2]$ and $F = [-1, 1]$ be two strategy spaces and let $e : F \to E$ and $f : E \to F$ be two decision rules defined by

$$e(y) = \begin{cases} \{-1\} & if \;\; y < 0 \\ E & if \;\; y = 0 \\ \varnothing & if \;\; y > 0 \end{cases} , \quad f(x) = \begin{cases} \{-1\} & if \;\; x < 1 \\ \varnothing & if \;\; x = 1 \\ \{1\} & if \;\; x > 1 \end{cases} ,$$

for every x in E and y in F. Emil has no reaction strategies if Frances chooses a strategy $y > 0$: then, any positive Frances' strategy is disarming for Emil. Instead, Frances has no reaction strategy if Emil plays 1: the Emil's strategy 1 is disarming for Frances.

Remark. For the previous example, consider the graphs of the two correspondences e and f in the cartesian products $F \times E$ and $E \times F$, respectively, and the graph of the reciprocal correspondence of e and that of the correspondence f in the same space $E \times F$. It is easily seen (geometrically and algebraically) that the intersection of the graph of the reciprocal of e with the graph of f contains just the point $(-1, 1)$.

Remark (about the domain of a decision rule). From previous definitions we can gather that the set of Emil's strategies to which Frances can oppose a reaction is the domain of the correspondence f, $\mathrm{dom}f$. Similarly, the set of Frances' strategies to which Emil can oppose a reaction is the domain of the correspondence e, $\mathrm{dom}e$. Consequently, the set of Emil's disarming strategies is the complement of $\mathrm{dom}f$ with respect to E and the set of Frances' disarming strategies is the complement of $\mathrm{dom}e$ with respect to F.

A game with decision rules everywhere defined is said *a game without disarming strategies.*

The instance that a decision rule is univocal at any point can be interpreted in the context of game theory, as in the following definition.

Definition (obliged strategies). *Let (e, f) be a decision-form game. If, with respect to the decision rule f, there is only one Frances' reaction y to a certain Emil's strategy x, that is if $f(x)$ is the singleton $\{y\}$, such strategy y*

is called ***Frances's obliged strategy by the Emil's strategy*** x. *Analogous definition can be given for Emil's strategies.*

1.7 Subgames

We now introduce another fundamental notion, that of subgame.

Definition (of subgame). *Let $G = (e, f)$ be a decision-form game with strategy base (E, F) and let (E', F') be a* ***subbase*** *of (E, F), namely a pair of subsets of E and F, respectively. We call* ***subgame of G with strategy base*** *(E', F') the pair (e', f') of the restrictions of the decision rules e and f to the pairs of sets (F', E') and (E', F'), respectively. It is important to remember that e' is the correspondence from F' to E' which associates with every strategy y' in F' the part $e(y') \cap E'$. In other words, it sends every strategy y' of F' into the corresponding Emil's reaction strategies to y' which belong to E'. We also call the subgame (e', f') the* ***restriction of the game G to the strategy pair*** *(E', F').*

Example (of subgame). Let $(\mathbb{R}, \mathbb{R})$ be the strategy base of the game $G = (e, f)$, defined by $e(y) = y^2$ and $f(x) = x^2$, for every couple of real numbers x and y. The subgame $G' = (e', f')$, with base $([-2, 2], [0, 1])$ is defined by

$$e(y) = y^2, \quad f(x) = \begin{cases} x^2 & if \ \ x \in [-1, 1] \\ \varnothing & if \ \ x \notin [-1, 1] \end{cases},$$

for each x in $[-2, 2]$ and y in $[0, 1]$. Even though in the game G there were no disarming strategies, its restriction to the subbase $([-2, 2], [0, 1])$ detects disarming strategies.

1.8 Rules induced by utility functions

In this section we introduce a standard method to define a decision rule when a player has a preference (preorder) on the bistrategy space induced by an utility function.

Definition (decision rule induced by a utility function). *Let* $(u_1, \geq)$ *be an Emil's utility function on the bistrategy space* $E \times F$*, that is a function* $u_1 : E \times F \to \mathbb{R}$ *endowed with the usual upper order of the real line. We call* **Emil's** ***best reply decision rule induced by the utility function*** $(u_1, \geq)$*, the rule* $B_1 : F \to E$ *defined by*

$$B_1(y) = \max_{u_1(.,y)}(E),$$

for every Frances' strategy y*. In other words, Emil's reaction set to a Frances' strategy* $y \in F$*, with respect to the rule* B_1*, is the set of every Emil's strategy maximizing the section* $u_1(., y)$*. Symmetrically, let* $(u_2, \geq)$ *be a Frances' utility function on the bistrategy space* $E \times F$*, that is a real function* u_2 *defined upon the bistrategy space* $E \times F$ *together with the canonical upper order of the real line. We call* ***Frances' best reply decision rule induced by the utility function*** $(u_2, \geq)$*, the rule* $B_2 : E \to F$ *defined by*

$$B_2(x) = \max_{u_2(x,.)}(F),$$

for each Emil's strategy x*. In other words, Frances' reaction set to the Emil's strategy* $x \in E$*, with respect to the rule* B_2*, is the set of every Frances' strategy maximizing the section* $u_2(x, .)$*.*

Memento. We write $\max_{u_1(.,y)}(E)$ to denote the set of maxima of the preordered space $(E, \leq_{u_1(.,y)})$, where by $\leq_{u_1(.,y)}$ we denote the preorder induced by the section $u_1(., y)$ on the set E. Such set of maxima is the set of maximum points (on E) of the function $u_1(., y)$, it is also denoted by $\mathrm{argmax}_E\, u_1(., y)$. There are symmetric notations for Frances.

Example (of induced rule). Let $E = [-1, 2]$ and $F = [-1, 1]$ be two strategy spaces and let $f : E \to F$ be the decision rule defined by

$$f(x) = \begin{cases} -1 & if \;\; x < 0 \\ F & if \;\; x = 0 \\ 1 & if \;\; x > 0 \end{cases},$$

for every Emil's strategy x in E. The rule f is induced by the utility function $u_2 : E \times F \to \mathbb{R}$ defined by $u_2(x, y) = xy$, for each bistrategy (x, y) of the game. Indeed, fix an Emil's strategy x, the section of partial derivative $\partial_2 u_2(x, .)$ coincide with the derivative $u_2(x, .)'$, therefore the function $u_2(x, .)$ is strictly increasing if $x > 0$, strictly decreasing if $x < 0$ and constant if $x = 0$, in particular:

1) if $x < 0$, the only Frances' strategy maximizing the function $u_2(x, .)$, on the compact interval $[-1, 1]$, is the strategy -1;

2) if $x > 0$, the only Frances' strategy maximizing the function $u_2(x, .)$, on the interval $[-1, 1]$, is the strategy 1;

3) if $x = 0$, each Frances' strategy maximizes the function $u_2(x, .)$, on the interval $[-1, 1]$, (since the value of the section $f_2(0, .)$ is zero in the whole domain).

Remark (about the never-best reply strategies). In the conditions of the above definition, an Emil's strategy x is called *never-best reply strategy with respect to the utility function* u_1 if and only if there is no $y \in F$ such that $x \in B_1(y)$. Moreover, a strategy x in E is said *non-reactive with respect to an Emil's decision rule* e if there is no y in F such that x lies in $e(y)$. The u_1 -never-best reply strategies are, so, the non-reactive strategies with respect to the decision rule B_1.

1.9 Rules induced by preorders

In this section we point out a generalization of the standard method to define decision rules of the previous section.

Note that, if $\geq_1$ is an Emil's preference on the bistrategy space $E \times F$ and if y is a Frances' strategy, the preorder $\geq_1$ induces, through y, a *section preorder* $\geq_1^y$ on E, that defined by

$$x_0 \geq_1^y x \quad \text{iff } (x_0, y) \geq_1 (x, y),$$

for each pair (x_0, x) of Emil's strategies.

Definition (decision rule induced by a preorder on the bistrategy space). *Let $\geq_1$ be an Emil's preference on the bistrategy space $E \times F$. We call **Emil's best reply decision rule induced by the preorder** $\geq_1$, the correspondence $B_1 : F \to E$ defined by*

$$B_1(y) = \max_{\geq_1^y}(E),$$

for each Frances' strategy y. In other words, the Emil's reaction set to the Frances' strategy $y \in F$ is the set of all those Emil's strategies maximizing the section preorder $\geq_1^y$. Similarly, let $\geq_2$ be a Frances' (utility) preorder on the *bistrategy space $E \times F$. We call **Frances' best reply** decision rule **induced by the utility preorder** $\geq_2$, the correspondence $B_2 : E \to F$ defined by*

$$B_2(x) = \max_{\geq_2^x}(F),$$

for each Emil's strategy x. In other words, the Frances' reaction set to the Emil's strategy $x \in E$ is the set of all those Frances' strategies maximizing the section preorder $\geq_2^x$.

Memento. We denote by $\max_{\geq_1^y}(E)$ the set of maxima in the preordered space $(E, \geq_1^y)$. Such set of maxima is as well the set of maximum points of the preorder $\geq_1^y$ and it may also be denoted by $\mathrm{argmax}_E \ \geq_1^y$. There are similar notations for Frances.

1.10 Cournot decision game

We apply the above standard method to build up standard decision rules for the case of an economic duopoly.

The context. Let us suppose Emil and Frances produce the same good and sell it on the same market.

Productive strategies and prices. Let $x, y \in \mathbb{R}_\geq$ be the amounts respectively produced by the players. The strategy sets are, a priori, upper unbounded. We assume that the price function, i.e. the inverse demand function $p : \mathbb{R}^2_\geq \to \mathbb{R}$, is an affine function of the total production (aggregate supply) $x + y$. Specifically, let us assume $p(x, y) = \alpha - \beta(x + y)$, for each productive bistrategy (x, y), with $\alpha \geq 0$ (price at aggregate supply 0) and $\beta > 0$ (marginal price with respect to the production $x + y$).

Cost functions. Assume the cost functions C_1 and C_2 of both producers are affine functions of the production, namely, $C_1(x) = \gamma x + \delta$ and $C_2(y) = \gamma y + \delta$, with $\gamma > 0$ (marginal cost) and $\delta \geq 0$ (fixed cost).

Net cost functions. Emil's net cost, corresponding to a bistrategy (x, y), is the cost $C_1(x)$ minus the revenue $p(x, y)x$, that is

$$f_1(x, y) = \gamma x + \delta - p(x, y)x = \beta x\,(x + y - u) + \delta,$$

and, symmetrically, Frances' one is

$$f_2(x, y) = \gamma y + \delta - p(x, y)y = \beta y\,(x + y - u) + \delta,$$

where we put $u = (\alpha - \gamma)/\beta$. The quantity u is called ***characteristic production or critic production of the considered duopoly***.

Reduction of the strategy spaces. *To stand a chance of obtaining a non-negative gain the two producers must exclude the strategies strictly greater than u.* Indeed, if $x > u$, then the value $x + y - u$ is positive, even more it is strictly greater than y, and thus the net cost $f_1(x, y)$ is positive. Analogously, one can proceed for Frances. The previous condition conducts to a reduction of the individual strategic possibilities: from the unbounded space $[0, +\infty[$ to the compact space $[0, u]$.

Cournot decision rules. Suppose that Frances produces y units of the good. An Emil's best reply to y is to produce $\overline{x}$ units to minimize his net cost function

$$f_1(\cdot, y) : x \mapsto x(x + y - u),$$

on the compact interval $[0, u]$. There exists a quantity $\overline{x}$ with minimum net cost, thanks to Weierstrass theorem, and it is unique, namely

$$\overline{x} = (1/2)(u - y).$$

Indeed, the derivative $f_1(\cdot, y)'(x)$ is negative for $x < \overline{x}$ and positive for $x > \overline{x}$. The Emil's best reply decision rule is

$$B_1 : [0, u] \to [0, u] : y \mapsto (1/2)(u - y).$$

Note that this rule sends Frances' strategy space onto the part $[0, u/2]$ of the Emil's strategy space. Analogously, Frances' best reply decision rule is

$$B_2 : [0, u] \to [0, u] : x \mapsto (1/2)\,(u - x)\,.$$

We call the two rules B_1 and B_2 *Cournot decision rules*, and the decision form game (B_1, B_2) *Cournot decision form game with bistrategy space* $[0, u]^2$.

1.11 A first price auction

In this section we study a first price auction as a decision-form game.

The context. Two players 1 and 2 take part to an auction to obtain an item in return for a payment.

Rules of the game. The auction has the following rules:

a) each player i makes a public evaluation v_i of the item;

b) if the two evaluations are equal and if no one of the two participants changes his own evaluation (or withdraws), the item will be drawed lots and the winner will pay an amount equal to his evaluation ;

c) if the evaluations are different, the two players will make simultaneously an offer for the item;

d) the bid b_i of the player i cannot exceed the evaluation v_i;

e) the item is assigned to the player that offers the greatest bid, or, in case of two same offers, to the player with the biggest evaluation;

f) the winner i^* pays his own bid b_{i^*} and receives the item.

Our aim is to describe the previous situation as a decision-form game, in case the auction actually takes place, that is when an evaluation is strictly greater than the other one.

Let us suppose that the first player evaluated the item more than the second one. The strategy spaces E and F of the two players are the spaces of the possible offers of the same players. The utility of the player i is zero, if he does not win; it is $v_i - b_i$, if he carries off the item paying b_i.

Strategy spaces and utility functions. Emil's and Frances' strategy spaces are the compact intervals $[0, v_1]$ and $[0, v_2]$, respectively. The utility functions of the two players are defined by

$$u_1(x, y) = \begin{cases} v_1 - x & if \;\; x \geq y \\ 0 & if \;\; x < y \end{cases} , \quad u_2(x, y) = \begin{cases} v_2 - y & if \;\; x < y \\ 0 & if \;\; x \geq y \end{cases} .$$

Decision rules. The best reply rules induced by the two utility functions are defined, respectively, by $B_1(y) = y$, for each y in $[0, v_2]$ and

$$B_2(x) = \begin{cases} \varnothing & if \;\; x < v_2 \\ F & if \;\; x \geq v_2 \end{cases} ,$$

for each x in E. As a matter of fact, if Emil offers a price x strictly smaller than v_2, Frances could carry off the prize, but she should maximize her own utility function on F, fixed the choice x of Emil, that is she has to maximize the section $u_2(x, .)$, which, when the Frances' offer is strictly greater than x (those that would assure her the item) is defined by

$$u_2(x, .)(y) = v_2 - y,$$

for every $y \in]x, v_2]$. Unfortunately, the supremum of $u_2(x, .)$ is the difference $x - y$, and such utility value is a shadow maximum (!), it is unreachable on F: therefore Frances has no best reply to the Emil's offer x. If, instead, $x \geq v_2$, the section $u_2(x, .)$ is constantly null, hence it assumes its maximum 0 on the whole F.

Best reply graphs. Emil's (inverse) best reply graph is the compact segment with end points $(0, 0)$ and (v_2, v_2). Frances' best reply graph is the compact interval $[v_2, v_1] \times F$.

Equilibrium. The two graphs intersect in the point (v_2, v_2) alone. An equilibrium solution, therefore, is that Emil awards the item and pays Frances' evaluation.

1.12 ε-best reply induced by a utility function

In this section we shall give a generalization of the concept of best reply.

Definition (ε-best reply induced by a utility function). *Let* $(u_1, \geq)$ *be an Emil's utility function on the bistrategy space* $E \times F$*, that is a function* $u_1 : E \times F \to \mathbb{R}$ *endowed with the usual upper order of the real line. For each positive real* ε*, we call* **Emil's ε-*best reply decision rule induced by the utility function*** $(u_1, \geq)$*, the rule* ${}^{\varepsilon}B_1 : F \to E$ *defined by*

$${}^{\varepsilon}B_1(y) = \{x \in E : u_1(x, y) \geq \sup_E u_1(., y) - \varepsilon\},$$

for every Frances' strategy y*. In other words, Emil's reaction set to a Frances' strategy* $y \in F$*, with respect to the rule* ${}^{\varepsilon}B_1$*, is the set of every Emil's strategy whose utility distance from the shadow utility* $\sup_E u_1(., y)$ *is less than* ε*. Symmetrically, we can do for Frances.*

Remark. The ε-best reply reaction set ${}^{\varepsilon}B_i(s)$ is always non-void by definition of supremum. Moreover, it contains the best reply $B_i(s)$.

Example. In the case of the above auction, we have

$$
{}^{\varepsilon}B_2(x) = \begin{cases}]x, x+\varepsilon] \cap F & if \;\; x < v_2 \\ F & if \;\; x \geq v_2 \end{cases},
$$

for each x in E, and hence Emil has no longer disarming strategies. Note, however, that, also in this case, there is only one equilibrium.

1.13 Example of different equilibria in a game

Scope of the section. In this section we associate with a normal-form game some decision-form games different from the canonical one (the pair of the best-reply rules). Each decision-form game which we shall consider represents a pair of player *behavioural ways.* In particular we introduce two types of behaviour: the *devote behaviour* and that *offensive behaviour.*

Let us introduce the devote behaviour.

Definition (devote responce). *We say that an Emil's action x is a* ***devote response to the Frances' strategy*** *y, in the game G, if x minimizes the Frances' partial loss function $f_2(.,y)$. We define* ***Emil's devotion decision rule*** *$L_1 : F \to E$ by $L_1(y) = \min_{f_2(.,y)} E$, for each y in F. In other terms, for any y, the responce-set $L_1(y)$ is the set of all Emil's strategies minimizing the partial loss function $f_2(.,y)$. Analogously, we can define the Frances' reaction-set $L_2(x)$, for every Emil's action x.We call the equilibria of the game (L_1, L_2)* ***devote equilibria of the loss game*** *G.*

Interpretation. *The decision-form game (L_1, L_2) represents the interaction of the two players when they are devoted each other.*

Example (of devote behaviours). We consider the two-player normal-form loss-game $G = (f, \leq)$ with strategy sets $E = F = [0, 1]$ and biloss (disutility) function defined by

$$
f(x, y) = (-4xy, x + y),
$$

for every bistrategy (x, y) of the square $[0, 1]^2$.

Devotion correspondences. We have $\partial_2 f_1(x, y) = -4x$, then there are two cases for the sign of the derivative: $x = 0$ and $x > 0$. If $x = 0$ then the partial loss function $f_1(x, .)$ is constant and then all Frances' strategies are devote to Emil's strategy 0. If $x > 0$ then the partial loss function $f_1(x, .)$ is strictly decreasing, and then the minimum value is reached with the strategy 1. Concluding, the devotion Frances' correspondence is defined by

$$L_2(x) = \begin{cases} F \text{ if } x = 0 \\ 1 \text{ if } x > 0 \end{cases} .$$

Concerning Emil's devotion, we have $\partial_1 f_2(x, y) = 1$, so the partial function $f_2(., y)$ is strictly increasing, for every $y \in F$, and then it assumes its minimum at 0. Concluding the Emil's devotion correspondence is defined by $L_1(y) = 0$, for every y in the Frances' strategy space.

Devotion equilibria. The set of all the devotion equilibria is the segment $[0_2, e_2]$, where 0_2 is the origin of the plane and e_2 is the second canonical vector of the plane: it is an infinite set.

About the devotion equilibria. The devotion equilibria are *non-cooperatively reachable*, playing Emil the *reaching-strategy* $x = 0$.

Remark. Concerning the efficiency, the devotion equilibrium e_2 is "strongly inefficient", since it lies upon the Pareto maximal boundary, on the contrary, the devotion equilibrium 0_2 is efficient, since it belongs to the Pareto minimal boundary, the remaining devotion equilibria are inefficient (this can be viewed with the complete study of the biloss-space).

We can made a confrontation of the devotion equilibria of the game considered in the example with the Nash equilibria.

Example (of properly non-cooperative behaviors). *Best reply correspondences.* We have $\partial_1 f_1(x, y) = -4y$, so there are two cases. First case. If $y = 0$, the section $f_1(., y)$ is constant and then $B_1(0) = E$. Second case. If $y > 0$ then $\partial_1 f_1(x, y) < 0$, the section $f_1(., y)$ is strictly decreasing and the minimum point of the section is 1, thus $B_1(y) = 1$, for every $y > 0$. Resuming, the Emil's best-reply correspondence is defined by

$$B_1(y) = \begin{cases} E \text{ if } y = 0 \\ 1 \text{ if } y > 0 \end{cases} .$$

Concerning the Frances' best reply, we have $\partial_2 f_2(x, y) = 1$, hence the section $f_2(x, .)$ is strictly increasing and the best reply is defined by $B_2(x) = 0$, for every $x \in E$. *Existence of Nash equilibria.* Kakutany's fixed point theorem assures the existence of at least a Nash equilibrium, Brouwer's fixed point theorem does not. *Nash equilibria.* The intersection of the graph of B_2 with the reciprocal graph of B_1 is

$$\mathrm{gr}\,(B_2) \cap \mathrm{gr}\,\left(B_1^-\right) = [A, B]\,,$$

so there are infinitely many Nash equilibria. All these equilibria are equivalent for Emil (Emil loss function is constantly equal zero on $[A, B]$) but not for Frances, so they are not equivalent. *About Nash equilibria.* The Nash equilibrium zone is reachable, playing Frances the reaching-strategy $y^N = 0$. The Nash equilibrium A is minimal, and it's the unique minimal equilibrium (good equilibrium), on the contrary, the Nash equilibrium B maximal (very bad equilibrium) the other ones are neither minimal nor maximal. The Nash equilibrium A is also a devotion equilibrium. By the way, we can ask if there are *dominant strategies.* Frances has one (and only one) dominant strategy: $y^D = 0$. Emil has one (and only one) dominant strategy: $x^D = 1$. *Dominant equilibria.* The Nash equilibrium B is a dominant Nash equilibrium.

Let us introduce the offensive behaviour.

Definition (offensive responce). *We say that an Emil's action x is an* ***offensive response to the Frances' strategy*** *y, in the loss game G with biloss function f, if x maximizes the Frances' partial loss function $f_2(., y)$. We define* ***Emil's offensive decision rule*** *$O_1 : F \to E$ by $O_1(y) = \max_{f_2(.,y)} E$, for each y in F. In other terms, for any y, the responce-set $O_1(y)$ is the set of all Emil's strategies maximizing the partial loss function $f_2(., y)$. Analogously, we can define the Frances' reaction-set $O_2(x)$, for every Emil's action x.We call the equilibria of the game (O_1, O_2)* ***offensive equilibria of the loss game*** *G.*

Interpretation. *The decision-form game (O_1, O_2) represents the interaction of the two players when they are offensive each other.*

Example(offensive correspondences and equilibria). We refer to the above example. We already saw that the players's (worst) offensive correspondences are defined by $O_1(y) = 1$, for every strategy $y \in F$, and

$$O_2\,(x) = \begin{cases} F \text{ if } x = 0 \\ 0 \text{ if } x > 0 \end{cases},$$

respectively. The intersection of the graph of O_2 with the reciprocal graph of O_1 is the unique offensive equilibrium B. *About the offensive equilibrium.* The unique offensive equilibrium is reachable non-cooperatively with the strategies $x^O = 1$ and $y^O = 1$, respectively. It is strongly-inefficient, since it lies on the Pareto maximal boundary. *Confrontation of the equilibria.* The unique Nash equilibrium that is a devotion equilibrium too is A. The unique Nash equilibrium that is an offensive equilibrium too is B. *Dominant offensive strategies.* Emil has the unique dominant offensive strategy 1. Frances has the unique dominant offensive strategy 0. *About the noncooperative solution.* The set of all Nash equilibria $[A, B]$ is controlled by Frances by the strategy 0. The equilibrium A is a focal point in the sense of Myerson: it is unique.

Chapter 2

Non-Reactivity

2.1 Introduction

The concept of solution by iterated elimination of non-reactive strategies, for two-player decision-form games was conceived by one of the authors and presented in [4], there (in [4]) two-player decision-form games were introduced. In this paper decision rules are used in the sense introduced by J. P. Aubin in [2] and [3], and they represent the *action-rationality*, the *behavioural way* itself, of each player in front of the competitive situation represented by the game. For different concepts of rationalizable solution, for instance solutions obtained by elimination of dominated strategies, the reader can see in [17], [18] and [15].

2.2 Preliminaries, notation and terminology

The context. We deal with *two-player games*. We shall consider two non-void sets E and F, viewed as the respective sets of strategies at disposal of two players. The aim is to form ordered pairs of strategies $(x, y) \in E \times F$, called *strategy profiles* or *bistrategies*, via the (individual or collective) selection of their components x and y, done by the two players in the sets E and F, respectively, in order that the strategy x of the first player is a good reaction to the strategic behavior y of the second player and vice versa.

Let us formalize our starting point.

Definition (strategy base and bistrategy space). *Let (E, F) be a pair of non-empty sets, we call it **strategy base of a two-player game**. The first set E is said* **the first player's** ***strategy set****; the second set F is said* **the second player's** ***strategy set****. Any element x of E is said a **first player's strategy** and any element y in F is said a **second player's strategy**. Every pair of strategies $(x, y) \in E \times F$ is said **a bistrategy of the strategy base** (E, F) and the cartesian product $E \times F$ is said **the bistrategy space of the base** (E, F).*

Interpretation and terminology. We call the two players of a game Emil and Frances: Emil, simply, stands for "first player"; Frances stands for "second player". Emil's aim is to choose a strategy x in the set E , Frances' aim is to choose a strategy y in F, in order to form a bistrategy (x, y) such that the strategy x is an Emil's good response to the Frances' strategy y and vice versa.

Definition (decision rule). *Let (E, F) be a strategy base of a two-player game. An **Emil's decision rule on the base** (E, F) is a correspondence from F to E, say $e : F \to E$. Symmetrically, a **Frances' decision rule on the base** (E, F) is a correspondence from E to F, say $f : E \to F$.*

Let us formalize the basic concept of our discourse.

Definition (decision-form game). *Let (E, F) be a strategy base of a two-player game. A **two-player decision-form game on the base** (E, F) is a pair (e, f) of decision rules of the players Emil and Frances, respectively, on the strategy base (E, F).*

Definition (of possible reaction and of capability of reaction). *Let (e, f) be a decision-form game. Let y be a Frances' strategy, the elements of the image of y by the correspondence e (that is, the elements of the set $e(y)$), i.e., the direct corresponding strategies of y by the rule e, are called **Emil's possible responses**, or* **Emil's** ***possible reactions, to the Frances' strategy***

y. Analogously, let x be an Emil's strategy, the elements of the image of x by the decision rule f (that is, the elements of the set $f(x)$), i.e. the direct corresponding strategies of x by the rule f, are said **Frances'** ***possible responses,*** *or* **Frances'** ***possible reactions, to the Emil's strategy*** *x. The set of Emil's possible reactions (responses) to the Frances' strategy y is said the* ***Emil's reaction set to the Frances' strategy*** *y. Finally, we say that* ***Emil can react to the Frances' strategy*** *y if the corresponding reaction set $e(y)$ is non-void.*

Definition (of a disarming strategy). *Let (e, f) be a game. The Emil's strategies x to which Frances cannot react, i.e. such that the image $f(x)$ is empty, are called* ***Emil's disarming strategies (for Frances)****. The Frances' strategies y to which Emil cannot react, namely such that the reaction set $e(y)$ is empty, are called* ***Frances' disarming strategies (for Emil)****.*

We now introduce another fundamental notion, that of subgame.

Definition (of subgame). *Let $G = (e, f)$ be a decision-form game with strategy base (E, F) and let (E', F') be a* ***subbase*** *of (E, F), namely a pair of subsets of E and F, respectively. We call* ***subgame of G with strategy base*** *(E', F') the pair (e', f') of the restrictions of the decision rules e and f to the pairs of sets (F', E') and (E', F'), respectively. It is important to remember that e' is the correspondence from F' to E' which associates with every strategy y' in F' the part $e(y') \cap E'$. In other words, it sends every strategy y' of F' into the corresponding Emil's reaction strategies to y' which belong to E'. We also call the subgame (e', f') the* ***restriction of the game G to the strategy pair*** *(E', F').*

2.3 Reactive strategies

In a decision-form game, if a certain player's strategy s does not react to any strategy of the other one, this strategy s can't be a reasonable action of the first player. For this reason, we are motivated to formalize, in the below definition, the concept of non-reactive strategy.

Definition (of a reactive strategy). *Let (e, f) be a two-player decision-form game. Let y_0 be a Frances' strategy, we call it* ***reactive (with respect to the decision rule f)*** *if it is a possible reaction to some Emil's strategy.*

In other words, a Frances' strategy y_0 is called reactive (with respect to f), if it belongs to the set $f(x)$, for some Emil's strategy x. A Frances' strategy is called ***non-reactive*** *if it is not reactive. Analogously, let x_0 be an Emil's strategy, we call it* ***reactive (with respect to the decision rule** e**)*** *if it is a possible reaction to some Frances' strategy. In other words, an Emil's strategy x_0 is called reactive (with respect to e), if it belongs to the set $e(y)$, for some Frances' strategy y. An Emil's strategy is called* ***non-reactive*** *if it is not reactive.*

Remark (on the sets of reactive strategies). Emil's and Frances' sets of respective reactive strategies are the two unions $\cup e := \bigcup_{y \in F} e(y)$ and $\cup f := \bigcup_{x \in E} f(x)$, i.e., the images of the correspondences e and f, respectively. Note that, for example, with the correspondence $e : F \to E$ it is, in a standard way, associated with the mapping $M_e : F \to \mathcal{P}(E)$, sending any Frances' strategy y into the reaction set $e(y)$. The mapping M_e is, therefore, a family of subsets of E indexed by the set F. Analogously, for the correspondence f, we can consider the family $M_f = (f(x))_{x \in E}$. So, the above two unions are the unions of the families M_e and M_f, respectively.

Example (of reactive and non-reactive strategies). Let (e, f) be a two-player decision-form game, let $E = [-1, 2]$ and $F = [-1, 1]$ be the strategy sets of the players and let the decision rules $e : F \to E$ and $f : E \to F$ be defined by

$$e(y) = \begin{cases} \{-1\} & if \ \ y < 0 \\ E & if \ \ y = 0 \\ \varnothing & if \ \ y > 0 \end{cases} , \quad f(x) = \begin{cases} \{-1\} & if \ \ x < 1 \\ \varnothing & if \ \ x = 1 \\ \{1\} & if \ \ x > 1 \end{cases} ,$$

for each bistrategy (x, y) of the game. All of Emil's strategies are reactive, since $\cup e = E$. Otherwise, only the Frances' strategies -1 and 1 are reactive, since $\cup f = \{-1, 1\}$.

2.4 Reduced games by elimination of non-reactive strategies

Definition (of a reduced game by elimination of non reactive strategies). *A game (e, f) is called* ***reduced by elimination of non-reactive***

strategies *if the images of the decision rules* e *and* f *are the strategy sets* E *and* F*, respectively. In other words, the game is reduced if the decision rules of the two players are onto.*

Example (of a non reduced game). Let (e, f) be a decision-form game, let $E = [-1, 2]$ and $F = [-1, 1]$ be the strategy sets of the two players and let the decision rules $e : F \to E$ and $f : E \to F$ be defined by

$$e(y) = \begin{cases} \{-1\} & if \ \ y < 0 \\ \{-1, 2\} & if \ \ y = 0 \\ \{2\} & if \ \ y > 0 \end{cases}, \quad f(x) = \begin{cases} \{-1\} & if \ \ x < 1 \\ \{0\} & if \ \ x = 1 \\ \{1\} & if \ \ x > 1 \end{cases},$$

for every bistrategy (x, y). The images of the rules e and f are the sets $\{-1, 2\}$ and $\{-1, 0, 1\}$; so, the game is not reduced by elimination of non-reactive strategies.

2.5 Elimination of non-reactive strategies

In a game, a rational behavior of the players is to use only reactive strategies, eliminating the non-reactive ones. So, they will play a subgame of the previous one, that we call reduction of the game by elimination of non-reactive strategies.

Before defining the reduction of a game we recall that, if $F : X \to Y$ is a correspondence and if X' and Y' are subset of X and Y, respectively, the restriction to the pair (X', Y') of F is the correspondence $F_{|(X',Y')}$ whose graph is $gr(F) \cap (X', Y')$.

Definition (the reduction of a game by elimination of non-reactive strategies). *Let* (e, f) *be a decision-form game on the strategy base* (E, F)*. We call* ***(first) reduction of the game*** (e, f) ***by elimination of non-reactive strategies*** *the subgame* (e', f') *on the subbase* $(e(F), f(E))$*, pair of images of the decision rules* e *and* f*, respectively. In other words, the* ***(first) reduction of the game*** (e, f) ***by elimination of non-reactive strategies*** *is the game whose decision rules are the restrictions* $e_{|(F',E')}$ *and* $f_{|(E',F')}$*, where* E' *and* F' *are the images of the rules* e *and* f*.*

Example (of reduction). Let (e, f) be the game, on the base $E = [-1, 2]$ and $F = [-1, 1]$, with decision rules $e : F \to E$ and $f : E \to F$ defined by

$$e(y) = \begin{cases} \{-1\} & if \;\; y < 0 \\ \{-1, 2\} & if \;\; y = 0 \\ \{2\} & if \;\; y > 0 \end{cases} \;\; , \quad f(x) = \begin{cases} \{-1\} & if \;\; x < 1 \\ \{-1, 0, 1\} & if \;\; x = 1 \\ \{1\} & if \;\; x > 1 \end{cases} \;\; ,$$

for every bistrategy (x, y) of the game. The images of the rules e and f are the sets $E_1 = \{-1, 2\}$ and $F_1 = \{-1, 0, 1\}$; so, the game is not reduced, since they don't overlap the spaces E and F, respectively. The (first) reduction of the game (e, f), by elimination of non reactive strategies, is the game whose decision rules $e_1 : F_1 \to E_1$ and $f_1 : E_1 \to F_1$ are defined by

$$e_1(y) = \begin{cases} \{-1\} & if \;\; y = -1 \\ \{-1, 2\} & if \;\; y = 0 \\ \{2\} & if \;\; y = 1 \end{cases} \;\; , \quad f_1(x) = \begin{cases} -1 & if \;\; x = -1 \\ 1 & if \;\; x = 2 \end{cases} \;\; .$$

Note that the subgame (e_1, f_1) is not reduced (since f_1 is not onto). The second reduction of the game (that is, the reduction of the first reduction), has the rules

$$e_2(y) = \begin{cases} \{-1\} & if \;\; y = -1 \\ \{2\} & if \;\; y = 1 \end{cases} \;\; , \quad f_2(x) = \begin{cases} -1 & if \;\; x = -1 \\ 1 & if \;\; x = 2 \end{cases} \;\; ,$$

on the base (E_2, F_2), where $E_2 = \{-1, 2\}$ and $F_2 = \{-1, 1\}$. In this case, both rules are onto and, so, the subgame $G_2 = (e_2, f_2)$ is reduced by elimination of non-reactive strategies.

2.6 Iterated elimination of non-reactive strategies

As we saw, the first reduction of a game can be non-reduced, so, we can consider the successive reductions to find a reduced subgame.

Definition (of k-th reduction by elimination of non-reactive strategies). *Let $G_0 = (e_0, f_0)$ be a game on a strategy base (E_0, F_0) and let k be a natural number. We define (recursively) the **k-th reduction, or reduction of order k, by elimination of non-reactive strategies of the***

game G_0 *as follows: the same game* G_0*, if* $k = 0$ *; the subgame* $G_k = (e_k, f_k)$ *on the base* (E_k, F_k)*, pair of the images of the decision rules of the* $(k-1)$ *reduction, i.e., pair of the sets* $e_{k-1}(F_{k-1})$ *and* $f_{k-1}(E_{k-1})$*, if* $k \geq 1$*. In other words, if* $k \geq 1$*, the decision rules* e_k *and* f_k *are the restrictions to the pairs* (F_k, E_k) *and* (E_k, F_k) *of the decision rules* e_{k-1} *and* f_{k-1}*, respectively. We say that a strategy* $x_0 \in E$ *survives the* k*-th elimination of non-reactive strategies if it belongs to* E_k*.*

Theorem (on the values of the reduced decision rules). *In the conditions of the above definition, for each strategy* s *of a player, which survived the* k*-th elimination of non-reactive strategies, the reaction set of the other player remains unchanged. In particular, if the game* G_0 *has not disarming strategies, all the reductions* G_k *has not disarming strategies.*

Proof. The first reduction (e_1, f_1) has strategy base $(e_0(F_0), f_0(E_0))$, so

$$e_1(y) = e_0(y) \cap e_0(F_0) = e_0(y),$$

for all Frances' strategy $y \in f_0(E_0)$. By induction we have

$$e_k(y) = e_0(y) \quad \text{AND} \quad f_k(x) = f_0(x),$$

for all k and for all bistrategy (x, y) in $E_k \times F_k$. ■

Definition (reducing sequence by elimination of non-reactive strategies). *Let* $G_0 = (e_0, f_0)$ *be a game on a strategy base* (E_0, F_0)*. We define* ***reducing sequence by elimination of non-reactive strategies of the game*** G_0 *the sequence of reduced subgames* $G = (G_k)_{k=0}^{\infty}$ *. In other words, it is the sequence with first term the game* G_0 *itself and with* k*-th term the* k*-th reduction of the game* G_0*.*

2.7 Solvability by iterated elimination

The reducing sequence allows us to introduce the concept of solvability and solution by iterated elimination of non-reactive strategies.

Definition (of solvability by iterated elimination of non-reactive strategies). *Let* $G_0 = (e_0, f_0)$ *be a decision-form game and let* G *be its reducing sequence by elimination of non-reactive strategies. The game* G_0 *is* *called **solvable by iterated elimination of non-reactive strategies** if there exists only one bistrategy common to all the subgames of the sequence G. In this case, that bistrategy is called **solution of the game G_0 by iterated elimination of non-reactive strategies**.*

Remark. The definition of solvability by iterated elimination of non-reactive strategies means that the intersection of the bistrategy spaces of all the subgames forming the reducing sequence, that is the intersection

$$\bigcap_{k=0}^{\infty} E_k \times F_k,$$

has one and only one element, which we said the solution of the game.

Remark. If the game G_0 is finite, it is solvable by iterated elimination of non-reactive strategies if and only if there exists a subgame of the sequence G with a unique bistrategy. In this case, that bistrategy is the solution, by iterated elimination of non-reactive strategies, of the game G_0.

2.8 Example of resolution

In the following example we present a simple resolution by iterated elimination of non-reactive strategies of an infinite game.

Example (solution by elimination of non-reactive strategies). Let $E = [-1, 2]$ and $F = [-1, 1]$, and let $e : F \to E$ and $f : E \to F$ be the decision rules defined by

$$e(y) = \begin{cases} \{-1\} & if \;\; y < 0 \\ E & if \;\; y = 0 \\ \varnothing & if \;\; y > 0 \end{cases} \quad , \quad f(x) = \begin{cases} \{-1\} & if \;\; x < 1 \\ \varnothing & if \;\; x = 1 \\ \{1\} & if \;\; x > 1 \end{cases} .$$

By elimination of non-reactive strategies, we obtain the subgame G_1, with strategy sets $E_1 = E$ and $F_1 = \{-1, 1\}$ and multifunctions $e_1 : F_1 \to E_1$ and $f_1 : E_1 \to F_1$ defined by

$$e_1(y) = \begin{cases} \{-1\} & if \;\; y = -1 \\ \varnothing & if \;\; y = 1 \end{cases} \quad , \quad f_1(x) = \begin{cases} \{-1\} & if \;\; x < 1 \\ \varnothing & if \;\; x = 1 \\ \{1\} & if \;\; x > 1 \end{cases} .$$

In the new game only the Emil's strategy -1 is reactive. Deleting all the others, we obtain an other subgame with strategy sets $E_2 = \{-1\}$ and $F_2 = F_1$ and multifunctions $e_2 : F_2 \to E_2$ and $f_2 : E_2 \to F_2$ defined by

$$e_2(y) = \begin{cases} \{-1\} & if \;\; y = -1 \\ \varnothing & if \;\; y = 1 \end{cases} \quad , \quad f_2(-1) = \{-1\} \, .$$

At last, Frances strategy 1 is, now, non-reactive, so, we have the trivial subgame with strategy sets $E_3 = F_3 = \{-1\}$ and multifunctions $e_3 : F_3 \to E_3$ and $f_3 : E_3 \to F_3$ defined by

$$e_3(-1) = \{-1\} \, , \quad f_3(-1) = \{-1\} \, .$$

We solved the game by iterated elimination of non-reactive strategies, and the solution is the unique survived bistrategy: the bistrategy $(-1, 1)$.

2.9 Iterated elimination in Cournot game

The game. Let $G_0 = (e_0, f_0)$ be the Cournot decision-form game with bistrategy space the square $[0, 1]^2$ and (univocal) decision rules defined, for every $x, y \in [0, 1]$, by $e_0(y) = (1 - y)/2$ and $f_0(x) = (1 - x)/2$. Set $x' = 1 - x$ and $y' = 1 - y$, the complement to 1 of the production strategies, for all bistrategy (x, y) of the game; briefly, we have $e_0(y) = y'/2$ and $f_0(x) = x'/2$.

The reducing sequence. Let $G = (G_k)_{k=0}^{\infty}$ be the reducing sequence by elimination of non-reactive strategies of the game G_0. The sequence G has as starting element the game G_0 itself. Let E_k and F_k be the strategy spaces of the k-th game $G_k = (e_k, f_k)$, for all natural k. The base of the game G_{k+1} is, by definition, the pair of images $e_k(F_k)$ and $f_k(E_k)$.

Reduction of the strategy spaces. The function e_k is strictly decreasing and continuous, so, the image of a real interval $[a, b]$ is the compact interval $[e_k(b), e_k(a)]$. The initial strategy spaces E_0 and F_0 are intervals, then, by induction, all the spaces $E_k = F_k$ are intervals. Let $[a_k, b_k]$ be the k-th strategy space E_k, then, concerning the $(k + 1)$-th, we have

$$F_{k+1} = e_k([a_k, b_k]) = [e_k(b_k), e_k(a_k)] = [e_0(b_k), e_0(a_k)] = [b'_k/2, a'_k/2] \, .$$

The interval $[b'_k/2, a'_k/2]$ does not coincide with the space $F_k = E_k$: the reduction G_k is not reduced.

Iterated reduction. We now study the sequence of the initial end-points of the interval family $(a_k, b_k)_{k=1}^{\infty}$. We have

$$a_{k+1} = e(b_k) = e(f(a_{k-1})).$$

The composite function $e \circ f$ is increasing and, moreover, the following inequalities $a_0 \leq a_1 \leq a_2$ hold true. So, the sequence a is increasing, and, then, it is convergent (as it is upper bounded by b_0). Moreover, being

$$a_{k+1} = \frac{1 - b_k}{2} = \frac{1 - a'_{k-1}/2}{2} = \frac{2 - a'_{k-1}}{4} = \frac{1 + a_{k-1}}{4},$$

and putting $a^* := \lim(a)$, we deduce

$$4a^* = 1 + a^*,$$

which gives $a^* = 1/3$. Similarly, we prove that b is a decreasing sequence and it converges to $1/3$.

Solution. Concluding, the unique point common to all the strategy intervals is the strategy $1/3$, in other terms we have

$$\bigcap_{k=0}^{\infty} [a_k, b_k] = \{1/3\}.$$

Then, *the game is solvable by elimination of non-reactive strategies and the solution is the bistrategy* $(1/3, 1/3)$.

2.10 Iterated elimination survival

In this section, we deal with the relations between solutions by iterated elimination of non-reactive strategies and game equilibria.

We introduce some definitions.

Definition (of survival the iterated elimination). *Let $G = (e, f)$ be a decision-form game. We say that a bistrategy **survives the iterated elimination of non-reactive strategies** if it belongs to all the bistrategy spaces of the reducing sequence of the game G.*

Terminology. We say that a base-game (E, F) is compact if E and F are both compact.

Theorem (existence of bistrategies surviving the iterated elimination). *Let $G = (e, f)$ be a game on a strategy base (E, F). Assume that (ε, φ) is a pair of topologies on the strategy sets of the base (E, F) and assume the game with closed-graph decision-rules and that there is at least a compact base of the reducing sequence of the game. Then, there exists at least one bistrategy surviving the iterated elimination of non-reactive strategies.*

Proof. Assume the subbase (E_k, F_k) be compact. Then, the images $F_{k+1} = f_k(E_k)$ and $E_{k+1} = e_k(F_k)$ are compacts, since e and f are with closed graph. By induction, every subbase (E_j, F_j) is compact, for $j > k$. So, the sequence of bistrategy space $(E_j \times F_j)_{j>k}$ is a sequence of compact sets with the finite intersection property; indeed, for every finite subset H of the set $\mathbb{N}(> k)$, that is the interval $]k, \to[_{\mathbb{N}}$, setting $h^* := \max H$, we have

$$\bigcap_{h \in H} E_h \times F_h = E_{h^*} \times F_{h^*}.$$

So, since the bistrategy space $E_k \times F_k$ is compact, that sequence has a non-void intersection. ■

Theorem (existence of a solution by iterated elimination). *Let $G = (e, f)$ be a game on a strategy base (E, F). Assume that (ε, φ) is a pair of complete metrics on the strategy sets of the base (E, F) and assume all the bistrategy spaces of the game are closed and with the sequence of their diameters vanishing. Then, there exists one and only one bistrategy surviving the iterated elimination of non-reactive strategies. So, under these assumptions, the game is solvable by elimination of non reactive strategies.*

Proof. It is a direct consequence of the nested closed principle in complete metric spaces (see, for instance, Kolmogorov - Fomin, Functional Analysis). ■

Corollary (existence of a solution, by iterated elimination, for sequentially continuous games). *Let $G = (e, f)$ be a sequentially continuous game on a compact base (E, F), with respect to a pair of complete metrics (ε, φ). Then, if the sequence of the diameters of the reduced bistrategy spaces is vanishing, the game is solvable by iterated elimination of non-reactive strategies.*

Proof. Strategy spaces of the game are compact and the decision rules are sequentially continuous, so, all the reduced bases are compact and, then, we can apply the previous theorem. ■

2.11 Survival of equilibria

Memento. Let $G = (e, f)$ be a decision-form game. Recall that a bistrategy (x^*, y^*) is called an equilibrium of the game G if $x^* \in e(y^*)$ and $y^* \in f(x^*)$.

The following theorem gives the first relation between solutions by iterated elimination of non-reactive strategies and game equilibria.

Theorem (survival of equilibria). *Let (x, y) be an equilibrium of a game (e_0, f_0). Then it survives iterated elimination of non-reactive strategies.*

Proof. By equilibrium definition, $x \in e_0(y)$ and $y \in f_0(x)$, that is, $(x, y) \in E_1 \times F_1$. Moreover, if $x \in e_k(y)$, then $x \in e_k(F_k) = E_{k+1}$, for every natural number k; analogously, if $y \in f_k(x)$, then $y \in f_k(E_k) = F_{k+1}$. By induction, we deduce that $(x, y) \in E_k \times F_k$, for each $k \in \mathbb{N}$. ■

Before the next theorem we need a lemma.

Lemma (characterization of Lipschitz continuity via bounded subsets). *Let (X, d) be a metric space, let $f : X \to X$ be a function and let L be a positive real number. Then, the following conditions are equivalent:*
1) f is Lipschitz continuous with a Lipschitz-constant L;
2) for any bounded subset B of the space, the following inequality holds

$$^{d}f(B) \leq L\, ^{d}B;$$

3) for any natural n and for any bounded subset B of the space, it is

$$^{d}f^{n}(B) \leq L^{n}\, ^{d}B.$$

Consequently, if the function f is a contraction, the sequence of diameters $(^{d}f^{n}(B))_{n=0}^{\infty}$ is vanishing.

Proof. 1) $\Rightarrow$ 2). It's easy to see, by induction, that, for every non-negative integer n,

$$^{d}f^{n}(B) \leq L^{n}\, ^{d}(B)\,.$$

Indeed, let us prove the inequality for $n = 1$. Let y and y' lie in $f(B)$, then there are two points x and x' in B such that $y = f(x)$ and $y' = f(x')$. By Lipschitz continuity, we have

$$\begin{aligned} d(y, y') &= d(f(x), f(x')) \leq \\ &\leq Ld(x, x') \leq \\ &\leq L\,{}^{d}B. \end{aligned}$$

Since the preceding inequality holds for all y and y' in $f(B)$, we deduce

$${}^{d}(f(B)) \leq L\,{}^{d}(B),$$

as we desired. If the result is true for $n - 1$, we have

$$\begin{aligned} {}^{d}(f^{n}(B)) &= {}^{d}(f(f^{n-1}(B))) \leq \\ &\leq L\,{}^{d}(f^{n-1}(B)) \leq \\ &\leq LL^{n-1}\,{}^{d}(B) \leq \\ &\leq L^{n}\,{}^{d}(B)\,. \end{aligned}$$

If $L < 1$, from the above inequality, since ${}^{d}(B) < \infty$, the result follows immediately. *3) ⇒ 2).* Obvious. *2) ⇒ 1).* Let x, y be two points of the space and let $B = \{x, y\}$. The set B is bounded and its diameter is the distance $d(x, y)$. The image $f(B)$ is the pair $\{f(x), f(y)\}$, whose diameter is the distance $d(f(x), f(y))$. From 2) it follows the Lipschitz -inequality $d(f(x), f(y)) \leq Ld(x, y)$. The Lemma is completely proved. ■

Theorem (existence of solution in contracting games). *Let $G_0 = (e_0, f_0)$ be a game on a strategy base (E_0, F_0). Assume (ε, φ) be a pair of complete metrics on the base (E_0, F_0) and let the game be univocal, contracting and with a bounded base (bounded with respect to the metrics). Then, there exists a unique bistrategy surviving the iterated elimination of non-reactive strategies, and it is the pair of the fixed points of the game rules.*

Proof. Existence of a survivor. Note that the intersection I of all the bistrategy spaces of the reduction sequences is non-void. Indeed, let $d : E \times F \to E \times F$ be the function defined by

$$d(x, y) = (e(y), f(x)),$$

for each ordered pair (x, y) in $E \times F$; the unique fixed point of the function d, that is the point (x^*, y^*) such that $d(x^*, y^*) = (x^*, y^*)$, belongs to the intersection I, thanks to the above theorem. *Uniqueness of the survivor.* We have just to prove that there is only one bistrategy surviving the iterated elimination. For, we claim that the sequence of diameters of the sequences E and F of the strategy spaces of the reduction sequence are vanishing. It is simple to prove that, for every natural n, we have

$$E_{2n} = (e \circ f)^n(E_0) \ , \quad E_{2n+1} = (e \circ f)^n(E_1).$$

Since the strategy base (E_0, F_0) is bounded, by the preceding lemma, the two subsequences $(E_{2n})_{n=1}^{\infty}$ and $(E_{2n+1})_{n=1}^{\infty}$ have the corresponding sequences of diameters vanishing. So, the intersection $\cap E$ can contain at most one point. Analogously, we can proceed for the sequence F, and the theorem is proved. ■

2.12 Nested compacts lemma

The following lemma will allow us to provide sufficient conditions in order that a solution of a game, by iterated elimination, is an equilibrium.

Notations and terminology. Recall that:

- A sequence of sets is said to be nested if each term of the sequence contains (widely) the following one.

Lemma (nested compacts lemma). *Let $F = (F_n)_{n=1}^{\infty}$ be a sequence of nested compact subsets of a metric space (F_0, φ) whose intersection contains one and only one point y_*. Then, the sequence of diameters of the family F is vanishing, that is*

$$\lim_{k \to \infty} {}^{\varphi}(F_k) = 0.$$

Consequently, each sequence y in the set F_0 such that $y_n \in F_n$, for each positive integer n, converges to the point y_*.

Proof. Let d be the sequence of diameters of the family F. Since the family F is nested, the sequence d is (widely) decreasing and it is bounded below by 0, so, it converges to its infimum d_*, that is to say to

$$d_* = \inf_{n \in \mathbb{N}} \sup_{y,z \in F_n} \varphi(y, z).$$

Since the set F_n is compact, for every index n, and since the metric φ is continuous, by Weierstrass theorem, there exist two sequences s and t in the compact F_0 such that $s_n, t_n \in F_n$, for all natural n, and such that

$$d_* = \inf_{n \in \mathbb{N}} \varphi(s_n, t_n).$$

Let us prove that the real d_* is zero. For all natural k, it is $\varphi(s_k, t_k) \geq d_*$. Moreover, since F_1 is compact, there exist two subsequences s' and t', extracted from the sequences s and t, respectively, converging in F_1 to points s_* and t_*, respectively. These subsequences are eventually contained in any closed F_k, and then, their limits are in any closed F_k, that is, in their intersection $\cap F = \{y_*\}$. This circumstance implies $s_* = t_* = y_*$, from which it must be

$$0 \leq d_* \leq \lim_{k \to \infty} \varphi(s'_k, t'_k) = \varphi(s_*, t_*) = \varphi(y_*, y_*) = 0.$$

Now, let y be a sequence in F_0 such that $y_k \in F_k$, for all k. For all k, then we have

$$\varphi(y_k, y^*) \leq {}^{\varphi}(F_k),$$

because both y_* and y_k belong to F_k. By the squeeze theorem, the sequence $(\varphi(y_k, y_*))_{k=1}^{\infty}$ is vanishing, then $y \to^{\varphi} y_*$, so, the lemma is proved. ■

Remark. The hypothesis of compactness is not avoidable. First of all we prove that the assumption of boundness is unavoidable. *Case of closed but not bounded sets.* Let F_0 be the real line $\mathbb{R}$ and $F_k = \{0\} \cup [k, \rightarrow[$, for all natural k. The sequence $(F_k)_{k=1}^{\infty}$ is a sequence of nested closed sets whose intersection is $\{0\}$, but all of the closed sets are not bounded. Then the diameter sequence must be vanishing. Now, we prove that the closedness needs, too. *Case of open bounded sets.* Consider the union

$$A_k =]0, 1/k[\cup B(1, 1/k),$$

for all natural $k \geq 1$, where $B(1, 1/k)$ is the open ball centered at 1 and with radius $1/k$. The sequence $A = (A_k)_{k=1}^{\infty}$ is a sequence of nested bounded open sets, whose intersection is (evidently) $\{1\}$. But, the diameter sequence converges to 1 and not to 0. Indeed, we have

$$^{d}(A_k) = 1 + 1/k,$$

for every natural integer k.

2.13 Non-reactive strategy and equilibria

Notations and terminology. Let $G = (e, f)$ be a game.

- If (X, μ) is a metric space and if S is a subset of X, we call diameter of S, with respect to μ, the following extremum

$$^{\mu}(S) := \sup_{x,y \in S} \mu(x, y).$$

- If the base of G is the pair (E, F) and (ε, φ) is a pair of metric on E and F, respectively, G is called a sequentially continuous game on the compact base (E, F) with respect to the pair of metrics if its rules are sequentially continuous, or if the graphs are closed in the product topology on the base $F \times E$ and $E \times F$, respectively.

Theorem (characterization of solvability). *Let (E_0, ε) and (F_0, φ) be compact metric spaces and let $G_0 = (e_0, f_0)$ be a decision-form game upon the base (E_0, F_0), without disarming strategies and sequentially continuous with respect to the pair of metrics (ε, φ). Then, the game G_0 is solvable by iterated elimination of non-reactive strategies if and only if the two diameter sequences of the strategy spaces of the reduced games are vanishing. Moreover, if the game is solvable, its solution is a game equilibrium.*

Proof. Let E and F be the sequences of the strategy spaces of the reducing sequence G of the game G_0 and let (x^*, y^*) be the solution by iterated elimination of non-reactive strategies. By definition of solution, it is

$$\bigcap_{k=1}^{\infty} (E_k \times F_k) = \{(x^*, y^*)\},$$

then, we have $\bigcap_{k=0}^{\infty} E_{k+1} = \{x^*\}$ and $\bigcap_{k=0}^{\infty} F_{k+1} = \{y^*\}$, or, in other terms

$$\bigcap_{k=0}^{\infty} f_k(E_k) = \{y^*\} \quad \text{et} \quad \bigcap_{k=0}^{\infty} e_k(F_k) = \{x^*\}.$$

Consequently, for all natural k, there exist a strategy $y_k \in F_k$ and a strategy $x_k \in E_k$ such that $x^* \in e_k(y_k)$ and $y^* \in f_k(x_k)$, that means $x^* \in e_0(y_k)$ e $y^* \in f_0(x_k)$. Since the correspondences e_0 and f_0 are sequentially continuous, all of their restrictions e_k and f_k are sequentially continuous. Moreover, since

G_0 has not disarming strategies and all the subgames of reducing sequence G are the restrictions to the images of the decisional rules of the previous game, these subgames have not disarming strategies. Therefore, the decision rules of the subgames are all sequentially continuous and with nonempty values and, because the initial strategy spaces E_0 and F_0 are compact, all the images of those decision rules are compact. Then, the sequences $x = (x_k)_{k=1}^{\infty}$ and $y = (y_k)_{k=1}^{\infty}$ converges to x^* and y^* respectively, thanks to the nested compacts lemma. Finally, because e_0 and f_0 are correspondences with closed graph, we have $x^* \in e_0(y^*)$ e $y^* \in f_0(x^*)$, ending the proof of the theorem. ■

Chapter 3

Reactivity

3.1 Introduction

In this paper we introduce the concept of *reactivity for two-player decision-form games* and concentrate upon it. Let $G = (e, f)$ be a decision-form game and let us christen our two player Emil and Frances, it is quite natural that if an Emil's strategy x can react to all the Frances' strategies to which an other strategy x' can react, then we must consider the strategy x *reactive* at least as the strategy x'; moreover, if the strategy x is reactive at least as x' and x can react to a Frances' strategy to which x' can not react, then Emil has to consider x strictly more reactive than x'. The previous simple considerations allow to introduce the *capacity of reaction*, or reactivity, of any Emil's strategy and to compare it with the capacity of reaction of the other Emil's strategies. In this direction, we introduce the *super-reactive strategies* of a player i, i.e. strategies of player i capable to reply to any opponent's actions to which the player i can reply: obviously these strategies (whenever they there exist) are the best ones to use, in the sense explained before. In a second time, we introduce the *reactivity comparison* between strategies and we observe that this relation is a preorder. Then, we define the concept of reactivity and explain the nature of the super-reactivity, this permits to define the concepts of *maximally reactive strategy*, *minimally reactive strategy,* and of *sub-reactive strategy.* The concept of sub-reactivity will give the opportunity to introduce the principal operative concepts of the paper, i.e. *the elimination of sub-reactive strategies*, the concept of *reducing sequence of a game by elimination of sub-reactive strategies* and, at last, the *solvability of a game by elimination of sub-reactive strategies* and the meaning of solution in the case of solvability.

3.2 Super-reactive strategies

Definition (of super-reactive strategy). *Let (e, f) be a two player decision-form game. An Emil's strategy x_0 is called* ***super-reactive with respect to the decision rule*** e *if it is a possible reaction to all the Frances' strategies to which Emil can react. In other terms, an Emil's strategy x_0 is called super-reactive if it belongs to the reaction set $e(y)$, for each Frances' strategy y belonging to the domain of the decision rule e. Analogously, a Frances' strategy y_0 is called* ***super-reactive with respect to the decision rule*** f *if it is a possible reaction to all the Emil's strategies to which Frances can react. In other terms, a Frances' strategy y_0 is called super-reactive if it belongs to the reaction set $f(x)$, for each Emil's strategy x in the domain of the decision rule f.*

Remark. The sets of all the Frances' and Emil's super-reactive strategies are so the two intersections

$$\begin{aligned} \cap^{\neq}(e) &= \bigcap_{y \in \mathrm{dom} e} e(y), \\ \cap^{\neq}(f) &= \bigcap_{x \in \mathrm{dom} f} f(x), \end{aligned}$$

respectively. If Frances has no disarming strategies toward Emil we have

$$\cap^{\neq}(e) = \cap e = \bigcap_{y \in F} e(y).$$

Analogously, if Emil has no disarming strategies toward Frances

$$\cap^{\neq}(f) = \cap f = \bigcap_{x \in E} f(x).$$

Obviously these two intersections can be empty.

We note here an elementary and obvious result.

Proposition. *Let (e, f) be a decision form game and let x_0 and y_0 be two non-disarming and super-reactive strategies of the first and second player respectively. Then the bistrategy (x_0, y_0) is an equilibrium of the game.*

It is straightforward that a game can have equilibria and lack in super-reactive strategy, as the following example shows.

Example (of game without super-reactive strategies). Let (e, f) be the decision form game with strategy spaces $E = [-1, 2]$ and $F = [-1, 1]$ and decision rules $e : F \to E$ and $f : E \to F$ defined by

$$e(y) = \begin{cases} -1 & \text{if } y < 0 \\ E & \text{if } y = 0 \\ 2 & \text{if } y > 0 \end{cases},$$

$$f(x) = \begin{cases} -1 & \text{if } x < 1 \\ F & \text{if } x = 1 \\ 1 & \text{if } x > 1 \end{cases}.$$

Emil has not super-reactive strategies, in fact

$$\cap e = \{-1\} \cap E \cap \{2\} = \varnothing.$$

Also Frances has no super-reactive strategies, in fact

$$\cap f = \{-1\} \cap F \cap \{1\} = \varnothing.$$

Note that this game has three equilibria.

We say that any equilibrium of a game is a *super-reactive equilibrium* when it is a super-reactive cross, i.e. when it is a pair of super-reactive strategies.

Example (of game with super-reactive strategies). Let (e, f) be the game with strategy spaces $E = [-1, 2]$ and $F = [-1, 1]$ and decision rules $e : F \to E$ and $f : E \to F$ defined by

$$e(y) = \begin{cases} [-1, 1] & \text{if } y < 0 \\ E & \text{if } y = 0 \\ [0, 2] & \text{if } y > 0 \end{cases},$$

$$f(x) = \begin{cases} -1 & \text{if } x < 1 \\ F & \text{if } x = 1 \\ \{-1, 1\} & \text{if } x > 1 \end{cases}.$$

Emil has infinite super-reactive strategies, in fact

$$\cap e = [-1, 1] \cap E \cap [0, 2] = [0, 1],$$

all the strategies x between 0 and 1 are super-reactive for Emil. Frances has only one super-reactive strategy, in fact

$$\cap f = \{-1\} \cap F \cap \{-1, 1\} = \{-1\} .$$

Note that this game has infinitely many equilibria, their set is the graph of the correspondence $f_1 : E \to F$ defined by

$$f_1(x) = \begin{cases} -1 & \text{if } x < 1 \\ F & \text{if } x = 1 \\ 1 & \text{if } x > 1 \end{cases} .$$

On the other hand, only the equilibria belonging to the segment $[0, 1] \times \{-1\}$ are super-reactive equilibria. Thanks to super-reactivity, in this game an equilibrium is non-cooperatively reachable; indeed, it is reasonable for Frances to play his unique super-reactive strategy -1 and for Emil to play one of his super-reactive strategies x in $[0, 1]$, consequently the game finishes in the equilibrium $(x, -1)$.

Remark (independence of the super-reactivity on the rival's rule). The Emil's (Frances's) super-reactive strategies depend only upon the Emil's (Frances's) decision rule, and not on both the decision rules.

Example (game with super-reactive strategies). Let E be the compact interval $[0, 1]$ and let F be the interval $[-1, 1]$, let $e : F \to E$ be the correspondence defined by $e(y) = [0, |y|]$, for each y in F. Frances has no disarming strategies toward Emil. The strategy 0 is the only Emil's super-reactive strategy, because

$$\cap e = \bigcap_{y \in F} [0, |y|] = \{0\} .$$

Let $f : E \to F$ be defined by $f(x) = [-x, x]$. Emil has no disarming strategies toward Frances. The strategy 0 is the only Frances' super-reactive strategy, because

$$\cap f = \bigcap_{x \in E} [-x, x] = \{0\} .$$

In this case we have again infinitely many equilibria, the points of the graph of the correspondence $f_1 : E \to F$ defined by $f_1(x) = \{-x, x\}$, but we have only one super-reactive equilibrium: the strategy profile $(0, 0)$.

3.3 Comparison of reactivity

The definition of super-reactive strategy can be generalized.

Definition (of comparison among reactivity). *Let (e, f) be a two player decision form game. Let x_0 and x be two Emil's strategies. We say that the strategy x_0 is* ***more reactive (in wide sense), with respect to the decision rule*** *e,* ***than the strategy*** *x, and we write $x_0 \geq_e x$, if x_0 is a possible reaction to all the Frances' strategies to which x can react. In other terms, a Emil's strategy x_0 is called more reactive than an other strategy x when x_0 belongs to the reaction set $e(y)$, for each strategy $y \in e^-(x)$. Analogously, let y_0 and y be two Frances' strategies. We say that y_0 is more* ***reactive, with respect to the decision rule*** *f,* ***than the strategy*** *y, and we write $y_0 \geq_f y$, if the strategy y_0 is a possible reaction to all the Emil's strategies to which y is a possible reaction. In other terms, a Frances' strategy y_0 is more reactive than y when y_0 belongs to the reaction set $f(x)$, for each strategy $x \in f^-(y)$.*

Memento (reciprocal correspondence). We remember that the set $e^-(x)$ is the set of Frances' strategies to which the strategy x can reply with respect to the decision rule e. In fact, the reciprocal image of the strategy x with respect to e is

$$e^-(x) = \{y \in F : x \in e(y)\},$$

therefore it is defined, exactly, as the set of the Frances' strategies y for which x is a possible response strategy. The reciprocal correspondence of e, i.e. the correspondence $e^- : E \to F : x \to e^-(x)$, associates to every Emil's strategy x the set of all those Frances's strategies for which x is a possible reaction. From here the interest in the determination of the correspondence e^-.

Example (of comparison of reactivity). Let (e, f) be the game with strategy spaces $E = [-1, 2]$ and $F = [-1, 1]$ and decision rules $e : F \to E$ and $f : E \to F$ defined by

$$e(y) = \begin{cases} \{-1\} & \text{if } y < 0 \\ E & \text{if } y = 0 \\ \{2\} & \text{if } y > 0 \end{cases} ,$$

$$f(x) = \begin{cases} \{-1\} & \text{if } x < 1 \\ F & \text{if } x = 1 \\ \{1\} & \text{if } x > 1 \end{cases} .$$

we want to determinate the reciprocal multifunctions of e and f. We have

$$e^-(x) = \begin{cases} [-1,0] & \text{if } x = -1 \\ \{0\} & \text{if } x \in \left]-1,2\right[\\ [0,1] & \text{if } x = 2 \end{cases},$$

$$f^-(y) = \begin{cases} [-1,1] & \text{if } y = -1 \\ \{1\} & \text{if } y \in \left]-1,1\right[\\ [1,2] & \text{if } y = 1 \end{cases}.$$

From here we can easily remark that Emil's strategies -1 and 2 are more reactive than all the other Emil's strategies in the interval $]-1,2[$, with respect to the rule e. For this aim, we have to prove that the strategies 1 and 2 belong to the set $e(y)$, for each strategy $y \in e^-(x)$. Let, so, x be in $]-1,2[$, we have $e^-(x) = \{0\}$, then the relation $y \in e^-(x)$ is equivalent to $y = 0$, but the image $e(0)$ is all E therefore include -1 and 2. Analogously, we prove that Frances' strategies -1 and 1 are more reactive than every other strategy $y \in]-1,1[$, with respect to f.

The following theorem expresses the reactivity comparison in conditional form.

Theorem. *In the conditions of the previous definition. An Emil's strategy x_0 is more reactive than an Emil's strategy x, with respect to the decision rule e, if for each Frances' strategy y, from $x \in e(y)$ we have $x_0 \in e(y)$. In symbols, the relation $x_0 \geq_e x$ holds if and only if*

$$(\forall y \in F)(x \in e(y) \Rightarrow x_0 \in e(y)).$$

Analogously, a Frances' strategy y_0 is more reactive than an other Frances' strategy y, with respect to the decision rule f, if for each Emil's strategy x, from $y \in f(x)$ we have $y_0 \in f(x)$. In symbols, the relation $y_0 \geq_f y$ holds if and only if

$$(\forall x \in E)(y \in f(x) \Rightarrow y_0 \in f(x)).$$

3.4 The reactivity preorder

It is immediate to verify that the relation of reactivity comparison determined by the decision rule f upon the strategy space F, defined, for each pair of strategies (y, y'), by $y \geq_f y'$, and that we denote by $\geq_f$, is a preorder. This justifies the following definition.

Definition (of reactivity preorder). *Let (e, f) be a decision form game upon the underlying strategy pair (E, F). The binary relation $\geq_f$ on the strategy set F is called **preorder of reactivity induced by the decision rule** f **on Frances' strategy space**. Symmetrically, the binary relation $\geq_e$ on the strategy space E is called **preorder of reactivity induced by the decision rule** e **on Emil's strategy space**.*

Remark (strict preorder of reactivity). Since the reactivity comparison $\geq_f$ is a preorder, it has an associated strict preorder, the preorder $>_f$ defined, as usual, for each pair of strategies (y_0, y) by $y_0 >_f y$ if and only if $y_0 \geq y$ and $y \ngeq y_0$. Analogous consideration holds for Emil.

Now we see an example of strict comparison of reactivity.

Example (of strict reactivity comparison). Let (e, f) be the game with strategy spaces $E = [-1, 2]$ and $F = [-1, 1]$ and with decision rules $e : F \to E$ and $f : E \to F$ defined by

$$e(y) = \begin{cases} -1 & \text{if } y < 0 \\ E & \text{if } y = 0 \\ 2 & \text{if } y > 0 \end{cases},$$

$$f(x) = \begin{cases} -1 & \text{if } x < 1 \\ F & \text{if } x = 1 \\ 1 & \text{if } x > 1 \end{cases}.$$

We apply the conditional characterization to prove (again) that Emil's strategies -1 and 2 are more reactive than all the strategies of the open interval $]-1, 2[$. In fact, for each Frances' strategy y, if a Emil's strategy $x \in]-1, 2[$ is in $e(y)$, y must necessarily be 0 (because it is the unique $e(y)$ that contains numbers different from -1 and 2 is $e(0)$), but, in that case, whether -1 or 2 belong to $e(y)$ (inasmuch, in reality, it is $e(0) = E$). We have demonstrated that $-1, 2 \geq_e x$, for each x in E. Now we prove that $-1, 2 >_e x$, for each $x \in]-1, 2[$ (i.e. that -1 and 2 are strictly more reactive strategies than every other Emil's strategy). It is sufficient prove that, to fix the ideas, the relation $2 \leq_e x$ is false; or, it is sufficient also show that a strategy y in F exists such that $2 \in e(y)$ and $x \notin e(y)$ (or a strategy which 2 reacts and x no). Let be $y = 1$, we have $e(y) = \{2\}$, then 2 is in $e(y)$ and any $x \in]-1, 2[$ no.

3.5 The reactivity of a strategy

Terminology (reciprocal decision rule). Let $f : E \to F$ be a Frances' decision rule. We can associate, in a natural way, with the correspondence f the Emil's decision rule

$$f^{-} : F \to E : y \mapsto f^{-}(y),$$

that we call *Emil's decision rule reciprocal of the Frances's decision rule* f. This reciprocal decision rule is canonically associated with the application of F into the set of subsets of E that associates with every Frances's strategy y the set of all Emil's strategies for which y is a possible reaction. With abuse of language, we will name this application *reciprocal function of the correspondence* f.

Theorem (characterization of the preorder of reactivity). *The (opposite) reactivity preorder* $\leq_f$ *is the preorder induced (in the usual sense) by the reciprocal function of the decision rule* f*, that is by the function*

$$F \to \mathcal{P}(E) : y \mapsto f^{-}(y),$$

endowing the set of the parts of E *(denoted by* $\mathcal{P}(E)$*) with the order of set inclusion* $\subseteq$*. In other terms, the opposite preorder of reactivity is the reciprocal image of the set inclusion order with respect to the reciprocal function of the decision rule* f*.*

Proof. Let $x \in E$ and $y \in F$ be strategies. The relation $y \in f(x)$ is equivalent to the relation $x \in f^{-}(y)$, therefore a Frances's strategy y_0 is more reactive than y if and only if $f^{-}(y) \subseteq f^{-}(y_0)$. ■

The above characterization allows to give the following definition.

Definition (of reactivity). *Let* (e, f) *be a decision form game. For each* x *in* E*, the reciprocal image of the strategy* x *by the correspondence* e*, that is the set* $e^{-}(x)$*, is called the* ***reactivity of*** x *with respect to* e*. Analogously, for each* y *in* F*, the reciprocal image of* y *by the decision rule* f *is called the* ***reactivity of*** y *with respect to* f*.*

3.6 Super-reactive strategies and maxima

The following obvious result characterizes super-reactive strategies of a player as maxima (upper optima) of the strategy space of the player with respect to the reactivity preorder induced by his decision rule. Therefore it allows to reduce the concept of super-reactive strategy to the concept of optimum.

Theorem (characterization of super-reactivity). *Let (e, f) be a decision form game. Every Frances' super-reactive strategy is a maximum of the preorder space $(F, \geq_f)$.*

Remark (on the nature of super-reactive strategies). After the realization of the true nature of super-reactive strategies, we can observe some of the previous examples in an other way. We have, in fact, seen that there are situations in which Frances has no super-reactive strategies, this simply means that the preorder space $(F, \geq_f)$ has no maxima, this does not surprise, in fact a preorder space has maxima only in very particular cases. Obviously, when a space has no maxima (as observed in preorder spaces theory) we have to look for other solutions of the corresponding decision problem (Pareto boundaries, cofinal and coinitial parts, suprema and so on) but we shall analyze this aspects in the following paragraphs.

We should notice that, in general, the space $(F, \geq_f)$ is not an ordered space, and therefore several maxima can exist (they must necessarily be indifferent between themselves by the theorem of indifference of optima in preordered spaces), as show the following example.

Example (of distinct and indifferent super-reactive strategies). Let (e, f) be the game with strategy spaces the two intervals of the real line $E = [a, b]$ and $F = [c, d]$ and decision rules $e : F \to E$ and $f : E \to F$ defined by

$$e(y) = \begin{cases} \{a, b\} & \text{if } y < 0 \\ E & \text{if } y = 0 \\ \{a, b\} & \text{if } y > 0 \end{cases},$$

$$f(x) = \begin{cases} \{c, d\} & \text{if } x < 1 \\ F & \text{if } x = 1 \\ \{c, d\} & \text{if } x > 1 \end{cases},$$

for each bistrategy (x, y) of the game. It is easy to realize that the two strategies a and b are super-reactive for Emil, and, because they are maxima of the set E with respect to the preorder $\geq_e$, they are indifferent. Let us see this directly. The set of Frances' strategies for which a is a possible reaction is $e^-(a) = F$, from which immediately follows that a is a maximum of the space $(E, \geq_e)$ (no Emil's strategy can be more reactive than a inasmuch the strategy a is a reaction for all Frances' strategies). Analogously, we can proceed for b (that is exactly in the same situation).

Remark (on the indifference in reactivity of strategies). We note that the reactivity indifference of two Emil's strategies x and x' is equivalent to the relation $e^-(x) = e^-(x')$. In fact, the preorder $\leq_e$ is induced by the function of E in $\mathcal{P}(F)$ defined by $x \mapsto e^-(x)$ with respect of the set inclusion, and therefore x and x' are equivalent in reactivity if and only if they have the same value in e.

3.7 Maximally reactive strategies

Definition (of maximally reactive strategy). *Let (e, f) be a game upon the underlying strategy pair (E, F). A Frances' strategy $y \in F$ is called **maximally reactive** if does not exist another Frances' strategy strictly more reactive than y (i.e., as we shall see later, if the strategy y is not a sub-reactive strategy). In other terms, a Frances' strategy is called maximally reactive if is (Pareto) maximal in the preorder space $(F, \geq_f)$. Analogously, an Emil's strategy is called maximally reactive if it is (Pareto) maximal in the preorder space $(E, \geq_e)$.*

Example (of maximally reactive strategy). Let (e, f) be the decision form game with strategy spaces $E = [-1, 2]$ and $F = [-1, 1]$ and decision rules $e : F \to E$ and $f : E \to F$ defined by

$$e(y) = \begin{cases} \{-1\} & \text{if } y < 0 \\ E & \text{if } y = 0 \\ \{2\} & \text{if } y > 0 \end{cases},$$

$$f(x) = \begin{cases} \{-1\} & \text{if } x < 1 \\ F & \text{if } x = 1 \\ \{1\} & \text{if } x > 1 \end{cases}.$$

The reciprocal correspondences of e and f are

$$e^-(x) = \begin{cases} [-1,0] & \text{if } x = -1 \\ \{0\} & \text{if } x \in\,]-1,2[\\ [0,1] & \text{if } x = 2 \end{cases},$$

$$f^-(y) = \begin{cases} [-1,1] & \text{if } y = -1 \\ \{1\} & \text{if } y \in\,]-1,1[\\ [1,2] & \text{if } y = 1 \end{cases}.$$

Hence we can easily note that the Emil's strategies -1 and 2 are maximally reactive. To fix the ideas, we study the strategy 2. It is sufficient to show that the subset $e^-(2)$ is not strictly included in any other image $e^-(x)$, and this is evident. We have seen before that these two maximal strategies are more reactive than all other Emil's strategies $x \in\,]-1,2[$, with respect to the rule e: therefore all the Emil's strategies, with the exception of the two maximal ones, are strictly less reactive than the maximal; moreover all Emil's strategies in $]-1,2[$ are indifferent between them (have same image through e^-), we see so that the interval $]-1,2[$ is even the set of all the minima of the preorder space $(E, \geq_e)$. Analogously, we can prove that the strategies -1 and 1 form the maximal boundary of the preorder space $(F, \geq_f)$.

3.8 Sub-reactive strategies

Definition (of sub-reactive strategy). *A strategy s is called **sub-reactive** if there exists a strategy s' strictly more reactive strategy than s. In other terms, a Frances' strategy is called sub-reactive if it is not (Pareto) maximal in the preorder space $(F, \geq_f)$. Analogously, an Emil's strategy is called sub-reactive if it is not Pareto maximal in the preorder space $(E, \geq_e)$.*

Example (of sub-reactive strategy). Let (e, f) be the game with strategy spaces $E = [-1, 2]$ and $F = [-1, 1]$ and decision rules $e : F \to E$ and $f : E \to F$ defined by

$$e(y) = \begin{cases} \{-1\} & \text{if } y < 0 \\ E & \text{if } y = 0 \\ \{2\} & \text{if } y > 0 \end{cases}, \quad f(x) = \begin{cases} \{-1\} & \text{if } x < 1 \\ F & \text{if } x = 1 \\ \{1\} & \text{if } x > 1 \end{cases}.$$

We have before seen that the two Emil's maximal strategies -1 and 2 are more reactive than all the other Emil's strategies $x \in\,]-1,2[$, with respect to the rule e: therefore all Emil's strategies, except the maximal, are sub-reactive.

3.9 Elimination of sub-reactive strategies

Definition (of reduced game by elimination of sub-reactive strategies). *A game (e, f) is called* ***reduced by elimination of*** **sub-reactive** ***strategies*** *if the maximal (Pareto) boundaries of the preorder spaces $(E, \geq_e)$ and $(F, \geq_f)$ coincide with the strategy sets E and F, respectively.*

Example (of not reduced game). Let (e, f) be the game with strategy spaces $E = [-1, 2]$ and $F = [-1, 1]$ and decision rules $e : F \to E$ and $f : E \to F$ defined by

$$e(y) = \begin{cases} \{-1\} & \text{if } y < 0 \\ E & \text{if } y = 0 \\ \{2\} & \text{if } y > 0 \end{cases} ,$$

$$f(x) = \begin{cases} \{-1\} & \text{if } x < 1 \\ F & \text{if } x = 1 \\ \{1\} & \text{if } x > 1 \end{cases} .$$

The maximal boundaries of the preorder spaces $(E, \geq_e)$ and $(F, \geq_f)$ are the sets $\{-1, 2\}$ and $\{-1, 1\}$, therefore the game is not reduced by elimination of sub-reactive strategies.

Before to proceed by the following definition, we recall the notion of subgame of a decision form game.

Definition (of subgame). *Let (e, f) be a decision form game upon the pair (E, F) and let (E', F') be a sub-underlying pair of (E, F), i.e. a pair of subsets of E and F, respectively. We call* ***subgame of*** (e, f) ***with underlying pair*** (E', F') *the pair (e', f') having as components the restrictions of the rules e and f to the pairs of sets (F', E') and (E', F'), respectively. We remember that, for example, e' is the correspondence from F' to E' which sends a strategy y' of F' into the intersection $e(y') \cap E'$. In other terms, e' sends every strategy y' of F' into all Emil's reaction strategies to y' which are in E'.*

Definition (reduction of a game by elimination of sub-reactive strategies). *Let $G = (e, f)$ be a decision form game with underlying pair (E, F). We call* ***reduction of the game*** (e, f) ***by elimination of*** **sub-reactive** ***strategies*** *the subgame (e', f') of G with underlying pair the pair of the maximal Pareto boundaries $\overline{\partial}_e E$ and $\overline{\partial}_f F$ of the preorder spaces $(E, \geq_e)$*

and $(F, \geq_f)$. In other terms, the ***reduction of the game*** *(e, f)* ***by elimination of the*** **sub-reactive** ***strategies*** *is the game of decision rules the restrictions $e_{|(F',E')}$ and $f_{|(E',F')}$, where E' and F' are the maximal Pareto boundaries $\overline{\partial}_e E$ and $\overline{\partial}_f F$ of the preordered spaces $(E, \geq_e)$ and $(F, \geq_f)$.*

Example (of reduction). Let (e, f) be the game with strategy spaces $E = [-1, 2]$ and $F = [-1, 1]$ and decision rules $e : F \to E$ and $f : E \to F$ defined by

$$e(y) = \begin{cases} \{-1\} & \text{if } y < 0 \\ E & \text{if } y = 0 \\ \{2\} & \text{if } y > 0 \end{cases},$$

$$f(x) = \begin{cases} \{-1\} & \text{if } x < 1 \\ F & \text{if } x = 1 \\ \{1\} & \text{if } x > 1 \end{cases}.$$

The maximal boundaries of the preordered spaces $(E, \geq_e)$ and $(F, \geq_f)$ are the sets $E_1 = \{-1, 2\}$ and $F_1 = \{-1, 1\}$, therefore the game is not reduced because they don't coincide with respective spaces. The reduction of (e, f) by elimination of sub-reactive strategies is the game with decision rules $e_1 : F_1 \to E_1$ and $f_1 : E_1 \to F_1$ defined by

$$e_1(y) = \begin{cases} -1 & \text{if } y = -1 \\ 2 & \text{if } y = 1 \end{cases},$$

$$f_1(x) = \begin{cases} -1 & \text{if } x = -1 \\ 1 & \text{if } x = 2 \end{cases}.$$

Example (of reduced game). We note that the game (e_1, f_1) of previous example is reduced. In fact, the reciprocals of e_1 and f_1 are defined by

$$e_1^-(x) = \begin{cases} \{-1\} & \text{if } x = -1 \\ \{1\} & \text{if } x = 2 \end{cases},$$

$$f_1^-(y) = \begin{cases} \{-1\} & \text{if } y = -1 \\ \{2\} & \text{if } y = 1 \end{cases}.$$

Maximal boundaries of the preordered spaces $(E_1, \geq_{e_1})$ and $(F_1, \geq_{f_1})$ are the sets $E_2 = \{-1, 2\}$ and $F_2 = \{-1, 1\}$, therefore the game is reduced because E_2 and F_2 coincide with the respective spaces. For an easy determination of the boundaries, we note that, for example, the preordered space $(E_1, \geq_{e_1})$ is isomorphic to the preordered space with two elements $(\{\{1\}, \{-1\}\}, \subseteq)$.

3.10 Iterated elimination of sub-reactivity

Definition (of reducing sequence of a game). *Let $G_0 = (e_0, f_0)$ be a game on a strategy base (E_0, F_0). We call **reducing sequence by elimination of*** **sub-reactive** ***strategies of*** *G_0 the sequence of subgames $G = (G_k)_{k=0}^{\infty}$, with first term the game G_0 and with k-th term the game $G_k = (e_k, f_k)$, such that the strategy base (E_k, F_k) of G_k be the pair of maximal boundaries of the preordered spaces $(E_{k-1}, \geq_{e_{k-1}})$ and $(F_{k-1}, \geq_{f_{k-1}})$, of $(k-1)$-th subgame, for each positive integer k. So, decision rules e_k and f_k are the restrictions to the pairs (F_k, E_k) and (E_k, F_k) of decision rules e_{k-1} and f_{k-1}, respectively.*

Definition (of solubility by iterated elimination of sub-reactive strategies). *Let $G_0 = (e_0, f_0)$ be a decision form game, and let G be its reducing sequence by elimination of sub-reactive strategies. The game G_0 is called **solvable by iterated elimination of*** **sub-reactive strategies** *if there exists only one bistrategy common to all subgames of the sequence G. In that case, that bistrategy is called the **solution*** **by iterated elimination of** ***sub-reactive strategies of the game*** *G_0.*

Remark. The definition of solubility by iterated elimination of sub-reactive strategies is so equivalent to contain the intersection $\bigcap_{k=1}^{\infty} E_k \times F_k$ only one element.

Remark. If the game G_0 is finite, it is solvable by iterated elimination of sub-reactive strategies if and only if there exists a subgame of the sequence G with only one bistrategy; in that case, that bistrategy is the solution by iterated elimination of sub-reactive strategies of the game G_0.

3.11 Relative super-reactivity

Definition (of relatively super-reactive strategy). *Let (e, f) be a two player game. Let E' be a set of Emil's strategies to which Frances can react and let y_0 be a Frances' strategy. The strategy y_0 is called **relatively super-reactive to*** E' ***(with respect to the decision rule*** f***)*** *if is possible reaction to all the Emil's strategies in E'. In other terms, a Frances' strategy y_0 is called relatively super-reactive to E' if belongs to the set $f(x)$, for each Emil's* *strategy x in E'. Analogously, let F' be a set of Frances' strategies to which Emil can react and x_0 a Emil's strategy. The strategy x_0 is called* ***relatively super-reactive for*** F' ***(with respect to the decision rule*** e***)*** *if it is a possible reaction to all the Frances' strategies in F'. In other terms, a Emil's strategy x_0 is called relatively super-reactive to F' if belongs to the set $e(y)$, for each Frances' strategy y in F'.*

Remark. So the sets of Emil and Frances' relatively super-reactive to F' and to E' strategies are the two intersections $\cap_{F'} e = \bigcap_{y \in F'} e(y)$ and $\cap_{E'} f = \bigcap_{x \in E'} f(x)$. Evidently these intersections can be empty.

Example (of relatively super-reactive strategies). Let (e, f) be the game with strategy spaces $E = [-1, 2]$ and $F = [-1, 1]$ and decision rules $e : F \to E$ and $f : E \to F$ defined by

$$e(y) = \begin{cases} -1 & \text{if } y < 0 \\ E & \text{if } y = 0 \\ 2 & \text{if } y > 0 \end{cases} ,$$

$$f(x) = \begin{cases} -1 & \text{if } x < 1 \\ F & \text{if } x = 1 \\ 1 & \text{if } x > 1 \end{cases} .$$

Emil has only a relatively super-reactive strategy with respect to not negative strategies and only relatively super-reactive strategy with respect to Frances' not positive strategies. In fact

$$\bigcap_{y \in [0,1]} e(y) = E \cap \{2\} = \{2\}$$

and

$$\bigcap_{y \in [-1,0]} e(y) = \{-1\} \cap E = \{-1\} .$$

Frances is in a situation similar with respect to Emil's greater or equal to 1 strategies and less or equal 1 strategies, in fact

$$\bigcap_{x \in [1,2]} f(x) = F \cap \{1\} = \{1\}$$

and

$$\bigcap_{x \in [-1,1]} f(x) = F \cap \{-1\} = \{-1\} .$$

Following theorem has an obvious proof.

Theorem (on reactivity). *Let x a Emil's strategy, most great set of Frances's strategies F' such that x let be relatively super-reactive to F' is the reactivity of x.*

Example (of reactivity). Let (e, f) be the game of above example. The reactivity of Emil's strategy 2 is the interval $[0, 1]$, the reactivity of Emil's strategy -1 is the interval $[-1, 0]$. Indeed, these intervals are the most big sets to which the above strategies can react, respectively.

3.12 Dominant strategies

Definition (of dominant strategy). *Let $(u_1, \geq)$ be an Emil's utility function on the bistrategy space $E \times F$ of a strategy base (E, F). An Emil's strategy x_0 in E is called* ***dominant with respect to*** **utility** ***function*** *u_1 if, for each x in E, the inequality*

$$u_1(x_0, y) \geq u_1(x, y),$$

holds, for each strategy y in F. In other terms, an Emil's strategy x_0 in E is called dominant if, for each other strategy x in E, the function inequality

$$u_1(x_0, .) \geq u_1(x, .)$$

holds true. Analogously, let $(u_2, \geq)$ be a Frances' utility function on the bistrategy space $E \times F$ of a strategy base (E, F). A strategy y_0 in F **is called dominant with respect to** ***utility*** **function** *u_2 if, for each y in F, the inequality*

$$u_2(x, y_0) \geq u_2(x, y),$$

holds, for each strategy x in E. In other terms, a Frances' strategy y_0 in F is called u_2-dominant if, for each other strategy y in F, the function inequality

$$u_2(., y_0) \geq u_2(., y)$$

holds true.

3.13 Dominant and super-reactive strategies

Let us see the first relationship between dominance and reactivity.

Theorem (characterization of dominant strategies). *Let $(u_1, \geq)$ and $(u_2, \geq)$ be respectively Emil's and Frances' utility functions and let B_1 and B_2 be the respective decision rules induced by the two functions u_1 and u_2. Then, an Emil's strategy x_0 is u_1-dominant if and only if it is B_1-super-reactive and, analogously, a Frances' strategy y_0 is u_2-dominant if and only if it is B_2-super-reactive.*

Proof. Let x_0 be a super-reactive strategy with respect to the decision rule B_1. Then, the strategy x_0 belongs to the reaction set $B_1(y)$, for each y in F. So, for each y in F, we have the equality

$$u_1(x_0, y) = \max u_1(., y),$$

that means

$$u_1(x_0, y) \geq u_1(x, y),$$

for each x in E and for each y in F. The vice versa can be proved following the preceding steps in opposite sense. ■

3.14 The preorder of dominance

Definition (of dominance). *Let $(u, \geq)$ be a normal-form game on the bistrategy space $E \times F$ of a strategy base (E, F). We say that **an** **Emil's** **strategy** x_0 **dominates (in wide sense) an other** **Emil's** **strategy** x **with respect to the** **utility** **function** u_1 if the partial function $u_1(x_0, \cdot)$ is greater (in wide sense) of the partial function $u_1(x, \cdot)$. In this case we write $x_0 \geq_{u_1} x$. We say that **an** **Emil's** **strategy** x_0 **dominates strictly an other** **Emil's** **strategy** x **with respect to the** **utility** **function** u_1 if the partial function $u_1(x_0, \cdot)$ is strictly greater than the partial function $u_1(x, \cdot)$. In this case we write $x_0 >_{u_1} x$. We say that **an** **Emil's** **strategy** x_0 **dominates strongly an other** **Emil's** **strategy** x **with respect to the** **utility** **function** u_1*

if the partial function $u_1(x_0, \cdot)$ is strongly greater than the partial function $u_1(x, \cdot)$. In that case we will write $x_0 \gg_{u_1} x$.

Memento (usual order on $\mathcal{F}(X, \mathbb{R})$). Let X be a non-empty set, we remember that a real function $f : X \to \mathbb{R}$ is called greater (in a wide sense) than an other function $g : X \to \mathbb{R}$, and we will write it $f \geq g$, if the inequality (in a wide sense)

$$f(x) \geq g(x),$$

holds for each x in X. The increase is called strict, and we will write it $f > g$, if f è greater (in a wide sense) than g but different. The f is called strongly greater than g, and we will write it $f \gg g$, if holds the strict inequality

$$f(x) > g(x),$$

for each x in X. The relation of majoring $\geq$ on the function's space $\mathcal{F}(X, \mathbb{R})$ is a order and is called *usual order on* $\mathcal{F}(X, \mathbb{R})$. We note that the relation $f \geq g$ is equivalent to the inequality

$$\inf (f - g) \geq 0.$$

Remark. We easily prove that the relation of dominance $\geq_{u_1}$ is a preorder on E. Actually, it is the reciprocal image of the usual order of the space of real functionals on F (the space $\mathcal{F}(F, \mathbb{R})$) with respect to the application $E \to \mathcal{F}(F, \mathbb{R})$ defined by $x \mapsto u_1(x, .)$.

Theorem (Characterization of the strict dominance for Weierstrass' functions). *Let $f_1 : E \times F \to \mathbb{R}$ be a Weierstrass' functional (that is, assume that there are topologies t_1 and t_2 on E and F respectively such that f_1 is continuous and defined on a compact, with respect to the product of the two topologies). Then, assuming the functional f_1 represent the Emil's disutility, the condition $x_0 \gg_{f_1} x$ is equivalent to the inequality*

$$\sup(f_1(x_0, .) - f_1(x, .)) < 0.$$

Proof. Necessity. Let the strong dominance $x_0 \gg_{f_1} x$ hold. Then the difference function $g = f_1(x_0, .) - f_1(x, .)$ is negative and moreover there exists (by the Weierstrass Theorem) a point y_0 in F such that the real $g(y_0)$ is the supremum of g, hence

$$\sup\, g = g(y_0) < 0.$$

Sufficiency (the Weierstrass' hypotheses are not necessary). If the supremum of g is negative, every value of g must be negative. ■

3.15 Dominance and reactivity

The following theorem explains the relationship between dominance and reactivity comparison.

Theorem (on the preorder of reactivity). *Let $(u_1, \geq)$ and $(u_2, \geq)$ be ,respectively, two Emil's and Frances' utility function, and let B_1 and B_2 be the decision rules induced by the two functions u_1 and u_2 respectively. Then, the reactivity preorder $\geq_{B_i}$ is a refinement of the preorder of dominance $\geq_{u_i}$.*

Proof. We show before that the preorder of reactivity refines the preorder of dominance. Let $x_0 \geq_{u_1} x$, then $u_1(x_0, .) \geq u_1(x, .)$, from which, if $y \in F$ and $x \in B_1(y)$ we have $x_0 \in B_1(y)$. In fact, $x \in B_1(y)$ means that

$$u_1(x, y) = \max u_1(., y)$$

but, because $u_1(x_0, y) \geq u_1(x, y)$, we have also

$$u_1(x_0, y) = \max u_1(., y),$$

i.e. $x_0 \in B_1(y)$. ■

The preorder of reactivity, in general, is at proper refinement of the preorder of dominance, as shows following example.

Example. Let (B_1, B_2) be Cournot decision game with bistrategic space $[0, 1]^2$ and net cost functions f_1 and f_2 defined by

$$f_1(x, y) = x\,(x + y - 1)\,,$$

and, symmetrically,

$$f_2(x, y) = y\,(x + y - 1)\,.$$

We easily see that every strategy in $[0, 1/2]$ is strictly more reactive than any strategy $x > 1/2$, in fact the reactivity of any strategy $x > 1/2$ is the empty set (it is a non-reactive strategy). In particular, we have $0 >_{B_1} 3/4$. On the other hand, the function $f_1(0, .)$ is the zero real functional on $[0, 1]$; on the contrary the partial function $f_1(3/4, .)$ is defined by

$$f_1(3/4, .)(y) = (3/4)(y - 1/4),$$

for each y in $[0, 1]$, since this last function has positive and negative values, it is incomparable with the zero function, with respect to usual order of the space of functions $\mathcal{F}(F, \mathbb{R})$. Consequently, the preorder $\geq_{B_1}$ is a *proper refinement* of the preorder $\geq_{f_1}$.

3.16 Non-reactivity and strong dominance

Another used concept in game theory is that of strongly dominated strategy (often it is known in literature as strictly dominated strategy, that for us is less demanding concept).

Definition (of strongly dominated strategy). *Let $(u, \geq)$ be a multi-utility function on the bistrategy space of a two player game. Let (E, F) be the pair of the strategy sets of the two players (a game base). We say that a strategy x_0 of E is* ***an*** **Emil's** ***strongly dominated strategy*** *if there exists an other strategy x_1 in E such that the strict inequality*

$$u_1(x_0, y) < u_2(x_1, y),$$

holds for each strategy y in F. In other terms, we say that a strategy $x_1 \in E$ ***dominates strongly*** *a strategy $x_0 \in E$, and is written $x_1 \gg_{u_1} x_0$, if the partial function $u_1(x_0, \cdot)$ is* ***strongly*** *less then the partial function $u_1(x_1, \cdot)$.*

The following theorem explains the relationships between the non-reactive strategies and the strongly dominated strategy.

Theorem (strongly dominated strategies as never best response). *Let $(u_1, \geq)$ and $(u_2, \geq)$ be respectively Emil's and Frances' utility functions and let B_1 and B_2 be the decision rules induced by two functions u_1 and u_2 respectively. Then, if a strategy is strongly dominated with respect to u_i it is non-reactive with respect to B_i.*

Proof. Let x_0 be an Emil's u_1-strongly dominated strategy, then there is at least a strategy x in E such that $u_1(x_0, y) < u_1(x, y)$, for every y in F. Hence the strategy x_0 cannot be a best response to any strategy y in F, since x is a response to y strictly better than x_0, for every y in F; so the reactivity $B_1^-(x_0)$ is empty. ■

To be a strongly dominated strategy is more restrictive than to be a never best response strategy, as shows following example.

Example (an undominated and never-best response strategy). Let $E = \{1, 2, 3\}$ and $F = \{1, 2\}$ be the strategy sets of a two players normal-form games $(u, \geq)$, and let u_1 be the Emil's utility function defined by

$$\begin{aligned} u_1(1,1) &= u_1(1,2) = 0 \\ u_1(2,1) &= u_1(3,2) = 1, \\ u_1(2,2) &= u_1(3,1) = -1. \end{aligned}$$

We can summarize the function u_1 in a utility matrix m_1, as follows

$$m_1 = \begin{pmatrix} 0 & 0 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

It is evident in m_1 that Emil's strategy 1 (leading to the first payoff-row) is u_1-incomparable with the other strategies 2 and 3, and then it cannot be strongly dominated (neither strictly dominated). On the other hand, 1 is an Emil's never best response strategy (i.e., it is non-reactive with respect to the best reply rule B_1).

Chapter 4

Mixed Extensions

4.1 Introduction

The Brouwer fixed point theorem and the Kakutani fixed point theorem represent, together with the separation theorems, the main instruments to prove the existence of equilibria in decision form games. These theorems require the convexity of the strategy sets. This hypothesis is hardly paid: it excludes, for example, the quite natural situation of finite sets of strategies. In his famous book written with O. Morgenstern, John Von Neumann, changing perspective, conceived situations where the assumption of convexity becomes natural and where it is needed to extend the finite context providing new sharp solutions. This latter Von Neumann's intuition leads to *the canonical convexification of a strategy space.*

Definition (of canonical convexification). *Let E be a finite set of m elements. We identify the set E with the set $\underline{m}$ of the first m positive integers and define* ***canonical convexification of*** *E, in the euclidean space $\mathbb{R}^m$, or* ***canonical mixed extension of*** *E, the set*

$$\mathbb{M}_m := \{p \in \mathbb{R}^m : p \geq 0 \ \text{ et } \ \|p\|_1 = \Sigma p = 1\},$$

i.e., the canonical $(m-1)$-simplex of $\mathbb{R}^m$.

Remark. The canonical convexification of a strategy set E with m elements is clearly a compact and convex subset of $\mathbb{R}^m$.

Remark (canonical immersion). We can imbed the finite strategy set E in the canonical simplex $\mathbb{M}_m$, through the function μ mapping the i-th element of E (we mean the element corresponding with the integer i in the chosen identification of E with $\underline{m}$) into the i-th element μ_i of the canonical basis μ of the vector space $\mathbb{R}^m$, that is the mapping defined by

$$\mu : E \to \mathbb{M}_m : i \mapsto \mu\,(i) := \mu_i.$$

Obviously, the function μ is injective, and it is said the *canonical immersion of the finite set E into the canonical simplex $\mathbb{M}_m$*. There is no matter of confusion in the identification of the immersion μ with the canonical basis $(\mu_i)_{i=1}^m$ of the vector space $\mathbb{R}^m$, since this one is nothing but the family indexed by the set $\underline{m}$ and defined by $\mu(i) := \mu_i$ (recall that a family x of points of a set X is a surjective function from an index set I onto a subset of X, and it is denoted by $(x_i)_{i \in I}$).

Remark (canonical simplex as convex envelope of the canonical basis). We note again, that the canonical simplex $\mathbb{M}_m$ is the convex envelope of the canonical base μ of the vector space $\mathbb{R}^m$, so we have in symbols $\mathbb{M}_m = \mathrm{conv}(\mu)$.

Remark (canonical simplex as the maximal boundary of the unit $\|.\|_1$-ball). We note moreover, that the canonical simplex $\mathbb{M}_m$ is the maximal boundary (with respect to the usual order of the space $\mathbb{R}^m$) of the unit ball with respect to the standard norm $\|.\|_1 : x \mapsto \Sigma_{i=i}^m |x_i|$, so we have $\mathbb{M}_m = \overline{\partial} B_{\|\cdot\|_1}(0_m, 1)$.

Interpretation (elements of the canonical simplex). John von Neumann proposed to interpret the points of the canonical simplex $p \in \mathbb{M}_m$ as *mixed strategies*. According to this interpretation, a player does not choose a single strategy $i \in E$ but he instead plays all the strategies of his strategy set E, deciding only the probability distribution $p \in \mathbb{M}_m$ according to which any strategy must be played.

Remark (mixed strategies to hide intentions). By adopting a mixed strategy, a player hides his intentions to his opponents. Playing randomly the strategies at his own disposal, by choosing only the probabilities associated to each of them, he prevents his opponents by discovering the strategy that he is going to play, since he himself does not know it.

Remark (mixed strategies as beliefs about the actions of other players). Even if the two players do not desire to hide their own strategic intentions, Emil, for instance, does not know what strategy Frances will adopt, and vice versa. Emil can assume only the probability whereby Frances will play her strategies; so, actually, what Emil is going to face are not the pure strategies adopted by Frances but his own probabilistic beliefs about the Frances' strategies, i.e. the mixed strategies generated by the Frances' process of convexification.

Remark (dynamic). By convexifying the sets of strategies, we are no longer in the original static context, because this random game can be seen as a repeated game. The convexification is a first step towards a dynamic context.

Remark (cooperative game). This process of convexification can be adopted also in the context of cooperative games, where we can convexify the sets *of player coalitions.*

4.2 Mixed extension of vector correspondences

After we have convexify the strategy space of a player, we should extend in a consistent manner all the functions and correspondences defined on it. The following definition is a first step in this direction and it extends the correspondences defined on the strategy space of a player and with values in a vector space.

Definition (of canonical extension). *Let $\vec{X}$ be a vector space (carried by the set X), let $\underline{m}$ be the set of the first m natural numbers and let $c : \underline{m} \to X$ be a correspondence. We say* ***canonical extension of the correspondence*** *c (to the vector space $\mathbb{R}^m$) the multifunction ${}^{\mathrm{ex}}c : \mathbb{R}^m \to X$ defined by*

$$ {}^{\mathrm{ex}}c(q) := \sum_{i=1}^{m} q_i c(i), $$

for each vector q in $\mathbb{R}^m$.

Remark. Note that the above definition works, since in a vector space we can sum two subsets and multiply a subset by a scalar, obtaining other subsets of the space.

Example (extension of a function). Let E be the set of the first three natural numbers and $c : E \to \mathbb{R}^4$ the correspondence defined by $c(i) = i\mu_{i+1}$, for any element i in E, where μ is the canonical basis of the vector space $\mathbb{R}^4$. For each triple $q \in \mathbb{R}^3$, we have

$$\begin{aligned} {}^{\mathrm{ex}}c(q) &= \sum_{i=1}^{3} q_i c(i) = \\ &= \sum_{i=1}^{3} q_i (i\mu_{i+1}) = \\ &= (0, q_1, 2q_2, 3q_3). \end{aligned}$$

Remark. If μ is the canonical immersion of the set E into the canonical simplex $\mathbb{M}_m$ (defined above), the following diagram will commute

$$\begin{array}{l} \mathbb{R}^m \overset{{}^{\mathrm{ex}}c}{\to} X \\ \uparrow_\mu \;\; \nearrow_c \\ E \end{array} \quad .$$

We have the following obvious but interesting result.

Proposition. *Let $c : E \to X$ be a function from the finite set E into a vector space $\vec{X}$ (i.e. assume that the correspondence c maps each element of the set E into a unique element of X). Then, its canonical extension ${}^{\mathrm{ex}}c$ is a linear function from the vector space $\mathbb{R}^m$ into the vector space $\vec{X}$.*

Remark (the linearization process induced by a convexification). The process that associates with the function $c : E \to X$ the linear function ${}^{\mathrm{ex}}c : \mathbb{R}^m \to X$ can be thought as a process of linearization associated to the convessification process which transforms the finite set E into the convex compact set $\mathbb{M}_m$.

4.3 Mixed extension of finite decision form games

In this section we define the mixed extension of a finite decision-form game (e, f). To this purpose, once convexified the strategy spaces of the players, we should extend in a consistent manner the decision rules defined between them. The following definition provides the extension of a decision rule in this case.

Definition (canonical extension of a decision rule). *Let $G = (e, f)$ be a game with a strategy carrier (E, F), let E be the set of the first m natural numbers and F the set of the first n natural numbers. We say **canonical extension of the decision rule** $e : F \to E$ to the pair of spaces $(\mathbb{R}^n, \mathbb{R}^m)$ the correspondence ${}^{\mathrm{ex}}e : \mathbb{R}^n \to \mathbb{R}^m$ defined by*

$$
{}^{\mathrm{ex}}e(q) := \sum_{j=1}^{n} q_j \mu_{e(j)},
$$

for each q in $\mathbb{R}^n$, where μ represents the canonical immersion of the set E into the vector space $\mathbb{R}^m$.

Remark. Note, for instance in the univocal case, that the vector ${}^{\mathrm{ex}}e(q)$ is a linear combination of the canonical vectors μ_i of $\mathbb{R}^m$. Therefore, if q is chosen in the $(n-1)$-canonical simplex of the space $\mathbb{R}^n$, the vector ${}^{\mathrm{ex}}e(q)$ will be a convex combination of the vectors of the canonical base of $\mathbb{R}^m$ and therefore, it will belong to the $(m-1)$-canonical simplex of $\mathbb{R}^m$. In other words, if q is a Frances' mixed strategy then the vector ${}^{\mathrm{ex}}e(q)$ will be an Emil's mixed strategy.

Definition (mixed extension of a decision-form game). *Let $G = (e, f)$ be a decision form game with a strategy carrier (E, F), where E is set of the first m natural numbers and F the set of the first n natural numbers. Assume $\mathbb{M}_m$ and $\mathbb{M}_n$ be the two convex spaces of mixed strategies of the two players, respectively. We say **mixed extension of the decision form game** G the decision form game ${}^{\mathrm{ex}}G := ({}^{\mathrm{ex}}e, {}^{\mathrm{ex}}f)$, where the decision rules are the multifunctions ${}^{\mathrm{ex}}e : \mathbb{M}_n \to \mathbb{M}_m$ and ${}^{\mathrm{ex}}f : \mathbb{M}_m \to \mathbb{M}_n$ defined by*

$$
{}^{\mathrm{ex}}e(q) := \sum_{j=1}^{n} q_j \mu_{e(j)}, \quad {}^{\mathrm{ex}}f(p) := \sum_{i=1}^{m} p_i \nu_{f(i)},
$$

for each mixed strategy p in $\mathbb{M}_m$ and for each mixed strategy q in $\mathbb{M}_n$, where μ and ν are the canonical immersions of the Emil's and Frances' (finite) strategy spaces into the two canonical simplexes $\mathbb{M}_m$ and $\mathbb{M}_n$, respectively.

Interpretation in Decision Theory. If Emil assumes that Frances will adopt the mixed strategy $q \in \mathbb{M}_n$, he will capable to face all the Frances' pure strategies, i.e. the full strategy system ν (canonical base of $\mathbb{R}^n$), weighed by the system of weights q. Therefore, a rational move for Emil is to play all his own possible reactions to the strategies ν_j, i.e. to play the full reaction system $(\mu_{e(j)})_{j=1}^n$, *using the same distribution of the weights q used by Frances*; in this way Emil will obtain the mixed strategy ${}^{\mathrm{ex}}e(q) := \sum_{j=1}^n q_j \mu_{e(j)}$.

4.4 Extension of finite univocal games

Before to proceed we define a useful tool that will allows us to construct immediately the mixed extension of a decision rule between finite strategy spaces.

Definition (the matrix of a function between finite sets). *Let m and n be two natural numbers and let $f : \underline{m} \to \underline{n}$ be a function from the set $\underline{m}$ of the first m strictly positive natural numbers into the set $\underline{n}$ of the first n strictly positive natural numbers. We say **matrix of the function** f the matrix with m columns and two rows having as first row the vector $(i)_{i=1}^m$, i.e. the m-vector having for i-th component the integer number i, and as second row the vector $(f(i))_{i=1}^m$, i.e. the real m-vector having as i-th component the image $f(i)$ of the integer number i under the function f.*

Example (with univocal rules). Let $\underline{n}$ be the set of the first n strictly positive integers and let $e : \underline{3} \to \underline{2}$ and $f : \underline{2} \to \underline{3}$ the Emil's and Frances' decision rules, respectively, with corresponding matrices

$$M_e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}, \quad M_f = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}.$$

Note that the game $G = (e, f)$ has no equilibria, since the two elements of the set $\underline{2}$ could not be equilibrium strategies for Emil. Indeed, we have for those two strategies the two corresponding evolutionary paths

$$1 \to^f 3 \to^e 2, \quad 2 \to^f 2 \to^e 1.$$

In order to obtain the mixed extension of the game G, we denote by b and b' the canonical bases of the spaces $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. By imbedding the two finite strategy spaces into their respective euclidean spaces, we can transform the two matrices M_e and M_f, obtaining their formal extensions

$${}^{\mathrm{ex}}M_e = \begin{pmatrix} b'_1 & b'_2 & b'_3 \\ b_1 & b_1 & b_2 \end{pmatrix}, \quad {}^{\mathrm{ex}}M_f = \begin{pmatrix} b_1 & b_2 \\ b'_3 & b'_2 \end{pmatrix}.$$

The mixed extensions of the decision rules are so defined, on the canonical simplexes $\mathbb{M}_2$ and $\mathbb{M}_3$ of the two vector spaces $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively, by

$$\begin{aligned} {}^{\mathrm{ex}}e &: \mathbb{M}_3 \to \mathbb{M}_2 : q \to q_1 b_1 + q_2 b_1 + q_3 b_2, \\ {}^{\mathrm{ex}}f &: \mathbb{M}_2 \to \mathbb{M}_3 : p \to p_1 b_3' + p_2 b_2'; \end{aligned}$$

therefore we have

$${}^{\mathrm{ex}}e(q) = (q_1 + q_2, q_3), \quad {}^{\mathrm{ex}}f(p) = (0, p_2, p_1).$$

Now, by imposing the conditions of equilibrium (recall that a bistrategy (x, y) of the game is an equilibrium if and only if $x = e(y)$ et $y = f(x)$) to the pair (p, q), we have

$$p =^{\mathrm{ex}} e(q) = (q_1 + q_2, q_3), \text{ et } q =^{\mathrm{ex}} f(p) = (0, p_2, p_1),$$

that is

$$\begin{cases} p_1 = q_1 + q_2 \\ p_2 = q_3 \end{cases} \quad \text{et} \quad \begin{cases} q_1 = 0 \\ q_2 = p_2 \\ q_3 = p_1 \end{cases};$$

from which we deduce immediately

$$\begin{cases} p_1 = q_1 + q_2 \\ p_2 = q_3 = q_2 = p_1 \\ q_1 = 0 \end{cases};$$

now, taking into account that the two vectors p and q are two probability distributions, we have $p = (1/2, 1/2)$ and $q = (0, 1/2, 1/2)$, so we have found the unique equilibrium $(p.q)$ in mixed strategies of the decision form game G.

4.5 Other univocal examples

Example (morra Chinese). Let the strategies of the two players be 1, 2 and 3 respectively (corresponding with the three strategies *scissors*, *stone* and *paper*). The decision rules of best reply of the two players in the morra Chinese, i.e. the decision rules which impose to reply to the moves of the other player in order to win, are the two decision rules $e : \underline{3} \to \underline{3}$ and $f : \underline{3} \to \underline{3}$ with associated matrices

$$M_e = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad M_f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix};$$

according to the above rules a player must reply to the strategy *scissors* by the strategy *stone*, to *stone* by the strategy paper and to paper by the strategy *scissors*. Note that the decision form game (e, f) has no equilibria, because the three Frances' strategies could not be equilibrium strategies. Indeed, we have the three evolutionary paths corresponding to any of the feasible strategies

$$1 \rightarrow^e 2 \rightarrow^f 3, \quad 2 \rightarrow^e 3 \rightarrow^f 1, \quad 3 \rightarrow^e 1 \rightarrow^f 2.$$

In order to obtain the mixed extension of the game, we denote by b the canonical basis of the vector space $\mathbb{R}^3$. By imbedding the two finite strategy spaces into their respective euclidean spaces, we can transform the two matrices M_e and M_f, obtaining their formal extensions

$$^{\mathrm{ex}}M_e = \begin{pmatrix} b_1 & b_2 & b_3 \\ b_2 & b_3 & b_1 \end{pmatrix}, \quad ^{\mathrm{ex}}M_f = \begin{pmatrix} b_1 & b_2 & b_3 \\ b_2 & b_3 & b_1 \end{pmatrix}.$$

The mixed extensions of the decision rules are defined on the canonical simplex $\mathbb{M}_3$ of the space $\mathbb{R}^3$ by

$$\begin{aligned} ^{\mathrm{ex}}e &: \mathbb{M}_3 \to \mathbb{M}_3 : q \to q_1 b_2 + q_2 b_3 + q_3 b_1, \\ ^{\mathrm{ex}}f &: \mathbb{M}_3 \to \mathbb{M}_3 : p \to p_1 b_2 + p_2 b_3 + p_3 b_1; \end{aligned}$$

therefore we have

$$^{\mathrm{ex}}e(q) = (q_3, q_1, q_2), \quad ^{\mathrm{ex}}f(p) = (p_3, p_1, p_2),$$

for any two mixed strategies p and q in the simplex $\mathbb{M}_3$. By imposing the condition of equilibrium to the pair (p, q), we have

$$\begin{cases} p_1 = q_3 \\ p_2 = q_1 \\ p_3 = q_2 \end{cases} \text{et} \begin{cases} q_1 = p_3 \\ q_2 = p_1 \\ q_3 = p_2 \end{cases},$$

from which we deduce

$$\begin{cases} p_1 = q_3 = p_2 \\ p_2 = q_1 = p_3 \\ p_3 = q_2 = p_1 \end{cases};$$

recalling that p and q are probability distributions (indeed they are elements of the canonical simplex $\mathbb{M}_3$), we find that the pair (p, q), with $p = q = (1/3, 1/3, 1/3)$, is the unique equilibrium in mixed strategies of the game.

Example. Let $\underline{n}$ be the set of the first n positive integers (> 0) and let $e : \underline{3} \to \underline{4}$ and $f : \underline{4} \to \underline{3}$ be the Emil's and Frances' decision rules corresponding to the matrices

$$M_e = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \end{pmatrix}, \quad M_f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 3 \end{pmatrix}.$$

Note that the game $G = (e, f)$ has no equilibria, because the three Frances' strategies could not be equilibrium strategies (for Frances). In fact, we have the three evolutionary orbits corresponding with any of the Frances' strategies

$$1 \to^e 4 \to^f 3, \quad 2 \to^e 3 \to^f 1, \quad 3 \to^e 2 \to^f 2.$$

To obtain the mixed extension of the game, we denote with b and b' the canonical bases of $\mathbb{R}^4$ and $\mathbb{R}^3$, respectively. By imbedding the two finite strategy spaces into their respective euclidean spaces, we can transform the two matrices M_e and M_f, obtaining their formal extensions

$$^{\mathrm{ex}}M_e = \begin{pmatrix} b'_1 & b'_2 & b'_3 \\ b_4 & b_3 & b_2 \end{pmatrix}, \quad ^{\mathrm{ex}}M_f = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b'_3 & b'_2 & b'_1 & b'_3 \end{pmatrix}.$$

The mixed extensions of the decision rules are defined on the canonical simplexes $\mathbb{M}_4$ and $\mathbb{M}_3$ of the vector spaces $\mathbb{R}^4$ and $\mathbb{R}^3$, respectively, by what follows

$$\begin{aligned} ^{\mathrm{ex}}e &: \mathbb{M}_3 \to \mathbb{M}_4 : q \to q_1 b_4 + q_2 b_3 + q_3 b_2, \\ ^{\mathrm{ex}}f &: \mathbb{M}_4 \to \mathbb{M}_3 : p \to p_1 b'_3 + p_2 b'_2 + p_3 b'_1 + p_4 b'_3; \end{aligned}$$

therefore we have

$$^{\mathrm{ex}}e(q) = (0, q_3, q_2, q_1), \quad ^{\mathrm{ex}}f(p) = (p_3, p_2, p_1 + p_4).$$

By imposing the conditions of equilibrium to the pair (p, q), we have

$$\begin{cases} p_1 = 0 \\ p_2 = q_3 \\ p_3 = q_2 \\ p_4 = q_1 \end{cases} \text{et} \begin{cases} q_1 = p_3 \\ q_2 = p_2 \\ q_3 = p_1 + p_4 \end{cases};$$

from which we deduce

$$\begin{cases} p_1 = 0 \\ p_2 = q_3 = p_4 = q_1 \\ p_3 = q_2 = p_2 \end{cases};$$

now, recalling that p and q are probability distributions, we have $p = (0, 1/3, 1/3, 1/3)$ and $q = (1/3, 1/3, 1/3)$, we thus have found the unique equilibrium (p, q) in mixed strategies of the game G.

Example. Let $\underline{n}$ be the set of the first n positive integers (> 0) and let $e : \underline{4} \to \underline{4}$ and $f : \underline{4} \to \underline{4}$ the Emil's and Frances decision rules with matrices

$$M_e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 1 & 2 \end{pmatrix}, \quad M_f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 3 & 4 \end{pmatrix}.$$

Note that the decision form game $G = (e, f)$ has the two "pure" equilibria $(1, 3)$ and $(2, 4)$. In fact, we have the following four evolutionary orbits corresponding to the Frances' strategies

$$1 \to^e 1 \to^f 3, \quad 2 \to^e 2 \to^f 4, \quad 3 \to^e 1 \to^f 3, \quad 4 \to^e 2 \to^f 4.$$

Anyway, we desire to see if there are mixed equilibria that are not pure equilibria. To obtain the mixed extension of the game G, we denote with b the canonical basis of $\mathbb{R}^4$. By imbedding the two finite strategy spaces into $\mathbb{R}^4$, we can transform the two matrices M_e and M_f into their formal extensions

$$^{\mathrm{ex}}M_e = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_1 & b_2 & b_1 & b_2 \end{pmatrix}, \quad ^{\mathrm{ex}}M_f = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_3 & b_4 & b_3 & b_4 \end{pmatrix}.$$

The mixed extensions of the decision rules are so defined on the simplex $\mathbb{M}_4$ of the space $\mathbb{R}^4$, by

$$\begin{aligned} ^{\mathrm{ex}}e &: \mathbb{M}_4 \to \mathbb{M}_4 : q \to q_1 b_1 + q_2 b_1 + q_3 b_1 + q_4 b_2, \\ ^{\mathrm{ex}}f &: \mathbb{M}_4 \to \mathbb{M}_4 : p \to p_1 b_3 + p_2 b_4 + p_3 b_3 + p_4 b_4; \end{aligned}$$

therefore we have

$$^{\mathrm{ex}}e(q) = (q_1 + q_3, q_2 + q_4, 0, 0), \quad ^{\mathrm{ex}}f(p) = (0, 0, p_1 + p_3, p_2 + p_4).$$

Now, by imposing the conditions of equilibrium to the strategy pair (p, q), we have

$$\begin{cases} p_1 = q_1 + q_3 \\ p_2 = q_2 + q_4 \\ p_3 = 0 \\ p_4 = 0 \end{cases} \text{et} \begin{cases} q_1 = 0 \\ q_2 = 0 \\ q_3 = p_1 + p_3 \\ q_4 = p_2 + p_4 \end{cases},$$

from which we deduce

$$\begin{cases} p_1 = q_3 \\ p_2 = q_4 \\ p_3 = 0 \\ p_4 = 0 \end{cases} \text{et} \begin{cases} q_1 = 0 \\ q_2 = 0 \\ q_3 = p_1 \\ q_4 = p_2 \end{cases};$$

recalling that p and q are probability distributions, we have $p = (a, a', 0, 0)$ and $q = (0, 0, a, a')$, with $a \in [0, 1]$ a probability coefficient and $a' := 1 - a$ its probability complement; we have thus finally found infinitely many equilibria (p_a, q_a) in mixed strategies for the game G.

4.6 Extension of the finite multivocal games

The useful concept of the matrix corresponding with a function between finite sets can be extended immediately to the multivocal case since it is enough to consider set valued matrices.

Definition (of matrix of a multifunction between finite sets). *Let $f : \underline{m} \to \underline{n}$ be a multifunction. We say matrix of f the set valued matrix with m columns and two rows which have as first row the vector $(i)_{i=1}^m$, i.e. the m-vector having as i-th component the integer number i, and as second row the vector $(f(i))_{i=1}^m$ of subsets of $\underline{n}$, i.e. the m-vector having as i-th component the image $f(i)$ (that is a set) of the number i under the correspondence f.*

Example (with multivocal rule). Let $\underline{n}$ be the set of first n positive integers (> 0) and let $e : \underline{2} \to \underline{3}$ and $f : \underline{3} \to \underline{2}$ be the Emil's and Frances' decision rules with associated matrices

$$M_e = \begin{pmatrix} 1 & 2 \\ 2 & \{1,3\} \end{pmatrix}, \quad M_f = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}.$$

Note that the game $G = (e, f)$ has two equilibria. In fact, we have the two evolutionary chains

$$1 \to^e 2 \to^f 1, \quad 2 \to^e 3 \to^f 2.$$

Therefore the game has the two equilibria $(2, 1)$ and $(3, 2)$. To obtain the mixed extension of the game G, we denote by b and b' the canonical bases of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Imbedding the two finite strategy spaces into their respective euclidean spaces, we can transform the two matrices, obtaining

$$^{\mathrm{ex}}M_e = \begin{pmatrix} b_1 & b_2 \\ b'_2 & \{b'_1, b'_3\} \end{pmatrix}, \quad ^{\mathrm{ex}}M_f = \begin{pmatrix} b'_1 & b'_2 & b'_3 \\ b_1 & b_1 & b_2 \end{pmatrix}.$$

the mixed extension of the decision rules are so defined on the two canonical simplexes $\mathbb{M}_2$ and $\mathbb{M}_3$ of the vector spaces $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively, by

$$\begin{aligned} ^{\mathrm{ex}}e &: \mathbb{M}_2 \to \mathbb{M}_3 : q \mapsto q_1 b'_2 + \{q_2 b'_1, q_2 b'_3\}, \\ ^{\mathrm{ex}}f &: \mathbb{M}_3 \to \mathbb{M}_2 : p \mapsto p_1 b_1 + p_2 b_1 + p_3 b_2; \end{aligned}$$

therefore we have

$$\begin{aligned} ^{\mathrm{ex}}e(q) &= \{(q_2, q_1, 0), (0, q_1, q_2)\}, \\ ^{\mathrm{ex}}f(p) &= (p_1 + p_2, p_3), \end{aligned}$$

for any two mixed strategies p and q. By imposing the conditions of equilibrium to the pair (p, q), we have

$$\begin{cases} p_1 = q_2 \\ p_2 = q_1 \\ p_3 = 0 \end{cases} \quad \text{et} \quad \begin{cases} q_1 = p_1 + p_2 \\ q_2 = p_3 \end{cases},$$

or

$$\begin{cases} p_1 = 0 \\ p_2 = q_1 \\ p_3 = q_2 \end{cases} \quad \text{et} \quad \begin{cases} q_1 = p_1 + p_2 \\ q_2 = p_3 \end{cases},$$

from which, recalling that p and q are probability distributions, we have $p = (0, 1, 0)$ and $q = (0, 1)$, or $p = (0, a, a')$ and $q = (a, a')$, for each $a \in [0, 1]$, where $a' = 1 - a$. We have thus found infinite equilibria in mixed strategies, among which there are the two equilibria in pure strategies (those already seen).

Part II

Normal Form Games and Applications

Chapter 5

Payoff space for C^1 games

5.1 Introduction

In the current literature the study of a game in normal form consists principally in the determination of the Nash equilibria in mixed strategies and in the analisys of their stability properties (see [17], [18] and [15]). This does not give a complete and global view of the game, since, for instance, it should be interesting to know the positions of the payoff profiles corresponding to the Nash equilibria in the payoff space of the game: but, the knowledge of these positions requires the knowledge of the entire payoff space. This need becomes inevitable when the problem to solve in the game is a bargaining one: in fact, the determination of a bargaining solution (or of compromise solutions) needs the analytical determination of the Pareto boundaries. In our paper we shall present a general method to find an explicit expression of the Pareto boundaries, via the determination of the entire topological boundary of the payoff space of the game. Resuming, the motivation of the paper resides upon the fact that a complete and deep study of a game in normal form requires the knowledge of the payoff space, or at least of its Pareto boundaries, especially when one passes to the cooperative phase of the game, since to find bargaining solutions or other compromise solutions, the knowledge of the Pareto boundaries is necessary.

Finally, we desire to note that this paper follows a general aim (we seek for also in [17], [18], [19], [20]): to construct solid theoretical bases for Economics and Finance by algebraic, topological and differentiable structures.

5.2 Preliminaries and notations

We shall consider n-person games in normal form. We give the definition used in this work for ease of the reader. The form of definition we give is particularly useful for our purpose.

Definition 1 (of game in normal form). *Let $E = (E_i)_{i=1}^n$ be an ordered family of non-empty sets. We call n-**person game in normal form upon the support** E each function $f :^\times E \to \mathbb{R}^n$, where $^\times E$ denotes the cartesian product $\times_{i=1}^n E_i$ of the family E. The set E_i is called the **strategy set of player** i, for every index i of the family E, and the product $^\times E$ is called the **strategy profile space**, or the n-strategy space, of the game.*

Terminology. With this choice of definition for games in normal form, we have to introduce some terminologies:

- the set $\{i\}_{i=1}^n$ of the first n positive integers is said *the set of the players* of the game;
- each element of the cartesian product $^\times E$ is said a *strategy profile* of the game;
- the image of the function f, i.e., the set of all real n-vectors of type $f(x)$, with x in the strategy profile space $^\times E$, is called the *n-payoff space*, or simply the *payoff space*, of the game f.

We recall, further, for completeness (and ease of the reader), the definition of Pareto boundary we shall use in the paper.

Definition 2 (of Pareto boundary). *The **Pareto maximal boundary of a game** f is the subset of the n-strategy space of those n-strategies x such that the corresponding payoff $f(x)$ is maximal in the n-payoff space, with* *respect to the usual order of the euclidean n-space $\mathbb{R}^n$. We shall denote the maximal boundary of the n-payoff space by $\overline{\partial}f(S)$ and the maximal boundary of the game by $\overline{\partial}_f(S)$ or by $\overline{\partial}(f)$. In other terms, the maximal boundary $\overline{\partial}_f(S)$ of the game is the reciprocal image (by the function f) of the maximal boundary of the payoff space $f(S)$. We shall use analogous terminologies and notations for the* ***minimal Pareto boundary****.*

5.3 The method

The context. We deal with a type of normal form game f defined on the product of n compact non-degenerate intervals of the real line, and such that f is the restriction to the n-strategy space of a C^1 -function defined on an open set of $\mathbb{R}^n$ containing the n-strategy space S (that, in this case, is a compact non-degenerate n -interval of the n-space $\mathbb{R}^n$).

Before to give the main result of the method, we recall some basic notions.

5.3.1 Topological boundary

We recall that the topological boundary of a subset S of a topological space $(X, \mathcal{T})$ is the set defined by the following three equivalent propositions:

- it is the closure of S without the interior of S: $\partial S = \mathrm{cl}(S) \backslash \mathrm{int}(S)$;
- it is the intersection of the closure of S with the closure of its complement
$$\partial S = \mathrm{cl}(S) \cap \mathrm{cl}(X \backslash S);$$
- it is the set of those points x of X such that every neighborhood of x contains at least one point of S and at least one point in the complement of S.

The key theorem of our method is the following one.

Theorem 1. *Let f be a C^1 function defined upon an open set O of the euclidean space $\mathbb{R}^n$ and with values in $\mathbb{R}^n$. Then, for every part S of the open*

O, the topological boundary of the image of S by the function f is contained in the union $f(\partial S) \cup f(C)$, where C is the critical set of f in S, that is the set of the points x of S such that the Jacobian matrix $J_f(x)$ is not invertible.

The full comprehension of the proof requires some important preliminary notions.

5.3.2 Local diffeomorphism and the local inversion theorem

Let X and Y be two open subsets of the euclidean space $\mathbb{R}^n$, let $f : X \to Y$ be a function and let x_0 be a point of X. The function f is said a *local homeomorphism* (respectively, *a local diffeomorphism*) *at the point* x_0 if there is an open neighborhood U of x_0 such that the restriction of f to the pair of subsets $(U, f(U))$ is a homeomorphism (respectively diffeomorphism).

The following fundamental theorem is a consequence of the Dini's theorem.

Theorem (local inversion theorem for C^1-functions). *Let X and Y be two open subsets of the euclidean space $\mathbb{R}^n$, and let $f : X \to Y$ be a C^1-function. Then, for every point $x_0 \in X$ such that the derivative $f'(x_0)$ is a bijective linear application, f is a local diffeomorphism at x_0.*

In the conditions of the above theorem, we shall say that a point x_0 of X is a regular point of the function f if the derivative $f'(x_0)$ is a bijective linear application.

Proof of theorem 1. The theorem derives from the local inversion theorem for C^1-functions, and it is based on the fact that a C^1 function f is a local diffeomorphism at every point (of its domain) in which f has invertible Jacobian matrix. More precisely, since f is a local diffeomorphism at the points not belonging to the critical part of f , it is also a local homeomorphism at those points, and then it sends the neighborhoods of a regular point x_0 onto neighborhoods of the image $f(x_0)$ and, consequentely, regular interior points to interior points of Y. So, let X be a payoff in the topological boundary of the image $f(S)$. The payoff X cannot be the transformation of a regular interior point of S, so it must belong to the part $f(\partial S)$ or to the part $f(C)$. ■

To determine the payoff space of the game we have to do some further topological remarks.

Remark 1. Taking into account that f is a continuous function (since it is C^1), the set $f(S)$ is compact since S is compact, and it is connected since S is connected.

Remark 2. The critical part C is a closed set, since it is the level of a continuous functional (the Jacobian functional $\det J_f$), so the complement of C in S is relatively open in S.

Remark 3 (on the openness of local homeomorphisms). Let X and Y be two open subsets of the euclidean n-space, let $f : X \to Y$ be a local homeomorphism and let O be an open subset of X. Clearly, $f(O)$ is open; indeed, let y_0 be a point of $f(O)$, we must prove that y_0 is an interior point. Let x_0 be a reciprocal image of y_0, by definition of local homeomorphism, there is a neighborhood U of x_0, such that $f(U)$ is an open neighborhood of y_0.

Remark 4 (transformation of simply connected parts). Let A be a relatively open and simply connected set in one of the connected components of $S\backslash C$, suppose that f is injective on A. Since the restriction to the pair $(S\backslash C, f(S\backslash C))$ is an open mapping, the restriction to the pair $(A, f(A))$ is an homeomorphism. Consequently, the image $f(A)$ is simply connected.

Conclusions. So, it is enough to determine the critical part of the game and transform it together with the sides of the n-strategy space, but it is necessary to taking into account the above topological considerations.

5.4 Example in dimension 2

Description of the game. We consider a loss-game $G = (f, \leq)$, with strategy sets $E = F = [0, 1]$ and biloss (disutility) function defined by

$$f(x, y) = (-4xy, x + y),$$

for every bistrategy (x, y) of the game.

Remark 5. This game can be viewed as the mixed extension of the finite bimatrix game

$$M = \begin{pmatrix} (-4, 2) & (0, 1) \\ (0, 1) & (0, 0) \end{pmatrix}.$$

Classification. The game is not linear, it is, utterly, bilinear. It is not symmetric (with respect to the players), since $f_1(x, y) \neq f_2(y, x)$, but it is symmetric with respect to the bistrategies, since $f_i(x, y) = f_i(y, x)$, for every player i. It is not invertible, since there are two different equivalent bistrategies: $f(1, 0) = f(0, 1) = (0, 1)$.

5.4.1 The critical space of the game

In the following we shall denote by A, B, C and D the vertices of the square $E \times F$, starting from the origin and going anticlockwise.

Jacobian matrix. The Jacobian matrix is

$$J_f(x, y) = \begin{pmatrix} -4y & -4x \\ 1 & 1 \end{pmatrix},$$

for every bistrategy (x, y). The Jacobian determinant is

$$\det J_f(x, y) = -4y + 4x,$$

for every pair (x, y).

Critical space. The critical zone is the subset of the bistrategy space of those bitrategies (x, y) verifying the equality $-y + x = 0$. In symbols, the critical zone is the segment

$$\mathcal{C}(f) = \left\{(x, y) \in [0, 1]^2 : x = y\right\} = [A, C].$$

Transformation of the critical space. Let us determine the image $f([A, C])$. The segment $[A, C]$ is defined by the relations

$$\begin{cases} x = y \\ y \in [0, 1] \end{cases}.$$

The value of the biloss function upon the generic point (y, y) is

$$f(y, y) = (-4y^2, 2y).$$

Setting

$$\begin{cases} X = -4y^2 \\ Y = 2y \end{cases},$$

we have

$$\begin{cases} X = -Y^2 \\ Y \in [0, 2] \end{cases}.$$

Thus, the image of the critical zone is the parabolic segment of equation $X = -Y^2$ with end points $A' = (0, 0)$ and $C' = (-4, 2)$.

5.4.2 The biloss (disutility) space

Transformation of the topological boundary of the bistrategy space. We start from the image $f([A, B])$. The segment $[A, B]$ is defined by the relations

$$\begin{cases} y = 0 \\ x \in [0, 1] \end{cases}.$$

The value of the biloss function upon the generic point of this segment is the biloss $f(x, 0) = (0, x)$. Setting

$$\begin{cases} X = 0 \\ Y = x \end{cases},$$

we have

$$\begin{cases} X = 0 \\ Y \in [0, 1] \end{cases}.$$

Thus the image of the segment $[A, B]$ is the segment of end points $A' = (0, 0)$ and $B' = (0, 1)$. *Image of* $f([D, C])$. The segment $[D, C]$ is defined by the relations

$$\begin{cases} y = 1 \\ x \in [0, 1] \end{cases}.$$

The image of the generic point is $f(x, 1) = (-4x, x + 1)$. Setting

$$\begin{cases} X = -4x \\ Y = x + 1 \end{cases},$$

we have

$$\begin{cases} X = 4 - 4Y \\ Y \in [1, 2] \end{cases}.$$

Thus the image is the segment of end points $D' = (0, 1)$ and $C' = (-4, 2)$. *Transformation* $f([C, B])$. The segment $[C, B]$ is defined by

$$\begin{cases} x = 1 \\ y \in [0, 1] \end{cases}.$$

The image of the generic point is $f(1, y) = (-4y, 1 + y)$. Setting

$$\begin{cases} X = -4y \\ Y = 1 + y \end{cases},$$

we obtain

$$\begin{cases} X = 4 - 4Y \\ X \in [-4, 0] \end{cases}.$$

So the image is the segment of end points $C' = (-4, 2)$ and $B' = (0, 1)$. Finally, let's determine the image $f([A, D])$. The segment $[A, D]$ is defined by

$$\begin{cases} x = 0 \\ y \in [0, 1] \end{cases}.$$

The image of the generic point is $f(0, y) = (0, y)$. Setting

$$\begin{cases} X = 0 \\ Y = y \end{cases},$$

we obtain

$$\begin{cases} X = 0 \\ Y \in [0, 1] \end{cases}.$$

So the image is the segment of end points $A' = (0, 0)$ and $D' = (0, 1)$.

Extrema of the game. The extrema of the game are

$$\alpha := \inf G = (-4, 2) \notin G,$$

and

$$\beta := \sup G = (0, 2) \notin G.$$

They are both shadow extremes.

Pareto boundaries. The Pareto boundaries of the biloss space are

$$\underline{\partial} f(E \times F) = f([A, C])$$

(the image of the critical zone of the game, that is a parabolic arc) and

$$\overline{\partial} f(E \times F) = [B', C'];$$

consequently the Pareto boundaries of the bistrategy space are

$$\underline{\partial}_f (E \times F) = [A, C]$$

and

$$\overline{\partial}_f (E \times F) = [B, C] \cup [D, C].$$

5.4.3 Cooperative phase: Selection of Pareto bistrategies

We shall examine the most common cooperative solutions.

Kalai Smorodinsky solution (elementary best compromise). The elementary best compromise biloss (X, Y) is the intersection of the segment joining the threat biloss $v^{\#}$ (see [2] and [3]) with the infimum of the game, thus it satisfies the system

$$\begin{cases} Y = (1/4)X + 1 \\ X = -Y^2 \\ X \in [-4, 0] \\ Y \in [0, 2] \end{cases},$$

leading to the resolvent equation $X^2 + 24X + 16 = 0$, its acceptable solution is $a = \sqrt{128} - 12$, so the biloss $K' = (a, a/4 + 1)$ is the best compromise biloss. The Kalai Smorodinsky solution is the unique corresponding bistrategy solving of the system

$$\begin{cases} -4xy = a \\ x + y = a + 1 \end{cases},$$

i.e., the strategy profile

$$K = \left(\frac{a+1}{2}, \frac{a+1}{2} \right).$$

Core best compromise. The core best compromise biloss is the intersection of the segment joining the threat biloss $v^{\#}$ with the infimum of the core, thus it satisfies the system

$$\begin{cases} Y = X + 1 \\ X = -Y^2 \\ X, Y \in [0, 1] \end{cases},$$

putting $\gamma = \left(-1 + \sqrt{5}\right)/2$ the solution is the biloss $P' = (-\gamma^2, \gamma)$, it is the unique core best compromise biloss. The core best compromise solution solves the system

$$\begin{cases} -4xy = -\gamma^2 \\ x + y = \gamma \end{cases},$$

taking into account that this solution must belong to the core, we known also that $x = y$, and then $x = y = \gamma/2$.

Nash bargaining solution with $v^{\#}$ as disagreement point. The possible Nash bargaining bilosses, with disagreement point represented by the conservative bivalue $v^{\#}$, are the possible solutions of the following optimization problem:

$$\begin{cases} \max \left(X - v_1^{\#}\right)\left(Y - v_2^{\#}\right) = \max X(Y-1) \\ \text{sub } X = -Y^2 \end{cases}.$$

The section of the objective Nash bargaining function upon the constraint is defined by

$$g(Y) = -Y^2(Y-1) = -Y^3 + Y^2,$$

for every Frances' loss Y. The derivative

$$g'(Y) = -3Y^2 + 2Y,$$

is non-negative when

$$Y(3Y-2) \leq 0,$$

that is on the interval $[0, 2/3]$, consequently the maximum point of g is the loss $Y = 2/3$, with corresponding Emil's loss $X = -4/9$ by the constraint. Concluding the point $F' = (-4/9, 2/3)$ is the unique Nash bargaining biloss. The set of Nash bargaining solutions is the reciprocal image of this biloss by the biloss function f.

Minimum aggregate loss (maximum collective utility). The possible bilosses with maximum collective utility are the possible solutions of the following optimization problem:

$$\begin{cases} \min \ (X + Y) \\ \text{sub } X = -Y^2 \end{cases}.$$

We immediately see that the unique biloss with these two properties is $C' = (-4, 2)$, with collective utility 2. The unique maximum utility solution of the game is then the corresponding bistrategy C.

5.5 Another example in dimension 2

We shall study the mixed extension of the finite game with payoff bimatrix

$$M = \begin{pmatrix} (0,0) & (0,1) \\ (1,0) & (a,b) \end{pmatrix},$$

where $a, b \in [0,1]$ and $a + b < 1$.

Payoff functions. The payoff functions of the mixed extension are defined on the biprobabilistic space $[0,1]^2$, by

$$\begin{aligned} f_1(p,q) &= p(1-q) + apq = p - (1-a)pq, \\ f_2(p,q) &= q(1-p) + bpq = q - (1-b)pq, \end{aligned}$$

for every probabilistic profile (p,q). The payoff function of the game is defined by

$$f(p,q) = (p - a'pq, q - b'pq) = (p(1-a'q), q(1-b'p)),$$

where $a' = 1 - a$ and $b' = 1 - b$ are the complements with respect to 1 of a and b, respectively. The complements with respect to 1 cannot be zero since $a + b < 1$.

Critical zone of the game. The Jacobian matrix of the function f at the bistrategy (p,q) is

$$J_f(p,q) = \begin{pmatrix} 1 - q + aq & -p + ap \\ -q + bq & 1 - p + bp \end{pmatrix},$$

i.e.,

$$J_f(p,q) = \begin{pmatrix} 1 - (1-a)q & -(1-a)p \\ -(1-b)q & 1 - (1-b)p \end{pmatrix} = \begin{pmatrix} 1 - a'q & -a'p \\ -b'q & 1 - b'p \end{pmatrix}.$$

The Jacobian determinant at (p,q) is

$$\begin{aligned} \det J_f(p,q) &= (1 - a'q)(1 - b'p) - a'b'pq = \\ &= 1 - a'q - b'p. \end{aligned}$$

It vanishes upon the line r of equation

$$a'q + b'p = 1.$$

This line r intersects the bistrategic space $[0,1]^2$ iff $a/b' \leq 1$, that is, iff $a+b \leq 1$; since we assumed $a+b<1$ this intersection must be non-empty, it shall be a segment. The end points of this segment are the points $H=(a/b',1)$ and $K=(1,b/a')$ (note that the relation $a/b' \leq 1$ is equivalent to the relation $b/a' \leq 1$). Consequently, the critical zone of the game is the segment $[H,K]$, and its first and second projections are, respectively, the interval $[a/b',1]$ and the interval $[b/a',1]$.

Remark 6. If $a/b' > 1$, that is $a+b>1$, the critical zone of the game is void.

Transformation of the critical zone. Let (p,q) be a bistrategy of the critical zone $[H,K]$, we have

$$\begin{aligned} f(p,q) &= (p-a'pq, q-b'pq) = \\ &= (p(1-a'q), q(1-b'p)) = \\ &= (b'p^2, a'q^2). \end{aligned}$$

Hence the first projection of the image of the critical zone is the interval

$$b'\left[(a/b')^2,1\right] = \left[a^2/b', b'\right].$$

The second projection is analogously $[b^2/a',a']$. The image of the critical zone is the set

$$f([H,K]) = \left\{(X,Y) \in \left[a^2/b',b'\right] \times \left[b^2/a',a'\right] : X = b'p^2, Y = a'q^2\right\}.$$

Note that the images of the two points H and K are, respectively,

$$\begin{aligned} f(H) &= f(a/b',1) = \\ &= ((a/b')(1-a'),(1-b'(a/b'))) = \\ &= ((a/b')a,(1-a)) = \\ &= (a^2/b',a'), \end{aligned}$$

and

$$\begin{aligned} f(K) &= f(1,b/a') = \\ &= ((1-b),(b/a')(1-b')) = \\ &= (b',b^2/a'). \end{aligned}$$

Explicit expression of the image of the critical zone. For a point $(b'p^2, a'q^2)$ of the image of the critical zone, we have

$$\begin{aligned}(b'p^2, a'q^2) &= (b'p^2, q(1-b'p)) = \\ &= \left(b'p^2, \frac{1}{a'}(1-b'p)^2\right) = \\ &= \left(X, (1/a')\left(1-\sqrt{b'X}\right)^2\right),\end{aligned}$$

where we put $X = b'p^2$. Hence the explicit equation of the image of the critical zone is

$$Y = \frac{1}{a'}\left(1 - 2\sqrt{b'X} + b'X\right),$$

where X is in the interval $[a^2/b', b']$.

Transformation of the sides of the bistrategic square. Let A, B, C and D be the four vertices of the bistrategic square $[0,1]^2$ starting from the origin and going anticlockwise. Let us transform the side $[B, C]$. The image of the generic point $(1, q)$ of the side $[B, C]$ is

$$\begin{aligned}f(1,q) &= (1-a'q, q-b'q) = \\ &= (1-a'q, bq) = \\ &= \left(1-\frac{a'}{b}Y, Y\right),\end{aligned}$$

where we put $Y = bq$. One has $Y \in [0, b]$ and $X = 1 - (a'/b)Y$ that is

$$Y = b/a' - (b/a')X.$$

Explicit equation of the Pareto boundary. The Pareto boundary, in the case $a/b' \leq 1$ is the set of payments pairs (X, Y) such that

$$Y = \begin{cases} 1-(b'/a)X & \text{iff } 0 \leq X \leq a^2/b' \\ (1/a')\left(1-2\sqrt{b'X}+b'X\right) & \text{iff } a^2/b' < X < b' \\ (b/a')-(b/a')X & \text{iff } b' \leq X \leq 1 \end{cases}.$$

If, on the contrary, $a > b'$, it is the union of the two segments $[(0,1), (a,b)]$ and $[(a,b), (1,0)]$.

Remark 7. Resuming, the interesting cases are those for which $a \leq b'$, when the boundary is the union of the two segments $[(0,1), f(H)]$, $[f(K), (1,0)]$ and of the arc Γ of equation

$$Y = (1/a')\left(1-\sqrt{b'X}\right)^2,$$

with end points $f(H) = (a^2/b', a')$ and $f(K) = (b', b^2/a')$.

Nash bargaining solutions. We have to maximize the function $G : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$G(X, Y) = (X - a)(Y - b),$$

for every pair (X, Y) of the plane, constrained to payments space $f\left([0,1]^2\right)$. There are two cases. If $a/b' \geq 1$ the game-payment corresponding to the Nash bargaining solution is $C' = (a, b)$, since the cone of the upper bounds of (a, b) intersects the payments space only in the point (a, b) itself. If $a/b' < 1$ the maximum is attained on the arc Γ - since, say A' and B' the payments $(1, 0)$ and $(0, 1)$, the segments $[A', C']$ and $[C', B']$ are not contained in the cone of the upper bounds of C' - and consequently the zone in which lies the maximum of the function G is contained in the remaining part of Pareto boundary, that is the arc Γ. Moreover, the function G vanishes on the frontier of the cone of upper bounds of the point C', and so the maximum shall be in the interior of the curve Γ (not at its end points) thus we can apply the Lagrange theorem.

Determination of the payments associated with Nash bargaining solution in the case $a/b' < 1$. The function G has the same values of the function g defined by

$$g(p, q) = G(b'p^2, a'q^2),$$

for every pair (p, q) of the segment $[H, K]$. The Lagrange function is defined by

$$\begin{aligned} L(p, q, \lambda) &= g(p, q) + \lambda(b'p + a'q - 1) = \\ &= (b'p^2 - a)(a'q^2 - b) - \lambda(b'p + a'q - 1). \end{aligned}$$

Applying the Lagrange theorem, we can conclude that the maximum point (p, q) of the g on the constraint must verify the system

$$\begin{cases} 2p(a'q^2 - b) - \lambda = 0 \\ 2(b'p^2 - a)q - \lambda = 0 \\ b'p + a'q - 1 = 0 \end{cases},$$

it conducts, at once, to a third degree equation in p or in q. Precisely, the equation

$$2t^3 - 3t^2 + (1 - b - a + 2ab)t + ab' = 0,$$

with $t = b'p$.

Solution in the particular case $a/b' < 1$ and $a = b$. If $a = b$, the Nash bargaining solution (p, q) must be a stationary point of the Lagrange function

$$L(p, q, \lambda) = a'p^2a'q^2 - \lambda(a'p + a'q - 1),$$

thus it solves the system

$$\begin{cases} 2pa'q^2 - \lambda = 0 \\ 2a'p^2q - \lambda = 0 \\ a'p + a'q - 1 = 0 \end{cases},$$

that is, the system

$$\begin{cases} q = p \\ a'p + a'q - 1 = 0 \end{cases},$$

so we immediately deduce the unique Nash bargaining solution

$$(p, q) = \left(\frac{1}{2a'}, \frac{1}{2a'} \right),$$

with payment

$$f(p, q) = \left(\frac{1}{4a'}, \frac{1}{4a'} \right).$$

Note that the payment $1/(4a')$ is greater or equal to a, for each $a \in [0, 1]$, and it is strictly greater than a, if $a \neq 1/2$, in fact the relation $1/(4a') > a$ is equivalent to the inequality $(2a - 1)^2 > 0$, and if $a/a' < 1$, we have just $a < 1/2$.

5.6 An example in dimension 3

The game. Let consider the three person game $f : [-1, 1]^3 \to \mathbb{R}^3$ defined by

$$f(x) = (x_1x_2, x_2x_3, -x_3x_1),$$

for every strategy triple x.

Critical part of the game. The Jacobian of the game at a strategy triple x is

$$J_f(x) = \begin{pmatrix} x_2 & x_1 & 0 \\ 0 & x_3 & x_2 \\ -x_3 & 0 & -x_1 \end{pmatrix}.$$

The Jacobian determinant is $D(x) = -2x_1x_2x_3$, for each strategy triple x. The functional D vanishes only upon the three coordinate planes.

Image of the critical part. The image of the critical part of the game is "the star" union of the three segments $[-e_i, e_i]$, with $i = 1, 2, 3$, where e_i is the i-th vector of the canonical basis of the 3-space $\mathbb{R}^3$. **Image of the topological boundary of the strategy space.** The images of the six sides of the strategy cube are pairwise coincident. The boundary of the n-payoff space is the union of the three supports of the parametric surfaces

$$s_i : [-1, 1]^2 \to \mathbb{R}^3,$$

defined by $s_3(y) = (y_1y_2, y_2, -y_1)$, $s_2(y) = (y_1, y_2, -y_1y_2)$ and $s_1(y) = (y_1, y_1y_2, -y_1)$, for each y in the square $[-1, 1]^2$.

We consider the game as a gain-game - rigorously we consider the function f endowed with the usual majoring order $\geq$ of the euclidean space - so we are interested in the part of the payoff in the first orthant.

Remark 8. It is evident that the payoff space is concave: for example, the point $(1/3)(1, 1, 1)$ does not belong to the payoff space, but it is convex combination of the canonical basis e, whose elements are in the payoff space.

Remark 9. If the game is with transferable utility, being the maximum cumulative utility of the game 1, the players can agree on the payoff $(1/3)(1, 1, 1)$, that is the barycentric payoff on the maximum utility triangle $\mathrm{conv}(e)$ (convex envelope of the canonical basis) of equation $u_1 + u_2 + u_3 = 1$.

Remark 10. If the game is without transferable utility, the players can agree to use mixed correlated strategy profiles. The convex hull of the 3-payoff space (which is the payoff space of the correlated mixed extension of the game) has the same plane, of equation $\sum u = 1$, as Pareto boundary, now the Kalai-Smorodinsky payoff in this situation is evidently $(1/3)(1, 1, 1)$.

Chapter 6

Study of a game with concave utility space

6.1 Introduction

We recall that for the complete study of a game we shall follow the following points of investigation, we shall:

0a) classify the game (linear, symmetric, invertible, symmetric in the strategies, ...);

0b) find the critical zone of the game and its image by f;

0c) determine the biloss space $f(E \times F)$;

0d) determine inf and sup of the game and see if they are shadow optima;

0e) determine the Pareto boundaries $\underline{\partial}_f(E \times F)$ e $\overline{\partial}_f(E \times F)$ and their images by f;

1a) specify the control of each player upon the boundaries $\underline{\partial}(G)$ and $\overline{\partial}(G)$;

1b) specify the noncooperative reachability and controllability of the Pareto boundaries;

1c) find the possible Pareto solutions and crosses;

1d) find devotion correspondences and devotion equilibria;

1e) specify the efficiency and noncooperative reachability of devotion equilibria;

2a) find best reply correspondences and Nash equilibria;

2b) study the existence of Nash equilibria (Brouwer and Kakutany);

2c) evaluate Nash equilibria: noncooperative reachability, position with respect to α and β, efficiency, devotion;

2d) find, if there are, dominant strategies;

2e) find strict and dominant equilibria, reduce the game by elimination of dominated strategies;

3a) find conservative values and worst loss functions of the players;

3b) find conservative strategies and crosses;

3c) find all the conservative parts of the game (in the bistrategy and biloss spaces);

3d) find core of the game and conservative knots;

3e) evaluate Nash equilibria by the core and the conservative bivalue;

4a) find the worst offensive correspondences and the offensive equilibria;

4b) specify noncooperative reachability of the offensive equilibria and their efficiency;

4c) find the worst offensive strategies of each player and the corresponding loss of the other;

4d) find the possible dominant offensive strategies;

4e) confront the possible non-cooperative solutions;

5a) find the elementary best compromises (Kalai-Smorodinsky solutions) and corresponding biloss;

5b) find the elementary core best compromise and corresponding biloss;

5c) find the Nash bargaining solutions and corresponding bilosses;

5d) find the solutions with closest bilosses to the shadow minimum;

5e) find the maximum collective utility solutions;

5f) confront the possible cooperative solutions.

6.2 The game

Description of the game. We consider a loss-game $G = (f, \leq)$ with strategy sets $E = F = [0, 1]$ and biloss (disutility) function defined by

$$f(x, y) = (-4xy, x + y),$$

for every bistrategy (x, y) of the game.

Remark. This game can be viewed as the mixed extension of the finite bimatrix game

$$M = \begin{pmatrix} (-4,2) & (0,1) \\ (0,1) & (0,0) \end{pmatrix}.$$

The conservative bivalue of the matrix game M is the pair $(0,1)$, in fact

$$\begin{array}{cccc} -4 & 0 & \overset{\max}{\to} & 0 \\ 0 & 0 & \overset{\max}{\to} & 0 \\ & & & \min(0,0)=0 \end{array} \quad , \quad \begin{array}{cccc} & 2 & 1 & \\ & 1 & 0 & \\ \downarrow_{\max} & 2 & 1 & \min(2,1)=1 \end{array}$$

0a) **Classification.** The game is not linear, it is, utterly, bilinear. It is not symmetric (with respect to the players), since $f_1(x, y) \neq f_2(y, x)$, but it is symmetric with respect to the bistrategies, since $f_i(x, y) = f_i(y, x)$, for every player i. It is not invertible, since there are two different equivalent bistrategies: $f(1, 0) = f(0, 1) = (0, 1)$.

6.3 The critical space of the game

0b) **Jacobian matrix.** The Jacobian matrix is

$$J_f(x, y) = \begin{pmatrix} -4y & -4x \\ 1 & 1 \end{pmatrix},$$

for every bistrategy (x, y). The Jacobian determinant is

$$\det J_f(x, y) = -4y + 4x;$$

for every pair (x, y).

Critical space. The critical zone is the subset of the bistrategy space of those bitrategies (x, y) verifying the equality $-y + x = 0$, that is $x = y$. In symbols, the critical zone is

$$\mathcal{C}(f) = \left\{(x, y) \in [0, 1]^2 : x = y\right\} = [A, C]$$

Transformation of the critical space. We must determine the image $f([A, C])$. The segment $[A, C]$ is defined by the relations

$$\begin{cases} x = y \\ y \in [0, 1] \end{cases} .$$

The value of the biloss function upon the generic point (y, y) is

$$f(y, y) = \left(-4y^2, 2y\right) .$$

Setting

$$\begin{cases} X = -4y^2 \\ Y = 2y \end{cases}$$

we have

$$\begin{cases} y = Y/2 \\ X = -4Y^2/4 \end{cases} \leftrightarrow \begin{cases} X = -Y^2 \\ Y \in [0, 2] \end{cases} .$$

Thus, the image of the critical zone is the parabolic segment of equation $X = -Y^2$ with end points $A' = (0, 0)$ and $C' = (-4, 2)$.

6.4 The biloss (disutility) space

0c) **Transformation of the topological boundary of the bistrategy space.** We start from the image $f([A, B])$. The segment $[A, B]$ is defined by

$$\begin{cases} y = 0 \\ x \in [0, 1] \end{cases} .$$

The value of the biloss function upon the generic point is $f(x, 0) = (0, x)$. Setting

$$\begin{cases} X = 0 \\ Y = x \end{cases} ,$$

we have

$$\begin{cases} X = 0 \\ Y \in [0, 1] \end{cases} .$$

Thus the image of the segment $[A, B]$ is the segment of end points $A' = (0, 0)$ and $B' = (0, 1)$. Image of $f([D, C])$. The segment $[D, C]$ is defined by

$$\begin{cases} y = 1 \\ x \in [0, 1] \end{cases} .$$

The image of the generic point is $f(x,1) = (-4x, x+1)$. Setting

$$\begin{cases} X = -4x \\ Y = x+1 \end{cases},$$

we have

$$\begin{cases} x = Y - 1 \\ X = -4Y + 4 \end{cases} \leftrightarrow \begin{cases} X = 4 - 4Y \\ Y \in [1,2] \end{cases}.$$

Thus the image is the segment of end points $D' = (0,1)$ and $C' = (-4,2)$. Transformation $f([C,B])$. The segment $[C,B]$ is defined by

$$\begin{cases} x = 1 \\ y \in [0,1] \end{cases}.$$

The image of the generic point is $f(1,y) = (-4y, 1+y)$. Setting

$$\begin{cases} X = -4y \\ Y = 1+y \end{cases},$$

we obtain

$$\begin{cases} y = Y - 1 \\ X = -4Y + 4 \end{cases} \leftrightarrow \begin{cases} X = 4 - 4Y \\ Y \in [-4,0] \end{cases}.$$

So the image is the segment of end points $C' = (-4,2)$ and $B' = (0,1)$. Finally, let's determine the image $f([A,D])$. The segment $[A,D]$ is defined by

$$\begin{cases} x = 0 \\ y \in [0,1] \end{cases}.$$

The image of the generic point is $f(0,y) = (0,y)$. Setting

$$\begin{cases} X = 0 \\ Y = y \end{cases},$$

we obtain

$$\begin{cases} X = 0 \\ Y \in [0,1] \end{cases}.$$

So the image is the segment of end points $A' = (0,0)$ and $D' = (0,1)$.

0d) **Extrema of the game.** The extrema of the game are

$$\alpha := \inf G = (-4,2) \notin G,$$

and

$$\beta := \sup G = (0,2) \notin G.$$

They are shadow extreme.

0e) **Pareto boundaries.** The Pareto boundaries are $\underline{\partial} f\,(E \times F) = \overset{\frown}{A'C'}$ and $\overline{\partial} f\,(E \times F) = [B', C']$ in the biloss space (where by $\overset{\frown}{A'C'}$ we denoted the image of the critical zone of the game, that is a parabolic arc) and $\underline{\partial}_f\,(E \times F) = [A, C]$ and $\overline{\partial}_f\,(E \times F) = [B, C] \cup [D, C]$ in the bistrategy space.

6.5 I phase: non-cooperative friendly phase

1a) **Pareto-controls.** Both Emil and Frances do not control Pareto minimal boundary. On the contrary, Emil controls part of the Pareto maximal boundary, the segment $[B, C]$, playing the strategy $x^P = 1$. Frances controls part of the Pareto maximal boundary, the segment $[D, C]$, playing the strategy $y^P = 1$.

1b) **Non-cooperative reachability of Pareto boundaries.** Both players can reach non cooperatively the Pareto maximal boundary playing the reaching-strategies $x^P = 1$ and $y^P = 1$, respectively. Neither Emil nor Frances can reach non cooperatively the Pareto minimal boundary (regrettably).

1c) **Non-cooperative Pareto solutions.** Do not exist non-cooperative minimal Pareto solutions. There is only a non-cooperative maximal Pareto solution $P = (1, 1) = C$, that is a control-cross.

1d) **Devotion correspondences.** We have $\partial_2 f_1(x, y) = -4x$, then there are two cases: $x = 0$ and $x > 0$. If $x = 0$ then the function $f_1(x, .)$ is constant and then all Frances' strategies are devote to Emil' strategy 0. If $x > 0$ then the function $f_1(x, .)$ is strictly decreasing, and then the minimum point is reached in the point 1. Concluding, the devotion correspondence of Frances is defined by

$$L_2\,(x) = \begin{cases} F \text{ se } x = 0 \\ 1 \text{ se } x > 0 \end{cases} .$$

Concerning Emil's devotion, we have $\partial_1 f_2(x, y) = 1$, so the function $f_2(., y)$ is strictly increasing for every $y \in F$ and then it assumes its minimum at 0. Concluding the Emil's devotion correspondence is defined by $L_1(y) = 0$, for every y in the Frances' strategy space.

Devotion equilibria. The set of all the devotion equilibria is the segment $[A, D]$: it is an infinite set.

1e) **About the devotion equilibria.** The devotion equilibria are non-cooperatively reachable playing Emil the reaching-strategy $x = 0$. Concerning the efficiency, the devotion equilibrium D is negatively inefficient, because it lies upon the Pareto maximal boundary, on the contrary, the devotion equilibrium A is efficient, since it belongs to the Pareto minimal boundary.

6.6 II phase: properly non-cooperative (egoistic phase)

2a) **Best reply correspondences.** We have $\partial_1 f_1 (x, y) = -4y$, so there are two cases. I case. If $y = 0$, the section $f_1(., y)$ is constant and then $B_1(0) = E$. II case. If $y > 0$ then $\partial_1 f_1(x, y) < 0$, the section $f_1(., y)$ is strictly decreasing and the minimum point of the section is 1, thus $B_1(y) = 1$. Resuming, the Emil's best-reply correspondence is defined by

$$B_1(y) = \begin{cases} E \text{ iff } y = 0 \\ 1 \text{ iff } y > 0 \end{cases}.$$

Concerning the Frances' best reply, we have $\partial_2 f_2(x, y) = 1$, hence the section $f_2(x, .)$ is strictly increasing and the best reply is defined by $B_2(x) = 0$ for every $x \in E$.

Nash equilibria. The intersection of the graph of B_2 with the reciprocal graph of B_1 is

$$\mathrm{gr}\,(B_2) \cap \mathrm{gr}\left(B_1^-\right) = [A, B]\,,$$

so there are infinitely many Nash equilibria. All these equilibria are equivalent for Emil (Emil loss function is constantly equal zero on $[A, B]$) but not for Frances, so they are not equivalent.

2b) **Existence of Nash equilibria.** Kakutany's fixed point theorem assures the existence of at least a Nash equilibrium, Brouwer's fixed point theorem does not.

2c) **About Nash equilibria.** The Nash equilibrium zone is reachable, playing Frances the reaching-strategy $y^N = 0$. The Nash equilibrium A is minimal, and it's the unique minimal equilibrium (good equilibrium), on the contrary, the Nash equilibrium B maximal (very bad equilibrium) the other ones are neither minimal nor maximal. The Nash equilibrium A is also a devotion equilibrium.

2d) **Dominant strategies.** Frances has one (and only one) dominant strategy: $y^D = 0$. Emil has one (and only one) dominant strategy: $x^D = 1$.

2e) **Dominant equilibria.** The Nash equilibrium B is a dominant Nash equilibrium.

6.7 III phase: defensive phase

3a) **Emil's conservative value.** We have

$$v_1^{\#} = \inf_{x \in E} \sup_{y \in F} (-4xy) = \inf_{x \in E} \left(f_1^{\#}(x) \right) = 0$$

Emil's worst loss function. By definition,

$$f_1^{\#}(x) = \sup_{x \in F} (-4xy) .$$

Let's compute the function: for the derivative $\partial_2 (-4xy) = -4x$ there are two cases.

I case. If $x = 0$ then $\partial_2 (-4xy) = 0$; and so the function $f_1(., y)$ is constant on E. Then $O_2(0) = [0, 1]$.

II case. If $x > 0$, then $\partial_2 (-4xy) < 0$, and so the function $f_1(., y)$ is strictly decreasing, consequently $O_2(x) = 0$, for every x in E different from 0. The worst loss function is then defined by

$$f_1^{\#}(x) = \begin{cases} 0 \text{ se } x = 0 \\ 0 \text{ se } x > 0 \end{cases} ,$$

for every x in E.

Frances' conservative value. We have

$$v_2^{\#} = \inf_{y \in F} \sup_{x \in E} (x + y) = \inf_{y \in F} (1 + y) = 1.$$

Frances' worst loss function. We deduce that: the Emil's worst offensive multifunction is defined by $O_1(y) = 1$ for each $y \in F$. The Frances' worst loss function is defined by $f_2^{\#}(y) = 1 + y$, for each y in F.

Conservative bivalue. The conservative bivalue is $v^{\#} = (0, 1) = B'$.

3b) **Players' conservative strategies.** Thus all the Emil's strategies are conservative, in other terms $E^{\#} = E$. The unique Frances' conservative strategy is $y^{\#} = 0$.

Conservative crosses. And finally, the set of all conservative crosses is the segment $c^{\#} = [A, B]$, since all the bistrategies of the type $(x, 0)$ is a conservative cross.

3c) **Conservative parts.** The conservative part of the biloss space is

$$f\left(E \times F\right)^{\#} = \operatorname{conv}\left(K', B', A'\right).$$

The Emil' conservative part is

$$\left(E \times F\right)_1^{\#} = \left\{(x, y) \in E \times F : f_1\left(x, y\right) \leq v_1^{\#}\right\},$$

that is the set of bistrategies (x, y) verifying the system

$$\begin{cases} -4xy \leq 0 \\ x, y \in [0, 1] \end{cases},$$

that is the set of the bistrategies (x, y) such that $xy \geq 0$, then all the bistrategies are conservative for Emil:

$$\left(E \times F\right)_1^{\#} = \operatorname{conv}\left(A, B, C, D\right).$$

Concerning Frances, we have

$$\left(E \times F\right)_2^{\#} = \left\{(x, y) \in E \times F : f_2(x, y) \leq v_2^{\#}\right\},$$

and this part is defined by the two relations

$$\begin{cases} x + y \leq 1 \\ x, y \in [0, 1] \end{cases},$$

consequently

$$\left(E \times F\right)_2^{\#} = \operatorname{conv}\left(A, B, D\right)$$

and then

$$(E \times F)^{\#} = \mathrm{conv}\,(A, B, D)\,.$$

3d) **Core.** The core of the biloss space is the segment of parabola with end points K' and A':

$$\mathrm{core}'(G) = \overset{\frown}{K'A'},$$

where $K' = (-1, 1)$. To determine the core of the game (in the bistrategy space) we have

$$K' = f\,(K)\,,$$

that is

$$\begin{cases} -4xy = -1 \\ x + y = 1 \end{cases},$$

the resolvent equation is

$$-4y + 4y^2 + 1 = 0,$$

which gives the following feasible solution $K = \left(\frac{1}{2}, \frac{1}{2}\right)$, then the core is

$$\mathrm{core}(G) = [A, K]\,.$$

Conservative knots. A possible conservative knot $N^{\#}$ verify $v^{\#} = f(N^{\#})$, that is, the system

$$\begin{cases} -4xy = 0 \\ x + y = 1 \end{cases},$$

which has the solutions $N_1^{\#} = (0, 1) = D$ and $N_2^{\#} = (1, 0) = B$.

3e) **About Nash equilibria.** There are infinitely many Nash equilibria, forming the segment $[A, B]$. All the equilibria are conservative but there is only one core equilibrium: the point A.

6.8 IV phase: offensive phase

4a) **Offensive correspondences and equilibria.** We already saw that the players's worst offensive correspondences are defined by $O_1(y) = 1$, for every strategy $y \in F$, and

$$O_2\,(x) = \begin{cases} F \text{ se } x = 0 \\ 0 \text{ se } x > 0 \end{cases},$$

respectively. The intersection of the graph of O_2 with the reciprocal graph of O_1 is the unique offensive equilibrium B.

4b) **About the offensive equilibrium.** The unique offensive equilibrium is reachable non-cooperatively with the strategies $x^O = 1$ and $y^O = 1$, respectively. It is negatively-efficient, since it lies on the Pareto maximal boundary.

4c) **Confrontation of the equilibria.** The unique Nash equilibrium that is a devotion equilibrium too is A. The unique Nash equilibrium that is an offensive equilibrium too is B.

4d) **Dominant offensive strategies.** Emil has the unique dominant offensive strategy 1. Frances has the unique dominant offensive strategy 0.

4e) **About the noncooperative solution.** The set of all Nash equilibria $[A, B]$ is controlled by Frances through the strategy 0. The equilibrium A is a focal point in the sense of Meyerson: it unique.

6.9 V phase (cooperative): Selection of Pareto bistrategies

We shall examine the most common cooperative solutions.

5a) **Kalai Smorodinsky solution (elementary best compromise).** The elementary best compromise biloss is the intersection of the segment joining the threat biloss $v^{\#}$ with the infimum of the game, thus it satisfies the system

$$\begin{cases} Y = \frac{1}{4}X + 1 \\ X = -Y^2 \\ X \in [-4, 0] \\ Y \in [0, 2] \end{cases},$$

leading to the resolvent equation $X^2 + 24X + 16 = 0$, its acceptable solution is $a = -12 - \sqrt{128}$, so the biloss $K' = \left(a, \frac{a}{4} + 1\right)$ is the best compromise biloss. The Kalai Smorodinsky solution is the unique corresponding bistrategy solving of the system

$$\begin{cases} -4xy = a \\ x + y = a + 1 \end{cases},$$

i.e., the strategy profile $K = (\frac{a+1}{2}, \frac{a+1}{2})$.

5b) **Core best compromise.** The core best compromise biloss is the intersection of the segment joining the threat biloss $v^{\#}$ with the infimum of the core, thus it satisfies the system

$$\begin{cases} Y = X + 1 \\ X = -Y^2 \\ X, Y \in [0, 1] \end{cases},$$

putting $\gamma = (-1 + \sqrt{5})/2$ the solution is the biloss $P' = (-\gamma^2, \gamma)$, it is the unique core best compromise biloss. The core best compromise solution solves the system

$$\begin{cases} -4xy = -\gamma^2 \\ x + y = \gamma \end{cases},$$

taking into account that this solution must belong to the core, we known also that $x = y$, and then $x = y = \gamma/2$.

5c) **Nash bargaining solution with $v^{\#}$ as disagreement point.** The possible Nash bargaining bilosses, with disagreement point represented by the conservative bivalue $v^{\#}$, are the possible solutions of the following optimization problem:

$$\begin{cases} \max \left(X - v_1^{\#}\right)\left(Y - v_2^{\#}\right) = \max X(Y-1) \\ \text{sub } X = -Y^2 \end{cases}.$$

The section of the objective Nash bargaining function upon the constraint is defined by

$$g(Y) = -Y^2(Y-1) = -Y^3 + Y^2,$$

for every Frances' loss Y. The derivative

$$g'(Y) = -3Y^2 + 2Y,$$

is non-negative when

$$Y(3Y-2) \leq 0,$$

that is on the interval $[0, 2/3]$, consequently the maximum point of g is the loss $Y = 2/3$, with corresponding Emil's loss $X = -4/9$ by the constraint. Concluding the point $F' = \left(-\frac{4}{9}, \frac{2}{3}\right)$ is the unique Nash bargaining biloss. The set of Nash bargaining solutions is the reciprocal image of this biloss by the biloss function f.

5e) **Minimum aggregate loss (maximum collective utility).** The possible bilosses with maximum collective utility are the possible solutions of the following optimization problem:

$$\begin{cases} \min\ (X+Y) \\ \text{sub } X = -Y^2 \end{cases} .$$

We immediately see that the unique biloss with these two properties is $C' = (-4, 2)$, with collective utility 2. The unique maximum utility solution of the game is then the corresponding bistrategy C.

5f) **About the cooperative solutions.** The cooperative solutions we found are different and not equivalent among themselves. Assuming the games with transferable utility, certainly the maximum utility solution is a good solution; but in this last case, the players must face the bargaining problem of fair division of the maximum collective utility.

Part III

Algorithms for Normal Form Games

Chapter 7

An Algorithm for Payoff Space in C^1-Games

7.1 Introduction

Often in Game Theory the study of a normal-form game consists principally in the determination of the Nash equilibria in mixed strategies and in the analysis of their various stability properties (see for instance [17], [18] and [15]). Others (see for instance the books of J. P. Aubin [2] and [3]) feel the need to know the entire set of possibilities (consequences) of the players'actions, what we call the payoff space of the game; and moreover they introduce other form of non-cooperative solutions such as the pairs of conservative strategies. Nevertheless, only recently D. Carfì proposed a method to determine analytically the topological boundary of the payoff space and consequently to handle more consciously and precisely the entire payoff space. This method gives a complete and global view of the game, since, for instance, it allows to know the positions of the payoff profiles corresponding to the Nash equilibria in the payoff space of the game or the position of the conservative n-value of the game. The knowledge of these positions requires, indeed, the knowledge of the entire payoff space. Moreover, the knowledge of the entire payoff space becomes indispensable when the problem to solve in the game is a bargaining one: in fact, the determination of a bargaining solution (or of compromise solutions) needs the analytical determination of the Pareto boundaries or at least of the topological one. In the cited paper D. Carfì presented a general method to find an explicit expression of the topological boundary of the payoff-space of a Game and this latter boundary contains the two Pareto boundaries of the game.

7.2 Preliminaries and notations.

For the ease of the reader we recall some basic notions of Game Theory. We shall consider n-person games in normal form. The form of definition we will give is particularly interesting since it is nothing but the definition of a specific differentiable parametric ordered submanifold of the Euclidean space.

Games in normal form. Let $E = (E_i)_{i=1}^n$ be a finite ordered family of non-empty sets. We call n-person game in normal form upon the support E each pair $G = (f, \mathcal{R})$, where f is a function of the cartesian product ${}^\times E$ of the family E into the Euclidean space $\mathbb{R}^n$ and $\mathcal{R}$ is one of the two natural orders ($\leq$ or $\geq$) of the real n-dimensional Euclidean space $\mathbb{R}^n$. By ${}^\times E$ we mean the cartesian product $\times_{i=1}^n E_i$ of the finite family E. The set E_i is called the strategy set of player i, for every index i of the family E, and the product ${}^\times E$ is called the strategy profile space, or the n-strategy space, of the game. The set $\{i\}_{i=1}^n$ of the first n positive integers is said the set of players of the game G; each element of the cartesian product ${}^\times E$ is said a strategy profile of the game; the image of the function f, i.e., the set of all real n-vectors of type $f(x)$, with x in the strategy profile space ${}^\times E$, is called the n-payoff space, or simply the payoff space, of the game f.

Pareto boundaries. The Pareto maximal boundary of a game $G = (f, \mathcal{R})$ is the subset of the n-strategy space of those n-strategies x such that the corresponding payoff $f(x)$ is maximal in the n-payoff space, with respect to the usual order $\mathcal{R}$ of the euclidean n-space $\mathbb{R}^n$. We shall denote the maximal boundary of the n-payoff space by $\overline{\partial} f(S)$ and the maximal boundary of the game by $\overline{\partial}_f(S)$ or by $\overline{\partial}(G)$. In other terms, the maximal boundary $\overline{\partial}_f(S)$ of the game is the reciprocal image (by the function f) of the maximal boundary of the payoff space $f(S)$. We shall use analogous terminologies and notations for the minimal Pareto boundary. (For an introduction to Pareto Boundaries see [19].)

The method. We deal with a type of normal form game $G = (f, \mathcal{R})$ defined on the product of n compact non-degenerate intervals of the real line $\mathbb{R}$, and such that the payoff function f is the restriction to the n-strategy space of a C^1 -function defined on an open set of the Euclidean space $\mathbb{R}^n$ containing the n-strategy space S (that, in this case, is a compact non-degenerate n-interval of the Euclidean n-space $\mathbb{R}^n$). We recall that the topological boundary of a subset S of a topological space (X, τ) is the set defined by the following three equivalent propositions:

1. it is the closure of S without the interior of S: $\partial(S) = \mathrm{cl}(S) \backslash \mathrm{int}(S)$;
2. it is the intersection of the closure of S with the closure of its complement $\partial(S) = \mathrm{cl}(S) \cap \mathrm{cl}(X \backslash S)$;
3. it is the set of those points x of X such that any neighborhood of x contains at least one point of S and at least one point in the complement of S.

The key theorem of the method proposed by D. Carfì is the following one.

Theorem 7.2.1. *teo1 Let f be a C^1 function defined upon an open set O of the euclidean space $\mathbb{R}^n$ and with values in $\mathbb{R}^n$. Then, for every part S of the open O, the topological boundary of the image of S by the function f is contained in the union $f(\partial S) \cup f(C)$, where C is the critical set of the function f in S, that is the set of all points x of S such that the Jacobian matrix $J_f(x)$ is not invertible.*

7.3 Algorithm

In this section we present the algorithm that we used to determine numerically the payoff space of normal form C^1-games in 2 dimensions.

Let A, B, C, D be the vertices of the initial rectangular domain. The inputs are the coordinates of such vertices and the functions $f1(\cdot)$ and $f2(\cdot)$ that define the game f, so that $f(P) = (f1(P), f2(P))$ con $P \in \mathbb{R}^2$.
Denote by x_{min} and x_{max} (y_{min} and y_{max}) the minimum and maximum of vertex abscissae (ordinates).

`STEP 1. TRANSFORMATION OF THE TOPOLOGICAL BOUNDARY.`

Then the initial domain is the rectangle

$$R_{ABCD} = \{(x, y) : x_{min} \leq x \leq x_{max} \textit{ and } y_{min} \leq y \leq y_{max}\}.$$

The transformation of the topological boundary, is a new quadrilateral, of vertices (A', B', C', D'), where a point P' is the image of a point P, by means the following transformation

$$P' = (f1(P), f2(P)) \; \forall P \in \{A,\ B,\ C,\ D\}$$

`STEP 2. PAYOFF SPACE AND CRITICAL ZONE.`

We evaluate the Jacobian determinant of the game f.
If it is zero, then the payoff space agrees the transformation of topological boundary.
Otherwise, the Jacobian determinant is a function of x or y.
In this case, we solve the jabobian in the depending variable. The critical zone is defined by all the points of this transformation, that are in the initial domain R_{ABCD}, too.

The payoff space is the area delimited by the topological boundary and the critical zone. Note that, if the critical zone is void, the payoff space agrees the transformation of the topological boundary.

`STEP 3. PLOTS`

The outputs of the algorithm are the graphics of transformation of topological boundary,of critical zone (if it exists), of payoff space.

7.4 First game

Description of the game. We consider a loss-game $G = (f, <)$, with strategy sets $E = F = [0, 1]$ and biloss (disutility) function defined by

$$f\,(x, y) = (-4xy, x + y)$$

for every bistrategy (x, y) of the game.
The critical space of the game. In the following, we shall denote by $A,\ B,\ C$ and D the vertices of the square $E \times F$, starting from the origin and going anticlockwise.
Jacobian matrix. The Jacobian matrix is

$$J_f = \begin{pmatrix} -4y & -4x \\ 1 & 1 \end{pmatrix},$$

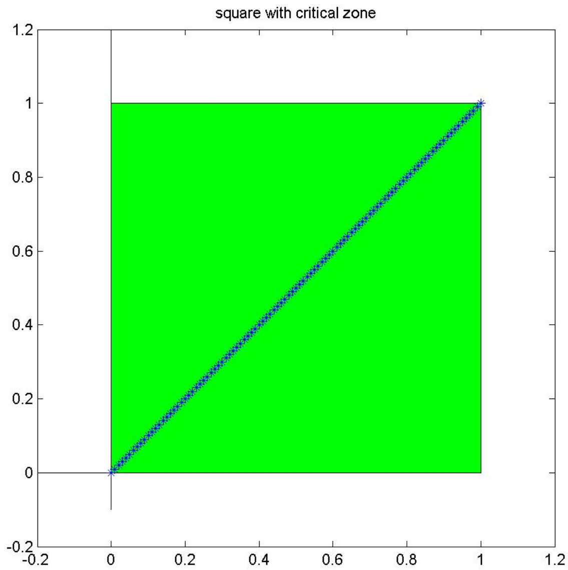


Figure 7.1: Bistrategy square with critical zone.

for every bistrategy (x, y). The Jacobian determinant is

$$detJ_f(x, y) = -4y + 4x$$

for every pair (x, y).
The **critical zone** is the subset of the bistrategy space of those bistrategies verifying the equality $-y + x = 0$. In symbols, the critical zone is the segment

$$\mathcal{C}(f) = \left\{(x, y) \in [0, 1]^2 : x = y\right\} = [A, C]$$

That, graphically, is
Transformation of the critical space. Let us determine the image $f([A, C])$. The value of the biloss function upon the generic point (y, y) of the segment $[A, C]$, is

$$f(y, y) = \left(-4y^2, 2y\right).$$

Thus the image of the critical zone is

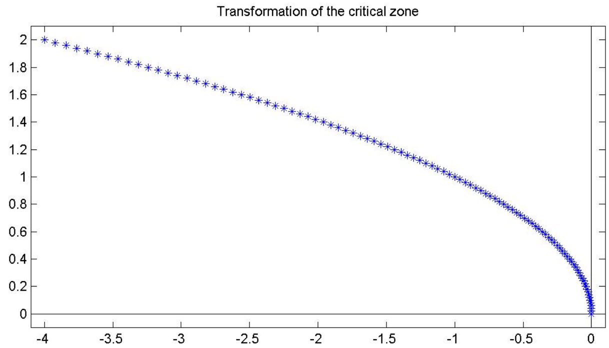


Figure 7.2: Plot of the transformation of the critical zone.

The biloss (disutility) space. Transformation of the topological boundary of the bistrategy space. We start from the image $f([A, B])$. The segment $[A, B]$ is defined by

$$\begin{cases} y = 0 \\ x \in [0, 1] \end{cases}$$

The value of the biloss function upon the generic point is $f(x, 0) = (0, x)$. That, graphically, is
Now we consider the image $f([D, C])$. The segment $[D, C]$ is defined by

$$\begin{cases} y = 1 \\ x \in [0, 1] \end{cases}$$

The value of the biloss function upon the generic point is $f(x, 1) = (-4x, x + 1)$. That, graphically, is

Let us determine the image $f([C, B])$. The segment $[C, B]$ is defined by

$$\begin{cases} x = 1 \\ y \in [0, 1] \end{cases}$$

The value of the biloss function upon the generic point is $f(1, y) = (-4y, 1 + y)$. That, graphically, is

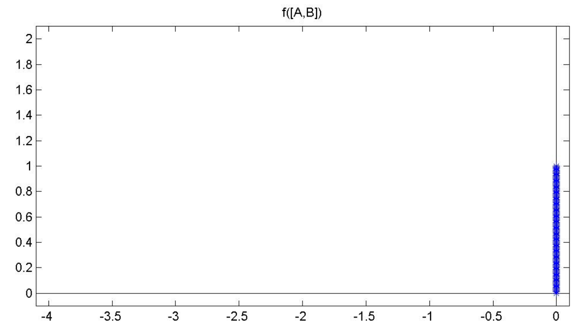


Figure 7.3: Plot of the transformation of the segment [A,B].

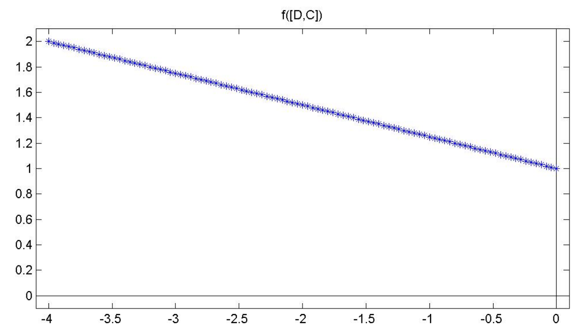


Figure 7.4: Plot of the transformation of the segment [D,C].

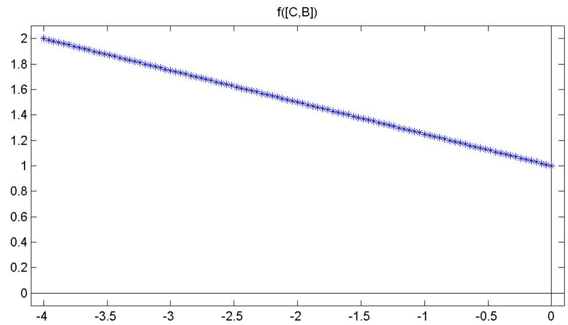


Figure 7.5: Plot of the transformation of the segment [C,B].

Finally, let us determine the image $f([A,D])$. The segment $[A,D]$ is defined by

$$\begin{cases} x = 0 \\ y \in [0,1] \end{cases}$$

The value of the biloss function upon the generic point is $f(0,y) = (0,y)$. That, graphically, is

So the image of the transformation of the topological boundary of the bistrategy space is

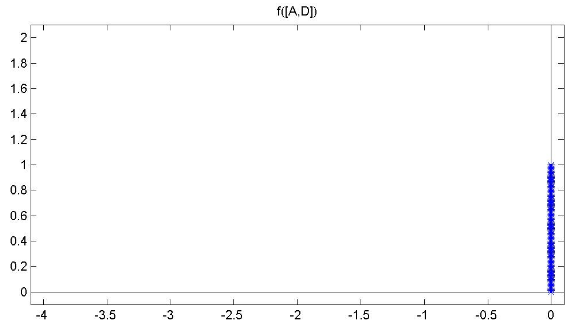


Figure 7.6: Plot of the transformation of the segment [A,D].

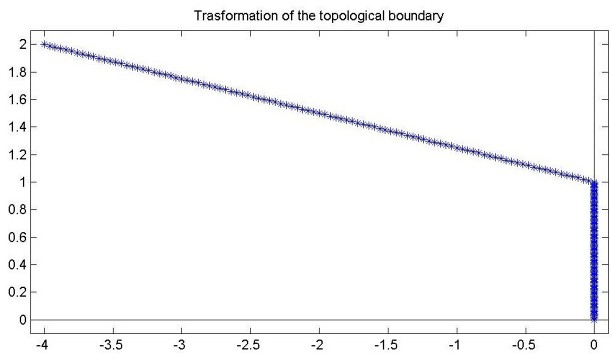


Figure 7.7: Plot of the transformation of the topological boundary of the bistrategy space.

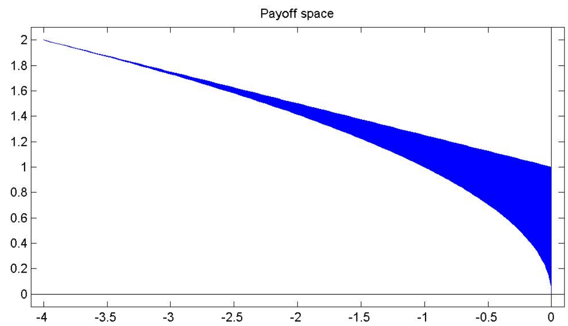


Figure 7.8: Plot of the payoff space.

At this point we can show the payoff space

7.5 Second game

Description of the game. We consider a loss-game $G = (f, <)$, with strategy sets $E = F = [0, 1]$ and biloss (disutility) function defined by

$$f(x, y) = \left(x - \frac{1}{2}xy, y - \frac{1}{2}xy\right)$$

for every bistrategy (x, y) of the game.
The critical space of the game. In the following we shall denote by A, B, C and D the vertices of the square $E \times F$, starting from the origin and going anticlockwise.
Jacobian matrix. The Jacobian matrix is

$$J_f = \begin{pmatrix} 1 - \frac{1}{2}y & -\frac{1}{2}x \\ -\frac{1}{2}y & 1 - \frac{1}{2}x \end{pmatrix},$$

for every bistrategy (x, y). The Jacobian determinant is

$$det J_f(x, y) = 1 - \frac{1}{2}x - \frac{1}{2}y$$

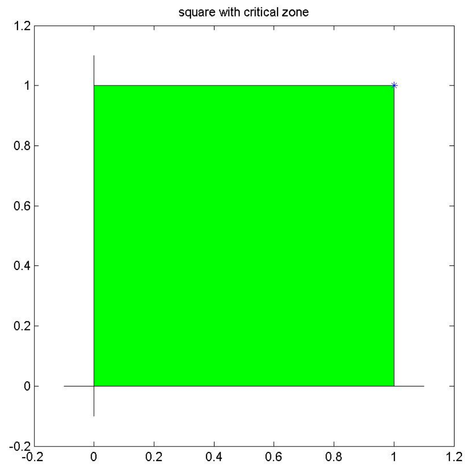


Figure 7.9: Bistrategy square with critical zone.

for every pair (x, y).
The **critical zone** is the subset of the bistrategy space of those bistrategies verifying the equality $y = 2 - x$. In symbols, the critical zone is

$$\mathcal{C}(f) = \left\{(x, y) \in [0, 1]^2 : y = 2 - x\right\} = (C)$$

That, graphically, is

Transformation of the critical space. Let us determine the transformation of the critical zone . It is defined by the relations

$$\begin{cases} y = 2 - x \\ x \in [0, 1] \end{cases}$$

The value of the biloss function upon he generic point $(x, 2 - x)$ is

$$f(x, 2 - x) = \left(x - x^2, 2 - x - x^2\right).$$

Thus the image of the critical zone is

The biloss (disutility) space. Transformation of the topological boundary of the bistrategy space. We start from the image $f([A, B])$. The segment $[A, B]$ is defined by

$$\begin{cases} y = 0 \\ x \in [0, 1] \end{cases}$$

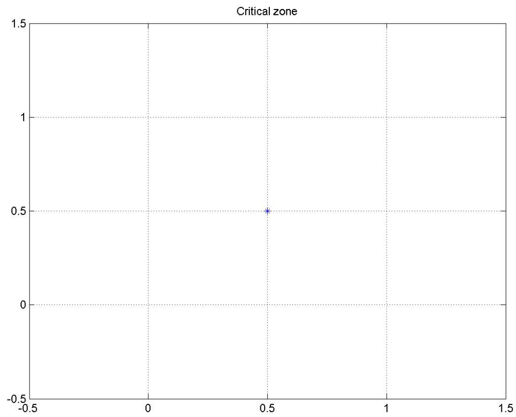


Figure 7.10: Plot of the transformation of the critical zone.

The value of the biloss function upon the generic point is $f(x,0) = (x,0)$. That, graphically, is

Now we consider the image $f([D,C])$. The segment $[D,C]$ is defined by

$$\begin{cases} y = 1 \\ x \in [0,1] \end{cases}$$

The value of the biloss function upon the generic point is $f(x,1) = \left(\frac{1}{2}x, 1 - \frac{1}{2}x\right)$. That, graphically, is

Let us determine the image $f([C,B])$. The segment $[C,B]$ is defined by

$$\begin{cases} x = 1 \\ y \in [0,1] \end{cases}$$

The value of the biloss function upon the generic point is $f(1,y) = \left(1 - \frac{1}{2}y, \frac{1}{2}y\right)$. That, graphically, is

Finally, let us determine the image $f([A,D])$. The segment $[A,D]$ is defined by

$$\begin{cases} x = 0 \\ y \in [0,1] \end{cases}$$

The value of the biloss function upon the generic point is $f(0,y) = (0,y)$. That, graphically, is
So the image of the transformation of the topological boundary of the bistrategy space is

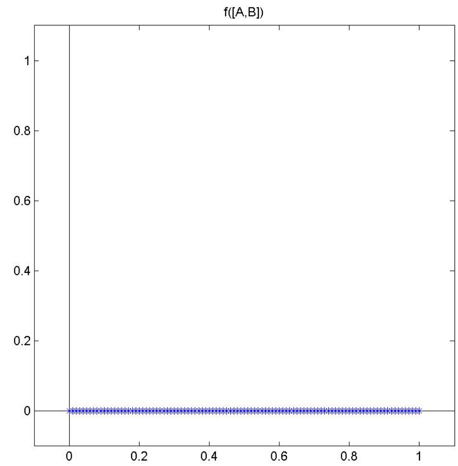


Figure 7.11: Plot of the transformation of the segment [A,B].

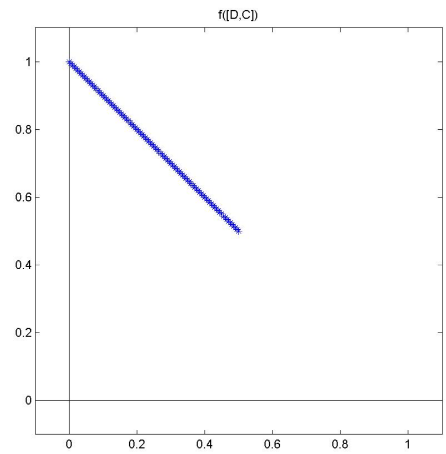


Figure 7.12: Plot of the transformation of the segment [D,C].

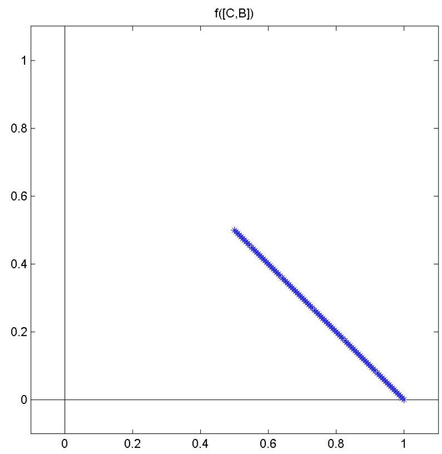


Figure 7.13: Plot of the transformation of the segment [C,B].

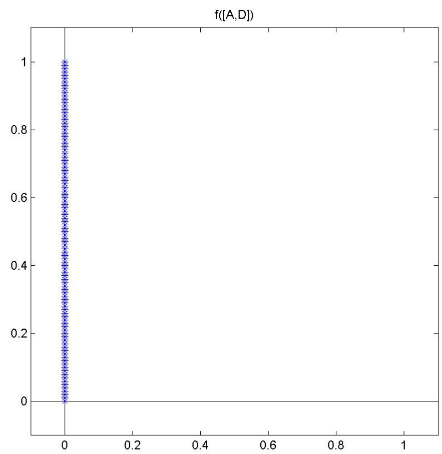


Figure 7.14: Plot of the transformation of the segment [A,D].

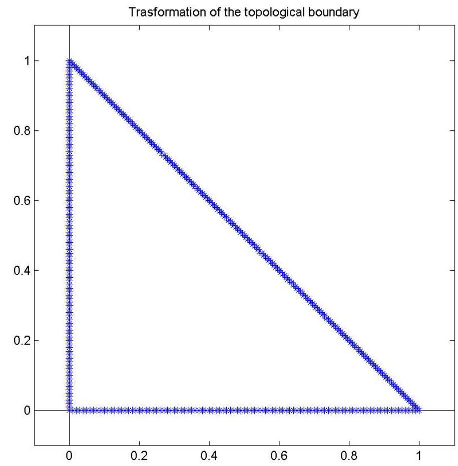


Figure 7.15: Plot of the transformation of the topological boundary of the bistrategy space.

At this point we can show the payoff space

7.6 Third game

Description of the game. We consider a loss-game $G = (f, <)$, with strategy sets $E = F = [0, 1]$ and biloss (disutility) function defined by

$$f(x, y) = (x, y + xy)$$

for every bistrategy (x, y) of the game.
The critical space of the game. In the following we shall denote by $A,\ B,\ C$ and D the vertices of the square $E \times F$, starting from the origin and going anticlockwise.
Jacobian matrix. The Jacobian matrix is

$$J_f = \begin{pmatrix} 1 & 0 \\ y & 1 + x \end{pmatrix},$$

for every bistrategy (x, y). The Jacobian determinant is

$$det J_f(x, y) = 1 + x$$

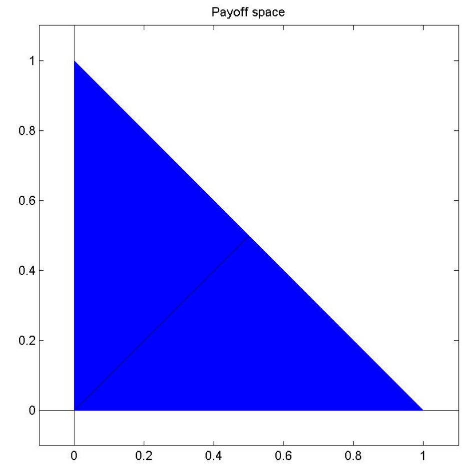


Figure 7.16: Plot of the payoff space.

for every pair (x, y). The **critical zone** is the subset of the bistrategy space of those bistrategies verifying the equality $x = -1$. So there are not points of the critical zone in the strategy sets. That, graphically, is

The biloss (disutility) space. Transformation of the topological boundary of the bistrategy space. We start from the image $f([A, B])$. The segment $[A, B]$ is defined by $y = 0$ and $x \in [0, 1]$. The value of the biloss function upon the generic point is $f(x, 0) = (x, 0)$. That, graphically, is

Now we consider the image $f([D, C])$. The segment $[D, C]$ is defined by $y = 1$ and $x \in [0, 1]$. The value of the biloss function upon the generic point is $f(x, 1) = (x, 1 + x)$. That, graphically, is

Let us determine the image $f([C, B])$. The segment $[C, B]$ is defined by $x = 1$ and $y \in [0, 1]$. The value of the biloss function upon the generic point is $f(1, y) = (1, 2y)$. Setting $X = 1$ and $Y = 2y$, we have $X = 1$ and $Y \in [0, 2]$. That, graphically, is

Finally, let us determine the image $f([A, D])$. The segment $[A, D]$ is defined by $x = 0$ and $y \in [0, 1]$. The value of the biloss function upon the generic point is $f(0, y) = (0, y)$. That, graphically, is

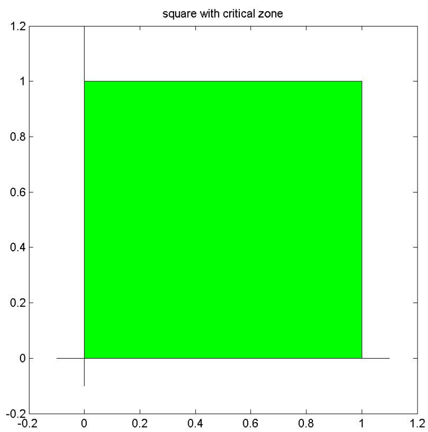


Figure 7.17: Bistrategy square with critical zone.

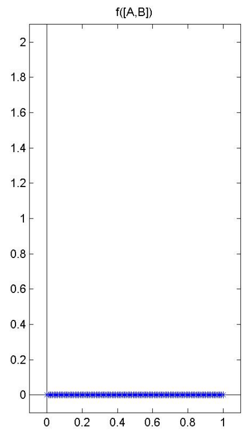


Figure 7.18: Plot of the transformation of the segment [A,B].

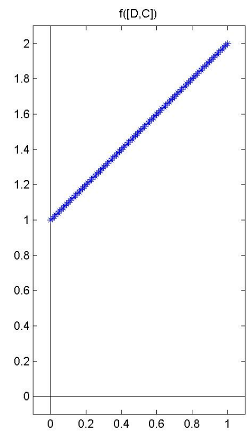


Figure 7.19: Plot of the transformation of the segment [D,C].

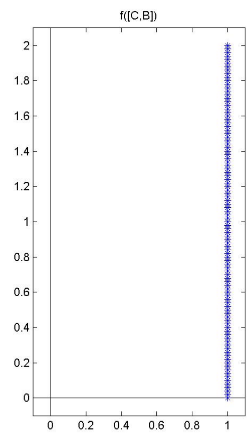


Figure 7.20: Plot of the transformation of the segment [C,B].

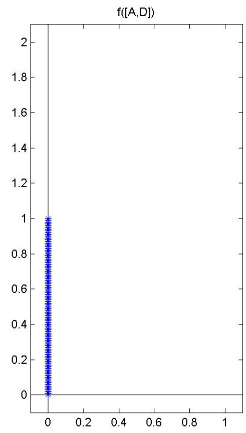


Figure 7.21: Plot of the transformation of the segment [A,D].

So the image of the transformation of the topological boundary of the bistrategy space is

At this point we can show the payoff space

7.7 Fourth game

Description of the game. We consider a loss-game $G = (f, <)$, with strategy sets $E = F = [0, 1]$ and biloss (disutility) function defined by

$$f(x, y) = \left(x - \frac{3}{4}xy, y - \frac{3}{4}xy \right)$$

for every bistrategy (x, y) of the game.
The critical space of the game. In the following we shall denote by A, B, C and D the vertices of the square $E \times F$, starting from the origin and going anticlockwise.
Jacobian matrix. The Jacobian matrix is

$$J_f = \begin{pmatrix} 1 - \frac{3}{4}y & -\frac{3}{4}x \\ -\frac{3}{4}y & 1 - \frac{3}{4}x \end{pmatrix},$$

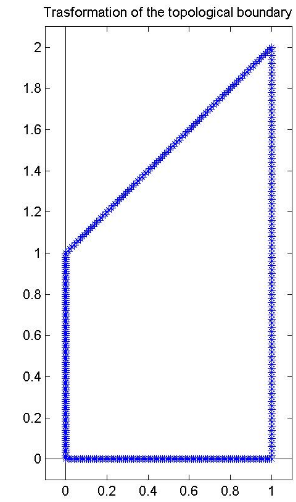


Figure 7.22: Plot of the transformation of the topological boundary of the bistrategy space.

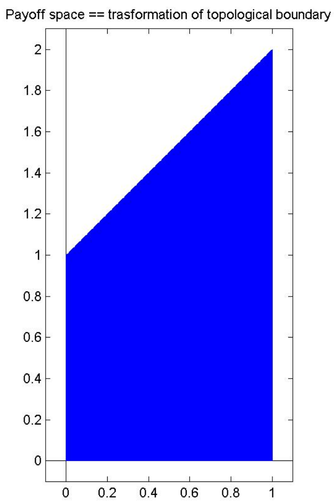


Figure 7.23: Plot of the payoff space.

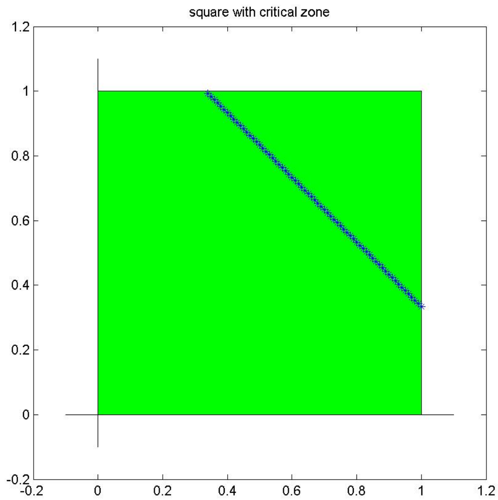


Figure 7.24: Bistrategy square with critical zone.

for every bistrategy (x, y). The Jacobian determinant is

$$detJ_f(x, y) = 1 - \frac{3}{4}x - \frac{3}{4}y$$

for every pair (x, y).
The **critical zone** is the subset of the bistrategy space of those bistrategies verifying the equality $y = \frac{4}{3} - x$. In symbols, the critical zone is

$$\mathcal{C}(f) = \left\{ (x, y) \in [0, 1]^2 : y = \frac{4}{3} - x \right\}$$

That, graphically, is

Transformation of the critical space. Let us determine the transformation of the critical zone . It is defined by the relations

$$\begin{cases} y = \frac{4}{3} - x \\ x \in [0, 1] \end{cases}$$

The value of the biloss function upon he generic point $\left(x, \frac{4}{3} - x\right)$ is

$$f\left(x, \frac{4}{3} - x\right) = \left(x - x^2, \frac{4}{3} - x - x^2\right).$$

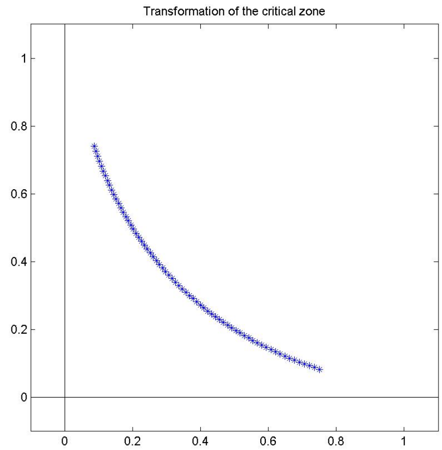


Figure 7.25: Plot of the transformation of the critical zone.

Thus the image of the critical zone is

The biloss (disutility) space. Transformation of the topological boundary of the bistrategy space. We start from the image $f([A, B])$. The segment $[A, B]$ is defined by $y = 0$ and $x \in [0, 1]$. The value of the biloss function upon the generic point is $f(x, 0) = (x, 0)$. That, graphically, is

Now we consider the image $f([D, C])$. The segment $[D, C]$ is defined by $y = 1$ and $x \in [0, 1]$. The value of the biloss function upon the generic point is $f(x, 1) = \left(\frac{1}{4}x, 1 - \frac{3}{4}x\right)$. That, graphically, is

Let us determine the image $f([C, B])$. The segment $[C, B]$ is defined by $x = 1$ and $y \in [0, 1]$. The value of the biloss function upon the generic point is $f(1, y) = \left(1 - \frac{3}{4}y, \frac{1}{4}y\right)$. That, graphically, is

Finally, let us determine the image $f([A, D])$. The segment $[A, D]$ is defined by $x = 0$ and $y \in [0, 1]$. The value of the biloss function upon the generic point is $f(0, y) = (0, y)$. That, graphically, is

So the image of the transformation of the topological boundary of the bistrategy space is

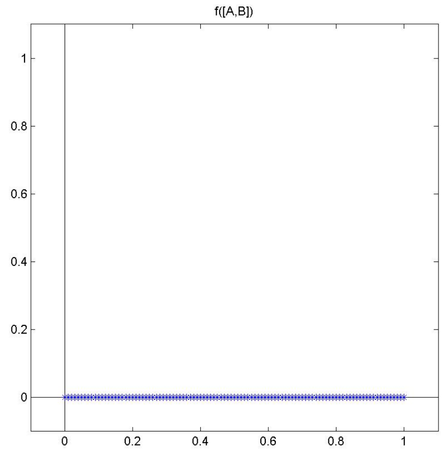


Figure 7.26: Plot of the transformation of the segment [A,B].

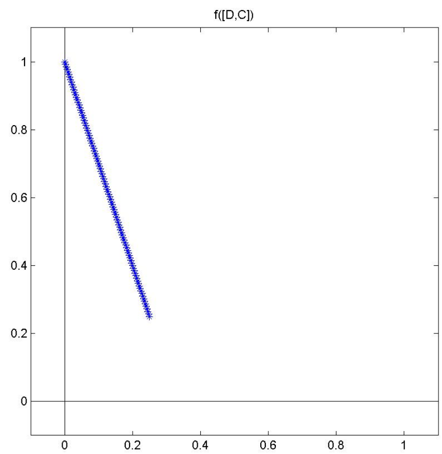


Figure 7.27: Plot of the transformation of the segment [D,C].

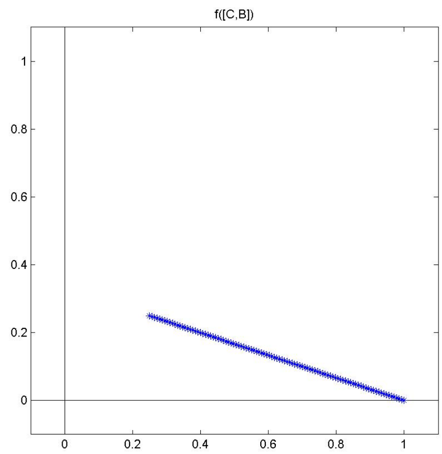


Figure 7.28: Plot of the transformation of the segment [C,B].

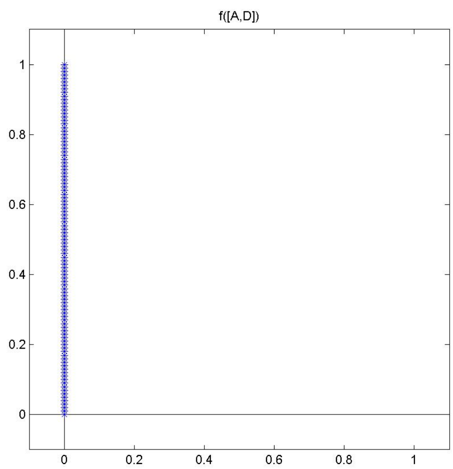


Figure 7.29: Plot of the transformation of the segment [A,D].

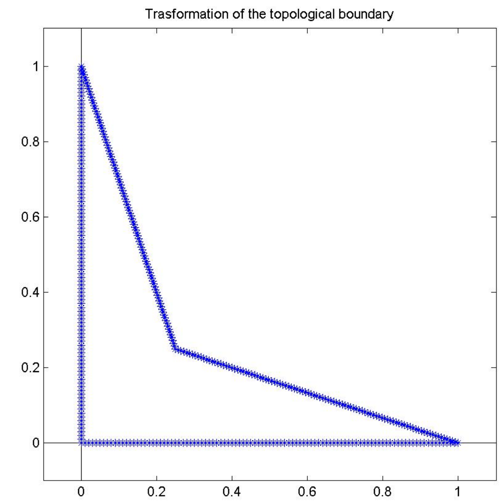


Figure 7.30: Plot of the transformation of the topological boundary of the bistrategy space.

At this point we can show the payoff space

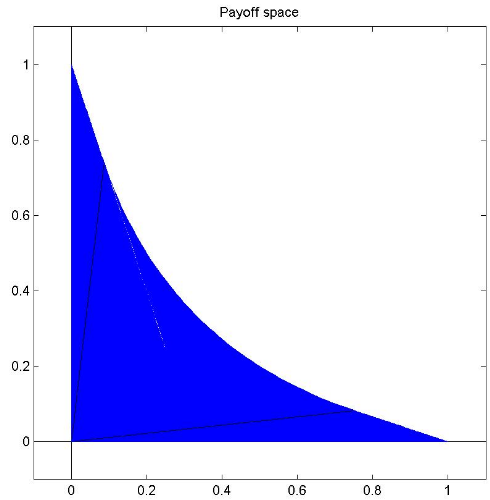


Figure 7.31: Plot of the payoff space.

Chapter 8

The payoff trajectories in C^1 parametric games

8.1 Introduction

The great achievement of the game theory arises form the wide variety of fields in which it has been applied, in order to model and analyze a large collection of human and animal behavior, but economic, political, sociological and psychological ones as well.
Our study pertain to normal-form C^1-games in$n-$dimensions that is $n-$players normal form games whose payoff functions are at least of class C^1 in a compact interval of the real line. This study includes also games depending on a parameter in a one dimensional set. In [5, 6, 8, 9, 10, 11], the authors analyze parametric games, where the parameter set is the coopetitive strategy one. It allows us to pass from the standard normal-form games to their coopetitive extension as illustrated in [3, 2, 17, 18].
In particular, in [17], a new procedure to determine the payoff space of such kind of games has been presented and it has been applied in [1] to numericcaly determine the payoff space for normal-form C^1 parametric games in two dimensions.
In this work, the method in [17] has been pointed out and assumed with the

aim of realizing an algorithm for the computational representation of the payoff trajectory in the case of normal-form C^1-parametric games.
To ease the reader, in the first section of the paper we bring to mind terminology and some definitions, while in the second part, the method proposed in [17] and applied in the development of our algorithm, is presented. Moreover, the particular case of two parametric games is shown in the third section.
The application of our algorithm to several examples concludes the paper.

8.2 Preliminaries on normal-form C^1 games

In order to help the reader and increase the level of readability of the paper, we recall some notations and definitions about $n-$person games in normal-form, presented yet in [17, 1].

Definition 1 (of game in normal-form). *Let $E = (E_i)_{i=1}^n$ be an ordered family of non-empty sets. We call n-**person game in normal-form upon the support** E each function*

$$f: \ {}^{\times}E \to \mathbb{R}^n,$$

*where ${}^{\times}E$ denotes the Cartesian product $\times_{i=1}^n E_i$ of the family E. The set E_i is called the **strategy set of player** i, for every index i of the family E, and the product ${}^{\times}E$ is called the **strategy profile space**, or the n-strategy space, of the game.*

Terminology. Together with the previous definition of game in normal form, we have to introduce some terminologies:

- the set $\{i\}_{i=1}^n$ of the first n positive integers is said *the set of the players* of the game;
- each element of the Cartesian product ${}^{\times}E$ is said a *strategy profile* of the game;
- the image of the function f, i.e., the set $f({}^{\times}E)$ of all real n-vectors of type $f(x)$, with x in the strategy profile space ${}^{\times}E$, is called the *n-payoff space*, or simply the *payoff space*, of the game f.

Moreover, we recall the definition of Pareto boundary whose main properties have been presented in [19].

Definition 2 (of Pareto boundary). *The **Pareto maximal boundary of a game** f is the subset of the n-strategy space of those n-strategies x such that the corresponding payoff $f(x)$ is maximal in the n-payoff space, with respect to the usual order of the euclidean n-space $\mathbb{R}^n$. If S denote the strategy space ${}^{\times}E$, we shall denote the maximal boundary of the n-payoff space by $\overline{\partial}f(S)$ and the maximal boundary of the game by $\overline{\partial}_f(S)$ or by $\overline{\partial}(f)$. In other terms, the maximal boundary $\overline{\partial}_f(S)$ of the game is the reciprocal image (by the function f) of the maximal boundary of the payoff space $f(S)$. We shall use analogous terminologies and notations for the **minimal Pareto boundary**.*

8.3 The method for C^1 games

In this paper, we deal with a type of normal-form game f defined on the product of n compact non-degenerate intervals of the real line, and such that f is the restriction to the n-strategy space of a C^1 function defined on an open set of $\mathbb{R}^n$ containing the n-strategy space S (which, in this case, is a compact non-degenerate n-interval of the n-space $\mathbb{R}^n$). Details are in [17, 18, 7], but in the following we recall some basic notations.

8.3.1 Topological boundary

We recall that the topological boundary of a subset S of a topological space (X, τ) is the set defined by the following three equivalent propositions:

- it is the closure of S minus the interior of S:

$$\partial S = \mathrm{cl}(S) \backslash \mathrm{int}(S);$$

- it is the intersection of the closure of S with the closure of its complement

$$\partial S = \mathrm{cl}(S) \cap \mathrm{cl}(X \backslash S);$$

- it is the set of those points x of X such that every neighborhood of x contains at least one point of S and at least one point in the complement of S.

The key theorem of our method is the following one.

Theorem 1. *Let f be a C^1 function defined upon an open set O of the euclidean space $\mathbb{R}^n$ and with values in $\mathbb{R}^n$. Then, for every part S of the open O, the topological boundary of the image of S by the function f is contained in the union*

$$f(\partial S) \cup f(C),$$

where C is the critical set of f in S, that is the set of the points x of S such that the Jacobian matrix $J_f(x)$ is not invertible. If, more, the function f is not continuous over a part H of O and C^1 elsewhere in O, the topological boundary of the image of S by the function f is contained in the union

$$f(\partial S) \cup f(C) \cup f(H),$$

where C is (again) the critical set of f in S.

8.4 Two players parametric games

In this section we introduce the definitions of parametric games.

Definition 3. *Let $E = (E_t)_{t\in T}$ and $F = (F_t)_{t\in T}$ be two families of non empty sets and let*

$$f = (f_t)_{t\in T}$$

be a family of functions

$$f_t : E_t \times F_t \to \mathbb{R}^2.$$

We define parametric game over the strategy pair (E, F) and with family of payoff functions f the pair

$$G = (f, >),$$

where $>$ is the usual strict upper order of the Euclidean plane $\mathbb{R}^2$. We define payoff space of the parametric game G the union of all the payoff spaces of the game family

$$g = ((f_t, >))_{t\in T},$$

that is the union of the payoff family

$$P = (f_t(E_t \times F_t))_{t\in T}.$$

We will refer to the above family as the (parametric) trajectory of the game G. Our algorithm allows us to represent both the dynamical evolution of P, in the sense of the dynamical evolution in time of our representation of P by the algorithm. Moreover, we numerically obtain the trace of this trajectory, i.e. the payoff space of the game G. We note also that the family P can be identified with the multi-valued path

$$p : T \to \mathbb{R}^2 : t \mapsto f_t(E_t \times F_t),$$

and that the graph of this path p is a subset of the Cartesian product $T \times \mathbb{R}^2$, precisely, the trace of the curve p.

In particular we are concentrated on the following specific kind of parametric game:

- parametric games in which the families E and F consist of only one set, respectively.

In the latter case we can identify a parametric game with a pair $(f, >)$, where f is a function from a Cartesian product $T \times E \times F$ into the plane $\mathbb{R}^2$, where T, E and F are three non-empty sets.

Definition 4. *When the triple (T, E, F) is a triple of subsets of normed spaces, we define the parametric game $(f, >)$* ***of class*** *C^1 if the function f is of class C^1.*

8.5 Numerical Results

In [1], a representation of the payoff space via disjoint union of each payoff space is given. In this work, we depicts the payoff trajectory or rather the dynamical evolution in time of the payoff family.

In order to compare the two algorithms, in the following we will analyze the same games illustrated in [1]. In details, we consider a (loss) parametric game $(h, <)$, with strategy sets $E = F = [0, 1]$, parameter set $T = [0, 1]^2$ and biloss (disutility) function

$$h : \times(T, E, F) \to \mathbb{R}^2$$

whose section

$$h_{(a,b)} : \times(E, F) \to \mathbb{R}^2$$

is defined by

$$h_{(a,b)}\,(x, y) = (x - (1 - a)xy, y - (1 - b)xy)\,,$$

for all $(x, y) \in E \times F$ and $(a, b) \in [0, 1]^2$.

The above game is the von Neuman convexification of the finite game represented by the following array

$$\begin{matrix} (a, b) & (1, 0) \\ (0, 1) & (0, 0) \end{matrix}.$$

Assume, now, that the parameter points (a, b) belong also to the 1-sphere $\mathbf{S}^1_p$, with respect to the p-norm, in the Euclidean plane $\mathbb{R}^2$, for some positive real p; that is, let us assume

$$a^p + b^p = 1,$$

for some positive real p. Consider, then, the restriction

$$g : S \times E \times F \to \mathbb{R}^2$$

of the function h to the parameter set

$$S = \mathbf{S}^1_p \cap T.$$

By projecting on the first factor of the product $S \times E \times F$, we can consider, instead of the parametric game g, with parameter set $\mathbf{S}^1_p \cap T$, the equivalent parametric game $(f, <)$, with parameter set $[0, 1]$ and a-payoff function f_a defined by

$$f_a(x, y) = \left(x - (1 - a)xy, y - (1 - (1 - a^p)^{1/p})xy\right),$$

for all $(x, y) \in E \times F$ and $a \in [0, 1]$. Here, by equivalent parametric game, we mean the existence of the bijection

$$j : S \to [0, 1] : (a, b) \mapsto a$$

whose inverse is the bijection

$$j^{-1} : [0, 1] \to S : a \mapsto (a, (1 - a^p)^{1/p}).$$

In the following sections we shall consider the following sub-cases:

1. $p = 1$:
$$f_a(x, y) = (x - (1 - a)xy, y - axy),$$
for all x, y and a in $[0, 1]$.

2. $p = 0.1$:

$$f_a(x,y) = \left(x - (1-a)xy, y - (1-(1-a^{0.1})^{10})xy\right),$$

for all x, y and a in $[0,1]$.

3. $p = 0.5$:

$$f_a(x,y) = \left(x - (1-a)xy, y - (1-(1-a^{0.5})^{2})xy\right),$$

for all x, y and a in $[0,1]$.

4. $p = 2$:

$$f_a(x,y) = \left(x - (1-a)xy, y - (1-(1-a^{2})^{0.5})xy\right),$$

for all x, y and a in $[0,1]$.

5. $p = 10$:

$$f_a(x,y) = \left(x - (1-xy), y - (1-(1-a^{10})^{0.1})xy\right),$$

for all x, y and a in $[0,1]$.

Moreover, we shall present the following games:

start=6

$$f_a(x,y) = \left(x + y + a, x - y + a^2\right),$$

for all $x, y \in [0,2]$ and $a \in [0,1]$.

stbrt=6

$$f_a(x,y) = (x + y + a, x - y + |a|),$$

for all $x, y \in [0,2]$ and $a \in [-1,1]$.

stcrt=6

$$f_a(x,y) = (x(1-x-y) + a, y(1-x-y) - (1/6)(a-3)^2 + 3/2),$$

for all $x, y \in [0,1]$ and $a \in [0,6]$.

8.6 First game $p = 1$

Let $E = F = [0, 1]$ be the strategy sets and let a be a real number fixed in the interval $[0, 1]$. Consider the a-biloss (disutility) function of the parametric game $(f, <)$, defined by

$$f_a(x, y) = (x - (1 - a)xy, y - axy),$$

for all (x, y) in $[0, 1]$.

The critical zone of the function f_a for every a in $[0, 1]$) is the set

$$\mathcal{C}(f_a) = \left\{(x, y) \in [0, 1]^2 : 1 - ax - (1 - a)y = 0\right\}.$$

The transformation of the topological boundary is given by the disjoint union of the family

$$(f_a(\partial(E \times F)))_{a \in T},$$

We obtain the representation of the Payoff Space of the parametric game as disjoint union of the family

$$(f_a(E \times F))_{a \in T}, \tag{8.1}$$

that is the disjoint union of the transformations of the payoff spaces, with respect to the parameter set. In this work we present a new numerical approach. Our algorithm does not represent the above disjoint union (see [1] for this representation), but it represent the simple union of the family 8.1, that is the genuine payoff space of the game G. In the figure 8.1, we show the obtained payoff space of the family 8.1. In particular, we can distinguish the transformation of the topological boundary in 8.1(a), the transformation of the critical zone in 8.1(b) and the entire payoff in 8.1(c).

8.7 Second game $p = 0.1$

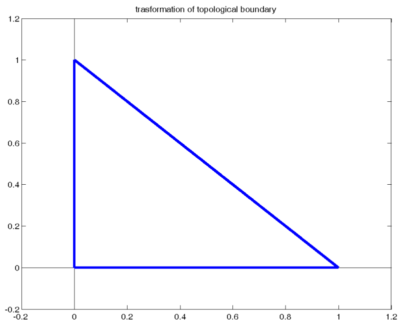
Let $E = F = [0, 1]$ be the strategy sets and let f_a be the a-biloss (disutility) function

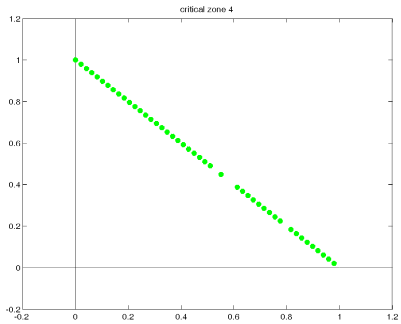
$$f_a\,(x, y) = \left(x - (1 - a)xy, y - (1 - (1 - a^{0.1})^{10})xy\right),$$

for all x, y a in [0, 1]. The critical zone of the a-biloss function is

$$\mathcal{C}(f_a) = \left\{(x, y) \in [0, 1]^2 : 1 - (1 - (1 - a^{0.1})^{10})x - (1 - a)y = 0\right\}.$$

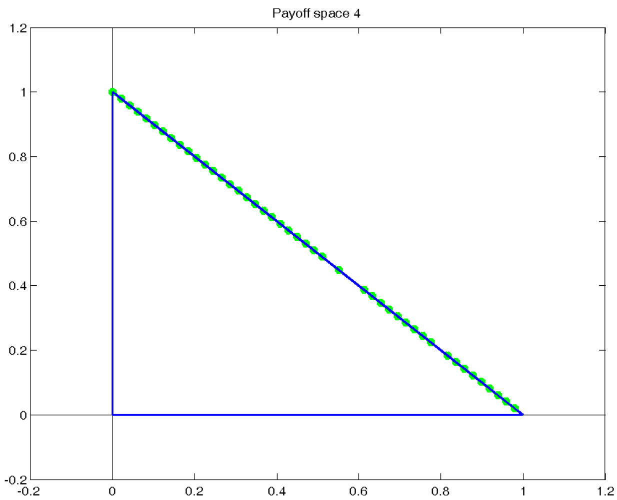
In the figure 8.2, the payoff space of the family 8.1 is depicted. In particular, we can distinguish the transformation of the topological boundary in 8.2(a), the transformation of the critical zone in 8.2(b) and the entire payoff in 8.2(c).





(a) Transformation of the topological boundary

(b) Transformation of the critical zone



(c) Payoff Space

Figure 8.1: The first game.

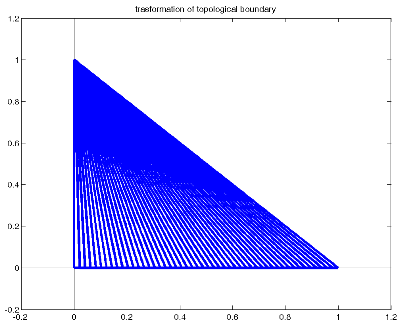
8.8 Third game $p = 0.5$

Let the strategy sets of the parametric game $G = (f, <)$ be $E = F = [0, 1]$ and let the a-biloss (disutility) function of G be defined by

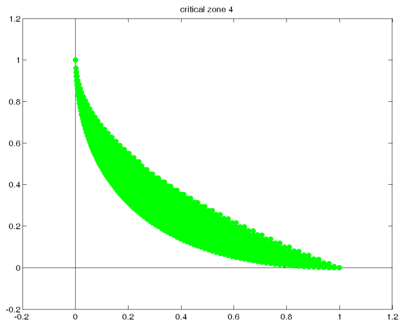
$$f_a(x, y) = \left(x - (1-a)xy, y - (1 - (1 - a^{0.5})^2)xy\right),$$

for all x, y and a in $[0, 1]$. The critical zones, in Figure **??**, are the sets

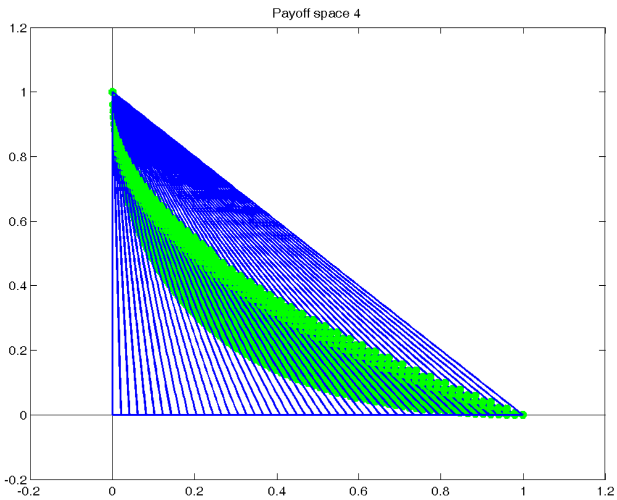
$$\mathcal{C}(f_a) = \left\{(x, y) \in [0, 1]^2 : 1 - (1 - (1 - a^{0.5})^2)x - (1 - a)y = 0\right\},$$



(a) Transformation of the topological boundary



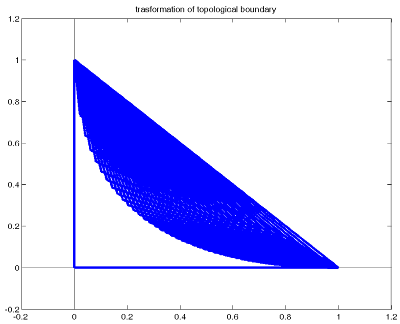
(b) Transformation of the critical zone

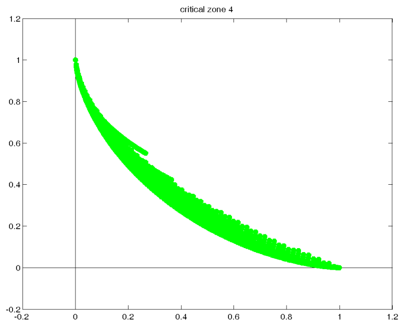


(c) Payoff Space

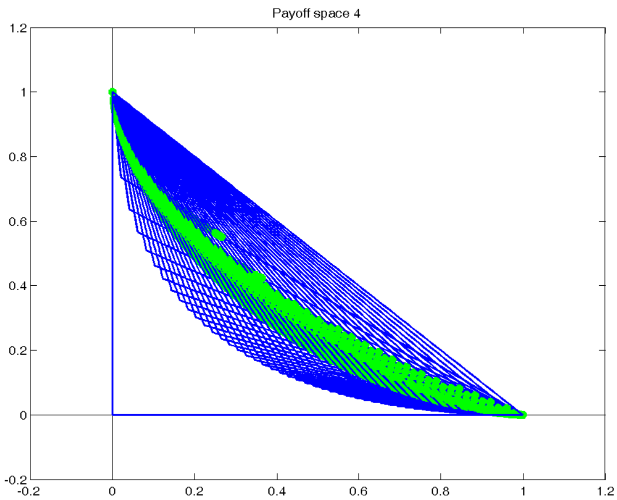
Figure 8.2: The second game.

with a varying in T. Also in this case, in the figure 8.3, we illustrate the obtained payoff space of the family 8.1, where the transformation of the topological boundary is given in 8.3(a), the transformation of the critical zone in 8.3(b) and the entire payoff in 8.3(c).





(a) Transformation of the topological boundary (b) Transformation of the critical zone



(c) Payoff Space

Figure 8.3: The third game.

8.9 Forth game $p = 2$

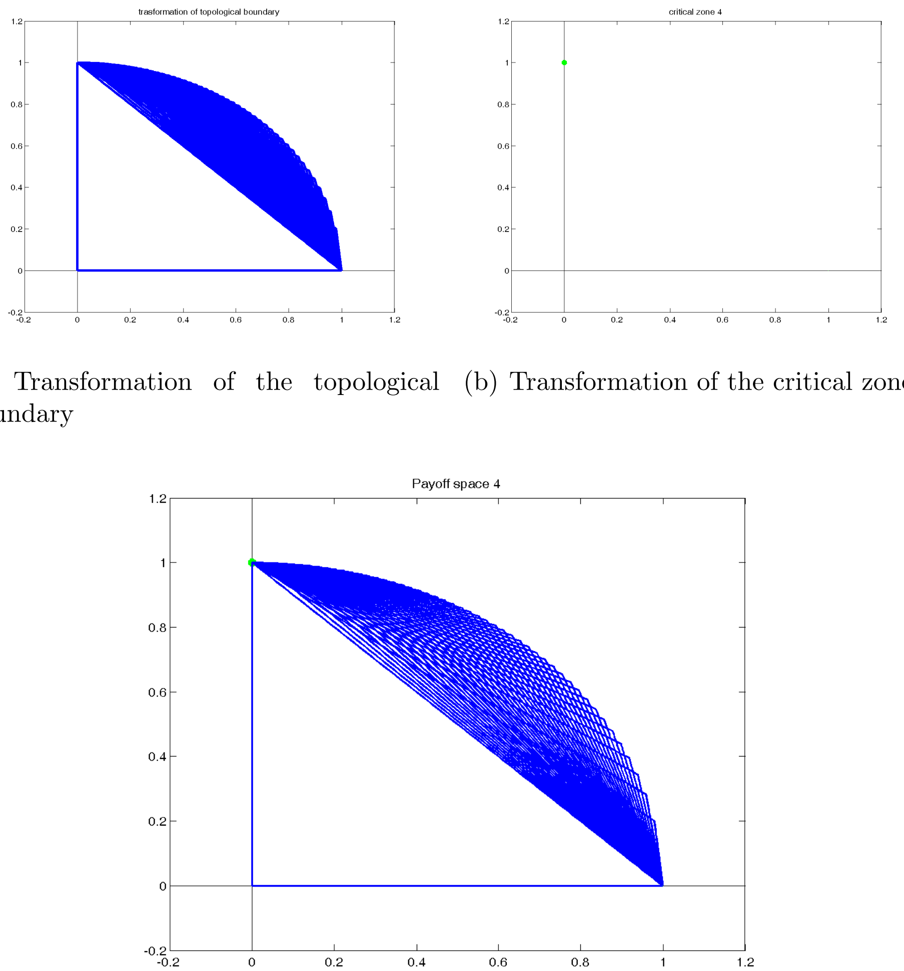
Let the strategy sets of the parametric game $G = (f, <)$ be $E = F = [0, 1]$ and let the a-biloss (disutility) function of G be defined by

$$f_a(x, y) = \left(x - (1 - a)xy, y - (1 - (1 - a^2)^{0.5})xy\right),$$

for all x, y and a in [0, 1].
The a-critical zone is

$$\mathcal{C}(f_a) = \left\{(x,y) \in [0,1]^2 : 1 - (1-(1-a^2)^{0.5})x - (1-a)y = 0\right\}.$$

In the figure 8.4, we show the payoff space of the family 8.1. We can distinguish the transformation of the topological boundary in 8.4(a), the transformation of the critical zone in 8.4(b) and the entire payoff in 8.4(c).



(a) Transformation of the topological boundary (b) Transformation of the critical zone

(c) Payoff Space

Figure 8.4: The forth game.

8.10 Fifth game $p = 10$

Let strategy sets be $E = F = [0, 1]$ and biloss (disutility) function be

$$f_a(x, y) = \left(x - (1-a)xy, y - (1 - (1-a^{10})^{0.1})xy\right),$$

for all (x, y) and a in $[0, 1]$. The critical zones, in Figure 8.5(b), are

$$\mathcal{C}(f_a) = \left\{(x, y) \in [0, 1]^2 : 1 - (1 - (1-a^{10})^{0.1})x - (1-a)y = 0\right\},$$

with a varying in T. Together with the transformation of the topological boundary 8.5(a), we obtain the payoff space 8.5(c).

8.11 Sixth game

In this section we present a new game, where strategy sets are $E = F = [0, 2]$, the parameter set is $T = [0, 1]$ and the a-biloss (disutility) function is defined by

$$f_a(x, y) = \left(x + y + a, x - y + a^2\right),$$

for all x, y in $[0, 2]$ and a in $[0, 1]$.

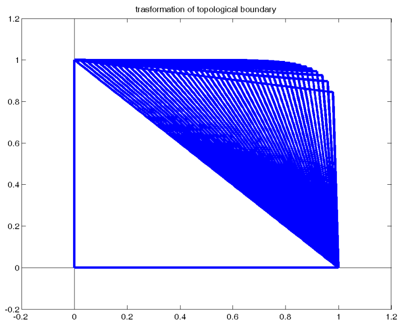
The critical zone is void, so the payoff spaces overlap the transformations of the topological boundary, in Figure 8.6.

8.12 Seventh game

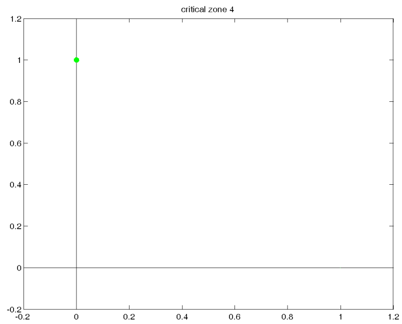
In this section we present a new game, where strategy sets are $E = F = [0, 2]$, the parameter set is $T = [-1, 1]$ and the a-biloss (disutility) function is

$$f_a(x, y) = (x + y + a, x - y + |a|),$$

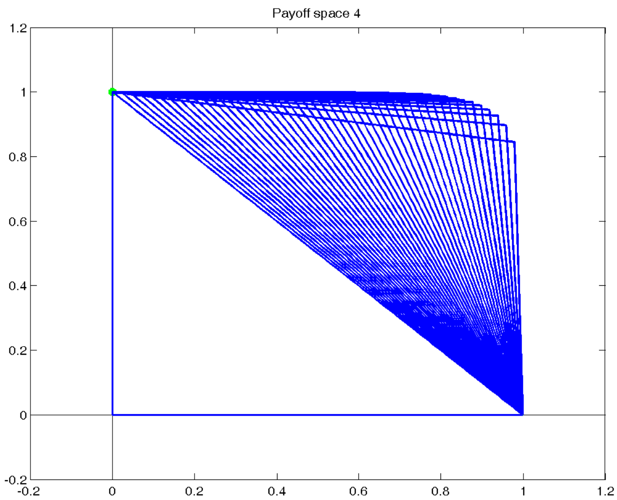
for all x, y, a in $[-1, 1]$.



(a) Transformation of the topological boundary



(b) Transformation of the critical zone



(c) Payoff Space

Figure 8.5: The fifth game.

8.13 Eight game

In this section we present a new game, where strategy sets are $E = F = [0, 1]$, the parameter set is $T = [-1, 1]$ and the a-biloss (disutility) function is

$$f_a(x, y) = (x(1 - x - y) + a, y(1 - x - y) - (1/6)(a - 3)^2 + 3/2),$$

for all x, y in $[0, 1]$, a in $[0, 6]$. In the figure 8.8, we show the obtained payoff space of the family 8.1. In particular, we can distinguish the transformation of the topological boundary in 8.8(a), the transformation of the critical zone

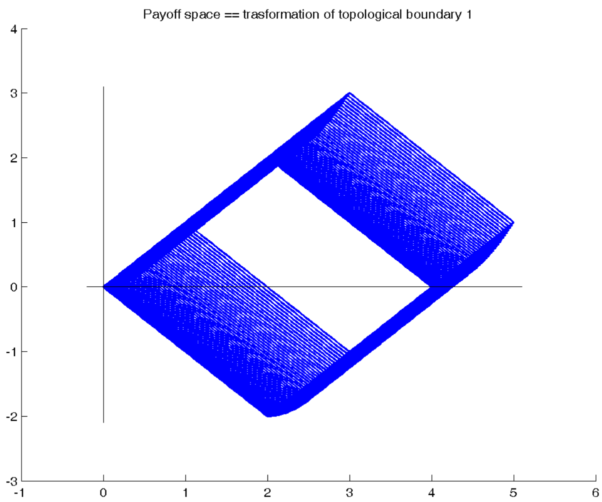


Figure 8.6: Payoff Space of the sixth game.

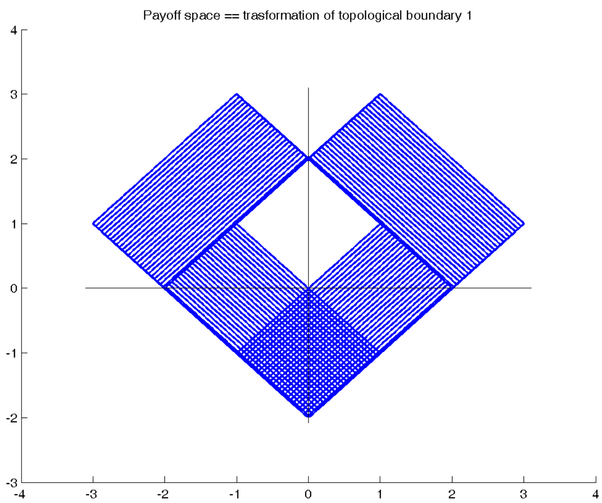
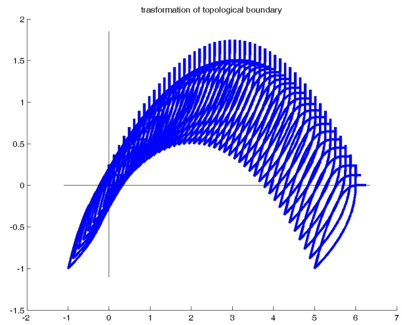
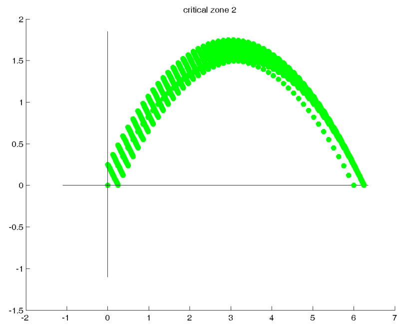


Figure 8.7: Payoff Space of the seventh game.

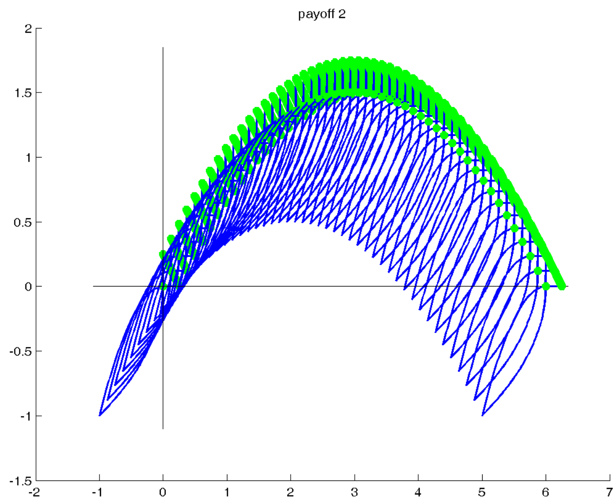
in 8.8(b) and the entire payoff in 8.8(c).



(a) Transformation of the topological boundary



(b) Transformation of the critical zone



(c) Payoff Space

Figure 8.8: The eighth game.

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