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Omar BENSLIMANE

Existence of solutions to elliptic problems in variable exponent spaces

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THESIS

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Omar BENSLIMANE

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Existence of solutions to elliptic Problems in variable exponent spaces

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Pr. Jaouad Bennouna	*Faculty of Sciences Dhar El Mahraz-Fez*	Supervisor
Pr. Ahmed Aberqi	*National School of Applied Sciences-Fez*	Co-Supervisor
Pr. Maria Alessandra Ragusa	*Universitá di Catania-Italy*	Rapporteur
Pr. Abdelhak Elmahi	*C.R.M.E.F. -Rabat*	Rapporteur
Pr. Youssef Akdim	*Faculty of Sciences Dhar El Mahraz-Fez*	Rapporteur
Pr. Mounir Mekkour	*Faculty of Sciences Dhar El Mahraz-Fez*	Member
Pr. Adnane Azzouzi	*Faculty of Sciences Dhar El Mahraz-Fez*	Member
Pr. Abdelmoujib Benkirane	*Faculty of Sciences Dhar El Mahraz-Fez*	Guest
Pr. Elhoussine Azroul	*Faculty of Sciences Dhar El Mahraz-Fez*	President

To my beloved mother.
To my beloved father.
To my beloved sisters.

I thank my parents **Said** and **Dounia**, my sisters **Zineb** and **Khaoula** whose work could not have been completed without their tireless support and encouragement. May they find in the completion of this work the culmination of their efforts and the expression of my most affectionate gratitude.

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Résumé

Au cours de la dernière décennie, une importante littérature traite de différents aspects des EDP dont la partie principale de l'opérateur a une croissance de type puissance, l'exemple principal étant le p-Laplacien. Il existe un large éventail de directions dans lesquelles le cas de la croissance polynomiale a été développé, notamment les approches à exposant variable, convexe, pondérée et à double phase.

L'objectif de cette thèse est d'appliquer ces approches à plusieurs espaces. Dans le premier chapitre, nous rappelons les Définitions, propositions, Lemmes et Théorèmes pertinents et nécessaires (et nous prouvons certains de ces Théorèmes) que nous utiliserons dans notre analyse. Dans le deuxième chapitre, nous nous intéressons à la solvabilité des résultats d'existence et d'unicité d'une classe d'équations elliptiques anisotropes avec le second terme, qui est un terme d'ordre inférieur et de croissance non polynomiale ; décrit par un N-uplet de N-fonctions satisfaisant la condition Δ_2 dans le cadre d'espaces de Sobolev-Orlicz anisotropes avec un domaine général. Ensuite, dans le troisième chapitre, nous nous concentrons dans l'étude de l'existence et de l'unicité d'une classe de problèmes elliptiques unilatéraux non linéaires $(\mathscr{P})$ dans un domaine général, gérés par un terme d'ordre bas et une croissance non polynomiale décrite par un N-uplet de N-fonction satisfaisant la condition Δ_2. De plus, le terme source est simplement intégrable. Parmi le large champ dans lequel le cas de la croissance polynomiale a été développé, nous avons introduit dans le quatrième chapitre une nouvelle classe de problèmes d'approximation correspondant à une équation d'obstacle quasilinéaire, qui implique un opérateur elliptique général à exposants variables sous forme de divergence, appelé opérateur d'obstacle à double phase à exposants variables, et sur la base du théorème du Mountain Pass, des outils de l'analyse non lisse, et de certaines hypothèses appropriées, nous prouvons l'existence de solutions faibles. L'étude de ce type de problèmes est à la fois importante et pertinente. D'une part, nous avons la motivation physique, puisque l'opérateur à double phase a été utilisé pour modéliser les solutions en régime permanent des problèmes de réaction-diffusion, qui se présentent en biophysique, en physique des plasmas et dans l'étude des réactions chimiques. D'autre part, ces opérateurs fournissent un paradigme utile pour décrire le comportement des matériaux fortement anisotropes, dont les propriétés de durcissement sont liées à l'exposant régissant la croissance du gradient changent radicalement avec le point, où le coefficient $\mu(\cdot)$ détermine la géométrie d'un composite constitué de deux matériaux différents. Dans le dernier chapitre, nous passons à un autre espace qui connaît actuellement un grand développement ; les variétés riemanniennes de Sobolev-Orlicz à exposant variable. Nous prouvons l'inégalité de Hölder, les résultats d'encastrement continu et compact. De plus, nous étudions l'existence de solutions non négatives non triviales pour une classe de problèmes à double phase où le terme source est une fonction de Carathéodory qui satisfait la condition de type Ambrosetti-Rabinowitz comme application.

Mots clés : Équation elliptique anisotropique, Problème d'obstacle, Solution entropique, Espaces de Sobolev-Orlicz anisotropes, Domaine général, Opérateur à double phase, Problèmes variationnels, Espaces d'Orlicz-Sobolev à exposant variable, Collecteur Riemannien de Sobolev-Orlicz, Collecteur de Nehari, Existence et unicité.

Abstract

Over the last decade, a large literature describes various aspects of PDEs whose main part of the operator has power-type growth with the leading example of the p-Laplacian. There is a wide range of directions in which the polynomial growth case has been developed, including variable exponent, convex, weighted and double phase approaches.

The purpose of this thesis is to apply those approaches to a several spaces. In the first chapter, we recall the relevant and the necessary Definitions, propositions, Lemmas and Theorems (and we prove some of these Theorems) that we will use in our analysis. In the second chapter, we are interested to the solvability of the existence and uniqueness results of a class of anisotropic elliptic equations with the second term, which is a low-order term and non-polynomial growth ; described by an N-uplet of N-function satisfying the Δ_2-condition in the framework of anisotropic Sobolev-Orlicz spaces with a general domain. Next, in the third chapter, we are focused in the study of the existence and uniqueness of a class of nonlinear unilateral elliptic problem $(\mathscr{P})$ in a general domain, managed by a low-order term and non-polynomial growth described by an N-uplet of N-function satisfying the Δ_2-condition. As well as, the source term is merely integrable. Among the wide scope in which the case of polynomial growth has been developed, we have introduced in the fourth chapter a new class of the approximating problems corresponding to a quasilinear obstacle equations, which involves a general variable exponents elliptic operator in divergence form, called double phase obstacle operator with variable exponents, and based on the mountain pass theorem, tools from non-smooth analysis, and some suitable assumptions, we prove the existence of weak solutions. The study of this type of problems is both significant and relevant. In the one hand, we have the physical motivation ; since the double phase operator has been used to model the steady-state solutions of reaction-diffusion problems, that arise in biophysic, plasma-physic and in the study of chemical reactions. In the other hand, these operators provide a useful paradigm for describing the behaviour of strongly anisotropic materials, whose hardening properties are linked to the exponent governing the growth of the gradient change radically with the point, where the coefficient $\mu(\cdot)$ determines the geometry of a composite made of two different materials. In the last chapter, we pass to another space which is currently undergoing a great development ; the Riemannian Sobolev-Orlicz manifolds with variable exponents. We prove the Hölder inequality, the continuous and compact embedding results. Furthermore, we study the existence of non-negative non- trivial solutions for a class of double-phase problems where the source term is a Caratheodory function that satisfies the Ambrosetti-Rabinowitz type condition as an application.

Key words : Anisotropic elliptic equation, Obstacle problem, Entropy solution, Sobolev–Orlicz anisotropic spaces, General domain, Double phase operator, Variational problems, Variable exponent Orlicz-Sobolev spaces, Sobolev-Orlicz Riemannian manifold, Nehari manifold.

Symbol Description

$\forall$	for all
$\exists$	there exists
$\equiv$	equivalent
$\sum$	summation
$\mathbb{N}$	set of natural numbers
$\mathbb{R}$	set of real numbers
N	positive integer greater than or equals to 1
$\mathbb{R}^N$	Euclidean space of N-dimensional vectors
Ω	open bounded subset of $\mathbb{R}^N$
$\partial\Omega$	boudary of Ω
$C(\Omega)$	the spaces of continuous functions on Ω
$C^\infty(\Omega)$	the spaces of infinitely differentiable functions on Ω
$C_0^\infty(\Omega)$	infinitely differentiable functions with compact support on Ω
ω	unbouded domain
$\mathring{W}_B^1(\Omega)$	Sobolev–Orlicz anisotropic space
B	N-function
M	Manifolds
$\overline{M}$	closure of M
$\mathscr{P}(M)$	the set of all measurable functions $p(\cdot): M \to (1,\infty)$
$\mathscr{P}^{log}(M)$	the set of log-Hölder continuous variable exponents
g	Riemannian metric
∇u	gradient of a function u
a.e	almost everywhere
$\longrightarrow$	strong convergence
$\rightharpoonup$	weak convergence
$\hookrightarrow$	continuous embedding
$\hookrightarrow\hookrightarrow$	compact embedding
X	arbitrary Banach space
X'	dual space of the Banach space X
supp f	support of function u
$\langle,\rangle$	scalar product of $\mathbb{R}^N$, duality between X and X'

Introduction

For several years, great efforts have been devoted to the study of nonlinear elliptic equations with an operator described by polynomial growth, which is motivated, for example, in the classical Sobolev space, not only by the description of many phenomena appearing in the applied sciences, due to the study of fluid filtration in porous media, constrained heating, elasto-plasticity, optimal control, financial mathematics, and others. Interested readers may refer to [14, 19, 23, 45] and the references therein for more background of applications. But also by the mathematical importance in the theory of this space. In addition, there is a vast literature describing various aspects of PDEs whose main part of the operator has a power-like growth with the preeminent example of the p-Laplacian. There is a wide range of directions in which the polynomial growth case has been developed, including variable exponent, convex, weighted, and double-phase approaches. For example, the double phase problem. Zhikov was the first who studied this type of problem in order to describe models of strongly anisotropic materials by studying the functional

$$u \longmapsto \int_{\Omega} (|\nabla u|^p + \mu(x)|\nabla u|^q)\, dx, \tag{1}$$

where the integrand switches two different elliptic behaviours. For more results see [105–107]. Then, several interesting works have been carried out on the double phase problem with a Dirichlet boundary condition. For a deeper comprehension, we refer the reader to [49, 65, 80, 82, 88, 100–103] and the references therein.

In the anisotropic Sobolev-Orlicz space ; a space almost as old as the classical Sobolev space as we know it today, constructed by Krasnosel'skii, and Rutickii [76] in 1968 and recently developed by Cianchi [48] in 2000. The most exhaustive studies on this space were developed in several interesting works. For example, Kozhevnikova in [74] established the existence of entropy solutions to the following problem :

$$(\mathscr{P}_1)\begin{cases} \displaystyle\sum_{i=1}^{N} (\, a_i(x,\nabla u)\,)_{x_i} = a_0(x,u) & \text{in } \ \Omega, \\ u(x) = \psi(x) & \text{on } \ \partial\Omega, \end{cases}$$

where Ω be an arbitrary domain in $\mathbb{R}^N, N \geq 2$.

$$a_0(x,s_0) = a_0(x,\psi) + b(x,s_0),$$

with $a_0(x,\psi) \in L^1(\Omega)$, the function $b(x,s_0)$ satisfies the Carathéodory condition and decreases in

$s_0 \in \mathbb{R}$, $b(x,\psi) = 0$ for all x in Ω, therefore $\forall x \in \Omega$, $s_0 \in \mathbb{R}$

$$b(x,s_0)\,(s_0 - \psi) > 0.$$

The author also supposed two other conditions ; the first one

$$\sup_{|s_0| \leq k} |\,b(x,s_0)\,| = G_k(x) \in L_{1,loc}(\Omega),$$

the second one, $\delta_0 > 0$ such as

$$b(x,\psi \pm \delta_0) \in L^1(\Omega).$$

In [75] Kozhevnikova proved again the existence of entropy solution to the following second-order anisotropic quasilinear elliptic equation :

$$(\mathscr{P}_2)\begin{cases} \sum_{i=1}^{N}(\,a_i(x,u,\nabla u)\,)_{x_i} - a_0(x,u,\nabla u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a_i(x,s_0,s)$, $i = 0,\cdots,N$are Carathéodory functions. In [47] Chmara and Maksymiuk, shown the existence of periodic solution for Euler-Lagrange equation, with the lagrangian consists of kinetic part (an anisotropic G-function), potential part $K - W$ and a forcing term, using the Mountain Pass Theorem. Then, they consider two situations : G satisfying $\Delta_2 \cap \nabla_2$ in infinity and globally, and finally, they give conditions on the growth of the potential near zero for both situations. For more results, we refer the reader to [4, 13, 15, 20, 26, 28, 29, 31, 36, 46, 66] and the references therein.

All these previous relevant contributions in which the exponent $p =$ cste, lead us to study the theories and applications of Lebesgue-Sobolev spaces with variable exponent, and the Sobolev-Orlicz with variable exponent in complete compact Riemannian manifolds in which the exponent $p(x)$ is a function. The first spaces was introduced by Orlicz [86] in 1931, and he was interested in the study of function spaces that contain all measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\rho(\lambda u) = \int_{\Omega} \phi(\lambda|u(x)|)\, dx,$$

for some $\lambda > 0$ and ϕ satisfying some natural assumptions, where Ω is an open set in $\mathbb{R}^N$. Then, in 1999, Fan and Zhao [61], presented some basic results on the generalized Lebesgue spaces $L^{p(x)}(\Omega)$ and generalized Lebesgue-Sobolev spaces $W^{1,p(x)}(\Omega)$. After that, the same authors in [60, 62] proved some embedding theorems for space $W^{1,p(x)}(\Omega)$ like the compact embedding theorems with symmetry of Strauss-Lions type, we refer also to [73] for more results. As a wide range of directions in which the polynomial growth case has been developed, we have the double phase problems with variable exponent, which were announced by Ragusa and Tachikawa in [91–95], they have achieved the regularity theory for minimizers of (1) with variable exponent. After that, these types of problems have been attacked by several researchers, we refer the reader to [27, 30, 44, 97, 98] for a deeper comprehension. When $p = p(x)$, the previous space is called by the Sobolev spaces with anisotropic variable exponent, was introduced by Mihăilescu, Pucci, and Rădulescu in [83, 84]. Moreover, one of the first contributions in this direction is due to Fragalà, Filippo, and Bernd in

[58]. The necessity of such theory appears naturally when we want to consider materials with inhomogeneities that behave differently in different directions in space. Since this subject is relatively new, only few papers have been published, see for example [6, 10, 37, 42, 51, 54, 56, 59, 71] and the references therein. Finally, we move on to Sobolev-Orlicz with variable exponent in complete compact Riemannian manifolds. This space is currently undergoing great development. Moreover, analysis proves to be a very powerful tool for solving geometrical problems. Conversely, geometry may help us to solve certain problems in analysis, as pointed out in Gaczkowski, Górka, and Pons [63, 64], they are presented many results in the theory of Sobolev space for compact and non-compact manifolds. For more details we quote [1, 2, 7, 8, 16, 30, 68] and the references therein.

Our Goal

Our objective in this thesis is to establish a more involved version of equations $(\mathscr{P}_1)$ and $(\mathscr{P}_2)$ given by $(\mathscr{P})$ (see chapter 2 below), which is managed by low-order term and non-polynomial growth ; described by an N-uplet of N-function satisfying the Δ_2-condition, the source f is merely integrable. Using an approximation procedure and some priori estimates, we proved the existence and uniqueness of entropy solution to the problem $(\mathscr{P})$ in the general domain. Next, we prove the existence and uniqueness of a class of nonlinear unilateral elliptic problems like

$$\begin{cases} A(u) + \sum_{i=1}^{N} b_i(x,u,\nabla u) = f & \text{in } \ \Omega\,, \\ u \geq \psi & \text{a.e in } \ \Omega\,, \end{cases}$$

in a general domain too, managed by a low-order term and non-polynomial growth described by an N-uplet of N-function satisfying the Δ_2-condition, and the obstacle ψ is a measurable function belongs to $L^\infty(\Omega) \cap \mathring{W}^1_B(\Omega)$. The source term is merely integrable. Moving on to the Sobolev-Orlicz space with variable exponent. We introduce a new class of the approximating problems corresponding to a quasilinear obstacle equation, which involves a general variable exponent elliptic operator in divergence form, called double phase obstacle operator with variable exponents, and based on the mountain pass theorem, tools from non-smooth analysis, and some suitable assumptions, we prove the existence of weak solutions. Finally, we go to another novel space ; the Sobolev-Orlicz spaces with variable exponents in complete compact Riemannian n-manifolds, and we prove the Hölder inequality, the continuous and compact embeddings. Moreover, using the method of Nehari manifold combined with the fibering maps, we prove the existence of non-negative non-trivial solutions to a class of double-phase problems where the source term is a Caratheodory function that satisfies the Ambrosetti-Rabinowitz type condition.

Outline

This thesis is composed of six chapters :

- In the first chapter, we give all necessary and relevant Defintions, Properties, Lemmas and Theorems (and we prove some of these theorems) that help us in our analysis.

— In the second chapter, we study the existence and the uniqueness solution of an anisotropic elliptic problems with the second term, which is a low-order term and non-polynomial growth ; described by an N-uplet of N-function satisfying the Δ_2-condition in the framework of anisotropic Sobolev-Orlicz spaces with a general domain.

— In the third chapter, we prove the existence and the uniqueness solution of an unilateral anisotropic elliptic problem, in general domain, managed by a low-order term and non-polynomial growth ; described by an N-uplet of N-function satisfying the Δ_2-condition.

— In the fourth chapter, we introduce a new class of the approximating problems corresponding to a quasi-linear obstacle equations, which involves a general variable exponents elliptic operator in divergence form, called double phase obstacle operator with variable exponents, and we demonstrate the existence of the solutions.

— In the Final chapter, we prove the Hölder's inequality, the continuous and compact embedding of the Sobolev-Orlicz spaces with variable exponents in complete compact Riemannian n-manifolds and as an application, we establish the existence of non-negative non-trivial solutions to a double phase problems using the Nehari manifold method.

Table des matières

1

Preliminaries

This chapter is devoted to recalling some definitions and proving some results that will play a role in this thesis.

1.1 Anisotropic Sobolev-Orlicz space

In this section, we briefly review some basic facts about Sobolev-Orlicz anisotropic space which we will need in our analysis in Chapters 2 and 3. A comprehensive presentation of Sobolev-Orlicz anisotropic space can be found in the work of M.A Krasnoselskii and Ja. B. Rutickii [76] and [48].

1.1.1 N-Functions

Definition 1.1.1. *We say that* $B : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *is a N-function if* B *is continuous, convex, with* $B(\theta) > 0$ *for* $\theta > 0$, $\frac{B(\theta)}{\theta} \to 0$ *when* $\theta \to 0$ *and* $\frac{B(\theta)}{\theta} \to \infty$ *when* $\theta \to \infty$.
This N-function B *admit the following representation :* $B(\theta) = \int_0^\theta b(t)\, dt$, *with* $b : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *which is an increasing function on the right, with* $b(0) = 0$ *in the case* $\theta > 0$ *and* $b(\theta) \longrightarrow \infty$ *when*

$\theta \longrightarrow \infty$.

Its conjugate is noted by $\bar{B}(\theta) = \int_0^{|\theta|} q(t)\, dt$ *with q also satisfies all the properties already quoted from b, with*

$$\bar{B}(\theta) = \sup_{\mu \geq 0} (\mu\, |\theta| - B(\mu)), \quad \theta > 0. \tag{1.1}$$

*The **Young's inequality** is given as follow*

$$\forall \theta, \mu > 0 \qquad \theta\, \mu \leq B(\mu) + \bar{B}(\theta). \tag{1.2}$$

Definition 1.1.2. *The N-function* $B(\theta)$ *satisfies the* Δ_2-*condition if* $\exists c > 0,\, \theta_0 \geq 0$ *such as*

$$B(2\,\theta) \leq c\, B(\theta) \qquad \theta \geq \theta_0. \tag{1.3}$$

This definition is equivalent to, $\forall k > 1,\ \exists\, c(k) > 0$ *such as*

$$B(K\,\theta) \leq c(K)\, B(\theta) \quad \textit{for} \quad \theta \geq \theta_0. \tag{1.4}$$

Definition 1.1.3. *The N-function* $B(\theta)$ *satisfies the* Δ_2-*condition as long as there exists positive numbers* $c > 1$ *and* $\theta_0 \geq 0$ *such as for* $\theta \geq \theta_0$ *we have*

$$\theta\, b(\theta) \leq c\, B(\theta). \tag{1.5}$$

Also, each N-function $B(\theta)$ *satisfies the inequality*

$$B(\mu + \theta) \leq c\, B(\theta) + c\, B(\mu) \qquad \theta,\, \mu \geq 0. \tag{1.6}$$

Proposition 1.1.1.

$$\theta\, B'(\theta) = \bar{B}(B'(\theta)) + B(\theta), \quad \theta > 0, \tag{1.7}$$

with B' *is the right derivative of the N-function* $B(\theta)$.

Proof. By (1.2), we take $\mu = B'(\theta)$, then we obtain

$$B'(\theta)\, \theta \leq B(\theta) + \bar{B}(B'(\theta)),$$

and by Ch. I [76], we get the result. □

1.1.2 Anisotropic Sobolev-Orlicz space

The Orlicz space $L_B(\Omega)$ is defined as equivalence classes modulo the equality a.e on Ω of real measurable functions on Ω such that

$$\int_\Omega B(u(x))\,dx < +\infty \qquad (\text{resp.} \int_\Omega B\left(\frac{u(x)}{k}\right)\,dx < +\infty \text{ for some } k > 0).$$

The Orlicz space $L_B(\Omega)$ provided with the norm of Luxemburg given by

$$||u||_{B,\Omega} = \inf\left\{ k > 0 / \int_\Omega B\left(\left|\frac{u(x)}{k}\right|\right)\,dx \leq 1\right\}. \tag{1.8}$$

According to [76] we obtain the inequalities

$$\int_\Omega B\left(\left|\frac{u(x)}{||u||_{B,\Omega}}\right|\right)\,dx \leq 1, \tag{1.9}$$

and

$$||u||_{B,\Omega} \leq \int_\Omega B(|u|)\,dx + 1. \tag{1.10}$$

Moreover, the **Hölder's inequality** holds and we have for all $u \in L_B(\Omega)$ and $v \in L_{\bar{B}}(\Omega)$

$$\left|\int_\Omega u(x)\,v(x)\,dx\right| \leq 2\,||u||_{B,\Omega}\,.\,||v||_{\bar{B},\Omega}. \tag{1.11}$$

Proposition 1.1.2. *[48, 76] If $P(\theta)$ and $B(\theta)$ are two N-functions such as $P(\theta) \ll B(\theta)$ and $meas\,\Omega < \infty$, then $L_B(\Omega) \subset L_P(\Omega)$, furthermore*

$$||u||_{P,\Omega} \leq A_0\,(meas\,\Omega)\,||u||_{B,\Omega} \qquad u \in L_B(\Omega). \tag{1.12}$$

And for all N-functions $B(\theta)$, if $meas\,\Omega < \infty$, then $L_\infty(\Omega) \subset L_B(\Omega)$ with

$$||u||_{B,\Omega} \leq A_1\,(meas\,\Omega)\,||u||_{\infty,\Omega} \qquad u \in L_B(\Omega). \tag{1.13}$$

Also for all N-functions $B(\theta)$, if $meas\,\Omega < \infty$, then $L_B(\Omega) \subset L^1(\Omega)$ with

$$||u||_{1,\Omega} \leq A_2\,||u||_{B,\Omega} \qquad u \in L_B(\Omega). \tag{1.14}$$

We define for all N-functions $B_1(\theta), \cdots, B_N(\theta)$ the space of Sobolev-Orlicz anisotropic $\mathring{W}^1_B(\Omega)$

as the adherence space $C_0^\infty(\Omega)$ under the norm

$$||u||_{\mathring{W}_B^1(\Omega)} = \sum_{i=1}^{N} ||u_{x_i}||_{B_i,\Omega}. \tag{1.15}$$

Definition 1.1.4. *A sequence $\{u_m\}$ is said to converge modularly to u in $\mathring{W}_B^1(\Omega)$ if for some $k > 0$ we have*

$$\int_\Omega B\left(\left|\frac{u_m - u}{k}\right|\right) dx \longrightarrow 0 \quad as \quad m \longrightarrow \infty. \tag{1.16}$$

Remark 1. *Since B satisfies the Δ_2-condition, then the modular convergence coincide with the norm convergence.*

Remark 2. *If the doubling condition is imposed on the modular function, but not on the conjugate, then the space for the solutions to exist is non-reflexive in general. For this raison we will assume in the remainder of this thesis that B satisfies the both conditions ; Δ_2-condition and ∇_2-condition, so the propositions 1.1.3 and 1.1.4 will remain true.*

Proposition 1.1.3. *[76] The Sobolev-Orlicz anisotropic space $\mathring{W}_B^1(\Omega)$ is complete and reflexive.*

Proposition 1.1.4. *[76] The Sobolev-Orlicz anisotropic $\mathring{W}_B^1(\Omega)$ is separable.*

The following lemmas are also true :

Lemma 1.1.1. *[67] For all $u \in \mathring{W}_{L_B}^1(\Omega)$ with $meas\,\Omega < \infty$, we have*

$$\int_\Omega B\left(\frac{|u|}{\lambda}\right) dx \leq \int_\Omega B(|\nabla u|)\, dx,$$

where $\lambda = diam(\Omega)$, is the diameter of Ω.

Note by $h(t) = \left(\prod_{i=1}^{N} \frac{B_i^{-1}(t)}{t}\right)^{\frac{1}{N}}$ and we assume that $\int_0^1 \frac{h(t)}{t}\, dt$ converge, so we consider the N-functions $B^*(z)$ defined by $(B^*)^{-1}(z) = \int_0^{|z|} \frac{h(t)}{t}\, dt$.

Lemma 1.1.2. *[72] Let $u \in \mathring{W}_B^1(\Omega)$. If*

$$\int_1^\infty \frac{h(t)}{t}\, dt = \infty, \tag{1.17}$$

then, $\mathring{W}_B^1(\Omega) \subset L_{B^*}(\Omega)$ and $||u||_{B^*,\Omega} \leq \frac{N-1}{N} ||u||_{\mathring{W}_B^1(\Omega)}$.

If

$$\int_1^\infty \frac{h(t)}{t}\, dt < \infty,$$

then, $\mathring{W}_B^1(\Omega) \subset L_\infty(\Omega)$ *and* $||\,u\,||_{\infty,\Omega} \leq \beta\,||\,u\,||_{\mathring{W}_B^1(\Omega)}$, *with* $\beta = \int_0^\infty \frac{h(t)}{t}\,dt$.

For each N-function $B_i(z) = \int_0^{|z|} b_i(t)\,dt$ obeys the further condition

$$\lim_{\alpha\to\infty}\inf_{\theta>0} \frac{b_i(\alpha\,\theta)}{b_i(\alpha)} = \infty, \quad i = 1,\cdots,N. \tag{1.18}$$

Example 1.

$$B(z) = |z|^b(\ln(|z|)+1), \qquad b > 1,$$

satisfy (1.18) *and* Δ_2-*condition.*

Lemma 1.1.3. *Suppose that* $(X,\mathscr{T},meas)$ *is a measurable set such that* $mes(X) < \infty$. *Let* $\theta : X \longrightarrow [0,+\infty]$ *be a measurable function such that* $meas\{\,x \in X : \theta(x) = 0\,\} = 0$. *Then, for any* $\varepsilon > 0$, *there exist* $\delta > 0$ *such that the inequality*

$$\int_\Omega \theta(x)\,dx \leq \delta$$

implies the following inequality

$$meas(\Omega) \leq \varepsilon.$$

With Ω *is a bounded domain.*

Proof. See Lemma 2, in [34]. □

1.2 Sobolev space on Riemannian manifolds

This section is devoted to the reminder of some basic definitions and properties of Sobolev spaces on Riemannian manifolds, as well as to the proof of some lemmas, which will help us in our analysis in chapter 5.

1.2.1 Definitions and Propositions

Definition 1.2.1. *[16] A Riemannian metric g on M is a smooth* $(2,0)-$*tensor field on M such that for any* $x \in M$, $g(x)$ *is a scalar product on* $T_x(M)$.

Definition 1.2.2. ***Christoffel symbols :****[16]*

$$\Gamma_{ij}^K(x) = \frac{1}{2}\left((\frac{\partial g_{mj}}{\partial x_i})_x + (\frac{\partial g_{mi}}{\partial x_j})_x - (\frac{\partial g_{ij}}{\partial x_m})_x \right).g(x)^{mk},$$

where the g^{ij}'s represents the inverse matrix of g_{ij}, such as $g_{im}\,g^{mj} = \delta_i^j$.

Definition 1.2.3. *[16] Given (M,g) a smooth Riemannian manifold, and $\gamma : [a,b] \longrightarrow M$ a curve of class C^1, the length of γ is defined by*

$$l(\gamma) = \int_a^b \sqrt{g(\gamma(t)).\left(\left(\frac{d\gamma}{dt}\right)_t.\left(\frac{d\gamma}{dt}\right)_t\right)}\,dt,$$

and $\forall x,y \in M$ we define the distance $d_g(x,y)$ as

$$d_g(x,y) = \inf\left\{\, l(\gamma) \,:\, \gamma : [a,b] \longrightarrow M \;\; such\ that\;\; \gamma(a) = x\ and\ \gamma(b) = y \,\right\}.$$

Definition 1.2.4. *[16] Let (M,g) be a smooth Riemannian manifold. For K integer, and $u : M \longrightarrow \mathbb{R}$ smooth, we denote by $\nabla^K u$ the K^{th} covariant derivative of u, and $|\nabla^K u|$ the norm of $\nabla^K u$ defined in a local chart by*

$$|\nabla^K u| = g^{i_1 j_1} \cdots g^{i_K j_K}\,(\nabla^K u)_{i_1 \cdots i_K}\,(\nabla^K u)_{j_1 \cdots j_K}.$$

Since $(\nabla u)_i = \partial_i u$, then

$$(\nabla^2 u)_{ij} = \partial_{ij} u - \Gamma_{ij}^K \partial_K u.$$

And For $p \geq 1$ a real, K a positive integer, we define the Sobolev space as follow

$$L^p(M) = \left\{ u : M \longrightarrow \mathbb{R}\ measurable \,/ \int_M |u|^p\, dv(g) < \infty \right\}.$$

And the functions space as

$$\mathscr{C}_K^p(M) = \left\{ u \in C^\infty \,/\, \forall j = 0, \cdots, K,\ \int_M |\nabla^j u|^p\, dv(g) < +\infty \right\},$$

where M is compact. By default $\mathscr{C}_K^p(M) = C^\infty(M)\ \forall K, \forall p \geq 1$.

• *For $u \in \mathscr{C}_K^p(M)$, we have*

$$||u||_{W^{K,p}} = \sum_{j=0}^{K} \left(\int_M |\nabla^j u|^p\, dv(g) \right)^{\frac{1}{p}}.$$

Definition 1.2.5. *[16] The Sobolev space $W^{K,p}(M)$ is the completion of $\mathscr{C}_K^p(M)$ with respect to $||\cdot||_{W^{K,p}}$, where*

$$||u||_{W^{K,p}} = ||\nabla u||_p + ||u||_p.$$

Proposition 1.2.1. *[16] Let $||\cdot||_p$ be the norm of $L^p(M)$ defined by*

$$||u||_p = \left(\int_M |u|^p\, dv(g)\right)^{\frac{1}{p}}.$$

So :

1/ Any Cauchy sequence in $\left(\mathscr{C}_K^p(M), ||\cdot||_{W^{K,p}}\right)$ is a Cauchy sequence in the Lebesgue space $\left(L^p(M), ||\cdot||_p\right)$.

2/ Any Cauchy sequence in $\left(\mathscr{C}_K^p(M), ||\cdot||_{W^{K,p}}\right)$ that converges to 0 in the Lebesgue space $\left(L^p(M), ||\cdot||_p\right)$ also converges to 0 in $\left(\mathscr{C}_K^p(M), ||\cdot||_{W^{K,p}}\right)$.

Proposition 1.2.2. *[16, 70] For any K integer, $W^{2,K}(M)$ is a Hilbert space when equipped with the equivalent norm*

$$||u|| = \sqrt{\sum_{j=0}^{K} \int_M |\nabla^j u|^2\, dv(g)},$$

the scalar product $\langle\cdot,\cdot\rangle$ associated to $||\cdot||$ is defined by

$$\langle u,v\rangle = \sum_{j=0}^{K} \int_M \langle\nabla^j u, \nabla^j v\rangle\, dv(g),$$

where, $\langle\cdot,\cdot\rangle$ is the scalar product on covariant tensor fields associated to g.

Proposition 1.2.3. *[16] If $p > 1$, $W^{K,p}(M)$ is reflexive.*

1.2.2 Some Embeddings results

Proposition 1.2.4. *[16, 70] Let (M,g) be a complete compact Riemannian n-manifold. Then, if the embedding $L_1^1(M) \hookrightarrow L^{\frac{n}{n-1}}(M)$ holds, then whenever the real numbers q and p satisfy*

$$1 \leq q < n,$$

and

$$q \leq p \leq q* = \frac{nq}{n-q},$$

the embedding $L_1^q(M) \hookrightarrow L^p(M)$ also holds.

Lemma 1.2.1. *[16, 70] Let (M,g) be a smooth compact Riemannian n-manifold. Given p,q two real numbers with $1 \leq q < p$ and given K,m two integers with $0 \leq m < K$. If $\frac{1}{p} = \frac{1}{q} - \frac{(k-m)}{n}$*

then, $W^{K,q}(M) \subset W^{m,p}(M)$.

So, $\exists c > 0$, $\forall u \in W^{K,q}(M)$

$$||u||_{W^{m,p}} \leq c||u||_{W^{K,q}}.$$

In particular, for any $q \in [1,n)$ *real,* $W^{1,q}(M) \subset L^p(M)$ *where* $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$.

So, $\exists c > 0$ *such that for any* $u \in \mathscr{D}(\mathbb{R}^N)$ *(the set of smooth functions with compact support in* $\mathbb{R}^N$*),*

$$||u||_p \leq c||u||_{W^{1,q}}. \qquad \text{with} \qquad \frac{1}{p} \geq \frac{1}{q} - \frac{1}{n}$$

Lemma 1.2.2. *If* (M,g) *is complete, then* $\forall p \geq 1$ $W_0^{1,p}(M) = W^{1,p}(M)$.

Proof. Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$h(\zeta) = \begin{cases} 1 & \text{when } \zeta \leq 0 \text{ ,} \\ 1-\zeta & \text{when } 0 \leq \zeta \leq 1 \text{ ,} \\ 0 & \text{when } \zeta \geq 1 \text{ ,} \end{cases}$$

and let $u \in C_1^p(M)$ where p is a real number greater than or equal to 1. Let x, $y \in M$ and set

$$u_i(y) = u(y)\,h(\,d_g(x,y) - i\,),$$

where d_g is the Riemannian distance associated to g, $i \in \mathbb{N}$ $\Big($ just note that, if (M,g) be a Riemannian manifold and $u : M \longrightarrow \mathbb{R}$ a Lipschitzian function on M which equals zero outside a compact subset of M, then $u \in W^{1,p}(M)$ $\forall p \geq 1 \Big)$. Thus, $u_i \in W^{1,p}(M)$ $\forall i$ and $u_i = 0$, hence, for all i we have $u_i \in W_0^{1,p}(M)$. Independently, we have $\forall i$

$$\left(\int_M |u_i - u|^p \, dv(g) \right)^{\frac{1}{p}} \leq \left(\int_{M \setminus B_x(i)} |u|^p \, dv(g) \right)^{\frac{1}{p}},$$

and

$$\left(\int_M |\nabla(u_i - u)|^p \, dv(g) \right)^{\frac{1}{p}} \leq \left(\int_{M \setminus B_x(i)} |\nabla u|^p \, dv(g) \right)^{\frac{1}{p}} + \left(\int_{M \setminus B_x(i)} |u|^p \, dv(g) \right)^{\frac{1}{p}}.$$

Where $B_x(i)$ is the geodesic ball of centre x and radius i. Hence,

$$\lim_{i\to\infty} u_i = u \quad \text{in} \quad W^{1,p}(M).$$

$\square$

Lemma 1.2.3. *[16] Given (M,g) a smooth, complete Riemannian manifold, the set $\mathscr{D}(M)$ of smooth functions with compact support in M is dense in $W^{1,p}(M)$ $\forall p \geq 1$.*

Remark 3. *By Proposition 2.11 in [16], we have*
If we suppose that $W^{1,1}(M) \subset L^{\frac{n}{n-1}}(M)$, then $\exists c > 0$ such as $\forall u \in W^{1,1}(M)$,

$$\left(\int_M |u|^{\frac{n}{n-1}}\, dv(g)\right)^{\frac{n-1}{n}} \leq c \int_M \big(|\nabla u| + |u|\big)\, dv(g).$$

Lemma 1.2.4. *Let (M,g) be a smooth complete Riemannian n-manifold. Suppose that $W^{1,1}(M) \subset L^{\frac{n}{n-1}}(M)$, then for any numbers real $1 \leq q < p$ and any integers $0 \leq m < K$ such that $\frac{1}{p} = \frac{1}{q} - \frac{K-m}{n}$, we have*

$$W^{K,q}(M) \subset W^{m,p}(M).$$

Proof. By Remark 3, if $W^{1,1}(M) \subset L^{\frac{n}{n-1}}(M)$, then there exists $c > 0$ such that $\forall u \in W^{1,1}(M)$, we have

$$\left(\int_M |u|^{\frac{n}{n-1}}\, dv(g)\right)^{\frac{n-1}{n}} \leq c \int_M \big(|\nabla u| + |u|\big)\, dv(g).$$

Let $q \in (1,n)$, $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$ and $u \in \mathscr{D}(M)$, and let $\varphi = |u|^{p-\frac{p}{n}}$. By Hölder's inequality we obtain that

$$\begin{aligned}
\left(\int_M |u|^p\, dv(g)\right)^{\frac{n-1}{n}} &= \left(\int_M |\varphi|^{\frac{n}{n-1}}\, dv(g)\right)^{\frac{n-1}{n}} \\
&\leq c \int_M \big(|\nabla\varphi| + |\varphi|\big)\, dv(g) \\
&= \frac{c\,p(n-1)}{n} \int_M |u|^{p'} |\nabla u|\, dv(g) + c \int_M |u|^{p-\frac{p}{n}}\, dv(g) \\
&\leq \frac{c\,p(n-1)}{n} \left(\int_M |u|^{p'q'}\, dv(g)\right)^{\frac{1}{q'}} \left(\int_M |\nabla u|^q\, dv(g)\right)^{\frac{1}{q}} \\
&\quad + c\left(\int_M |u|^{p'q'}\, dv(g)\right)^{\frac{1}{q'}} \left(\int_M |u|^q\, dv(g)\right)^{\frac{1}{q}}.
\end{aligned}$$

where, $\frac{1}{q} + \frac{1}{q'} = 1$ and $1 = \frac{1}{n} + \frac{p'}{p} - \frac{1}{p}$. Moreover, $p'q' = p$ since $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$. Hence, we deduce that

$$\left(\int_M |u|^p\, dv(g)\right)^{\frac{1}{p}} \leq \frac{c\,p(n-1)}{n}\left(\left(\int_M |\nabla u|^q\, dv(g)\right)^{\frac{1}{q}} + \left(\int_M |u|^q\, dv(g)\right)^{\frac{1}{q}}\right). \quad (1.19)$$

And by Lemma 1.2.3, we get the result. □

Remark 4. *The fact that $W^{1,1}(M) \subset L^{\frac{n}{n-1}}(M)$ implies that there is a lower bound for the volume of balls with respect to their center.*

Proposition 1.2.5. *[16, 70] • Since M is compact, M can be covered by finite numbers of charts $(\Omega_m, \varphi_m)_{m=1,\cdots,N}$, such that $\forall m$ the components g_{ij}^m of g in (Ω_m, φ_m) satisfy*

$$\frac{1}{2}\delta_{ij} \leq g_{ij}^m \leq 2\,\delta_{ij},$$

as bilinear forms.

• Since M is assumed to be compact, (M,g) has finite volume. Hence, for $1 \leq q \leq q'$, we have $L^{q'}(M) \subset L^q(M)$.

Lemma 1.2.5. *[16] Let (M,g) be a smooth compact Riemannian manifold of n-dimension. Given p,q two real numbers with $1 \leq q < n$ and $p \geq 1$ such that $\frac{1}{p} > \frac{1}{q} - \frac{1}{n}$. The inclusion $W^{1,q}(M) \subset L^p(M)$ is compact.*

Lemma 1.2.6. *[16] Let $q \geq 1$ be real and let $m < K$ be two integers.*

- *If $\frac{1}{q} - \frac{(K-m)}{n} > 0$. Then,*

$$W^{k,q}(\mathbb{R}^N) \subset W^{m,p}(\mathbb{R}^N),$$

where $\frac{1}{p} = \frac{1}{q} - \frac{(K-m)}{n}$.

- *If $\frac{1}{q} - \frac{(K-m)}{n} < 0$. Then,*

$$W^{k,q}(\mathbb{R}^N) \subset \mathscr{C}_B^m(\mathbb{R}^N),$$

where $\mathscr{C}_B^m(\mathbb{R}^N)$ denotes the space of functions $u : \mathbb{R}^N \longrightarrow \mathbb{R}$ of class $\mathscr{C}^m$ for which the norm,

$$||u||_{\mathscr{C}^m} = \sum_{|\alpha|=0}^{m} \sup_{x\in\mathbb{R}^N} |D_\alpha u(x)| \text{ is finite.}$$

Lemma 1.2.7. *[16] Let (M,g) be a smooth compact Riemannian n-manifold, $q \geq 1$ real and $m < K$ two integers. If $\frac{1}{q} < \frac{K-m}{n}$ then, $W^{k,q}(M) \subset \mathscr{C}^m(M)$. Where*

$$||u||_{\mathscr{C}^m} = \sum_{j=0}^{m} \max_{x\in M} |\nabla^j u(x)|.$$

1.3 Variable exponents Lebesgue spaces

In this section, we recall some properties of spaces $L^{q(x)}(\Omega)$ and $W_0^{1,q(x)}(\Omega)$ where Ω is an open subset of $\mathbb{R}^N$ which are called the Lebesgue spaces with variable exponents and the Sobolev spaces with variable exponents setting, which can be found in [9, 18, 61, 62] and references therein.

1.3.1 Definitions and Propositions

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, we define the Lebesgue space with variable exponent $L^{q(\cdot)}(\Omega)$ as the set of all measurable function $u : \Omega \longmapsto \mathbb{R}$ for which the convex modular

$$\rho_{q(\cdot)}(u) = \int_\Omega |u(x)|^{q(x)}\,dx,$$

is finite. If the exponent is bounded, i.e if $q^+ = ess\,sup\{q(x)/x \in \Omega\} < +\infty$, then the expression

$$||u||_{q(\cdot)} = \inf\left\{\lambda > 0 : \rho_{q(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1\right\},$$

defines a norm in $L^{q(\cdot)}(\Omega)$, called the Luxemburg norm.

Proposition 1.3.1. *The space $(L^{q(\cdot)}(\Omega), ||\,.\,||_{q(\cdot)})$ is a separable Banach space.*

Remark 5. *If $1 < q^- \leq q^+ < +\infty$, then $L^{q(\cdot)}(\Omega)$ is uniformly convex, where $q^- = ess\,inf\{q(x)/x \in \Omega\}$, hence reflexive, and its dual space is isomorphic to $L^{q'(\cdot)}(\Omega)$ where $\frac{1}{q(x)} + \frac{1}{q'(x)} = 1$.*

Proposition 1.3.2. *(* ***Hölder type inequality*** *) For all $u \in L^{q(\cdot)}(\Omega)$ and $v \in L^{q'(\cdot)}(\Omega)$, we have*

$$\left|\int_\Omega u\,v\,dx\right| \leq \left(\frac{1}{q^-} + \frac{1}{(q')^-}\right) ||u||_{q(\cdot)}||v||_{q'(\cdot)}.$$

Definition 1.3.1. *We define the variable exponent Sobolev space by*

$$W^{1,q(\cdot)}(\Omega) = \left\{u \in L^{q(\cdot)}(\Omega) \text{ and } |\nabla u| \in L^{q(\cdot)}(\Omega)\right\}.$$

which is a Banach space equipped with the following norm

$$||u||_{1,q(\cdot)} = ||u||_{q(\cdot)} + ||\nabla u||_{q(\cdot)} \qquad \forall u \in W^{1,q(\cdot)}(\Omega).$$

Proposition 1.3.3. *The space $(W^{1,q(\cdot)}(\Omega), ||\cdot||_{1,q(\cdot)})$ is separable and reflexive Banach space.*

We denote by $W_0^{1,q(\cdot)}(\Omega)$ the closure $C_0^\infty(\Omega)$ in $W^{1,q(\cdot)}(\Omega)$.

Proposition 1.3.4. *[60] (**Poincaré inequality**) If $q \in C_+(\overline{\Omega})$, then there is a constant $c > 0$ such that*

$$||\,u\,||_{q(x)} \leq c\,||\,|\,\nabla u\,|\,||_{q(x)}, \quad \forall u \in W_0^{1,q(x)}(\Omega).$$

Where, $C_+(\overline{\Omega}) = \{q/q \in C(\overline{\Omega}), q(x) > 1\, for\, x \in \overline{\Omega}\}$.
Consequently, $||\,u\,|| = ||\,|\,\nabla u\,|\,||_{q(x)}$ and $||\,u\,||_{1,q(x)}$ are equivalent norms on $W_0^{1,q(x)}(\Omega)$.

1.3.2 Theory of Sobolev-Orlicz spaces with variable exponents

In this part, we recall some relevant definitions and properties that will help us in our analysis in chapter 4. For the convenience of the readers and for the sake of completeness, we recall the proofs of some results.

Definition 1.3.2. *We define the following real valued linear space as follows*

$$L^{p(\cdot)}(\Omega) + L^{q(\cdot)}(\Omega) = \{u/u = v + w,\, v \in L^{p(\cdot)}(\Omega),\, w \in L^{q(\cdot)}(\Omega)\,\},$$

which is endowed with the norm

$$|u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} = \inf\{|v|_{L^{p(\cdot)}(\Omega)} + |w|_{L^{q(\cdot)}(\Omega)}\,/\,v \in L^{p(\cdot)}(\Omega),\, w \in L^{q(\cdot)}(\Omega),\, u = v + w\,\}. \quad (1.20)$$

Definition 1.3.3. *We defined the linear space as follows*

$$L^{p(\cdot)}(\Omega) \cap L^{q(\cdot)}(\Omega) = \{\,u\,/\,u \in L^{p(\cdot)}(\Omega)\; and\; u \in L^{q(\cdot)}(\Omega)\,\},$$

which is endowed with the norm

$$|u|_{L^{p(\cdot)}(\Omega)\cap L^{q(\cdot)}(\Omega)} = \max\{\,|u|_{L^{p(\cdot)}(\Omega)}, |u|_{L^{q(\cdot)}(\Omega)}\,\}.$$

We denote

$$\mathscr{D}_u = \{x \in \Omega \,/\, |u(x)| > 1\} \text{ and } \mathscr{D}_u^c = \{x \in \Omega \,/\, |u(x)| \leq 1\}.$$

Proposition 1.3.5. *Let $\Omega \subset \mathbb{R}^N$ and $u \in L^{p(\cdot)}(\Omega) + L^{q(\cdot)}(\Omega)$. Assume that $(\mathscr{A}_1)$ - (iv) are true. Then, the following properties hold :*

(a) If $\Omega' \subset \Omega$ is such that $|\Omega'| < +\infty$, then $u \in L^{p(\cdot)}(\Omega')$.

(b) If $\Omega' \subset \Omega$ is such that $u \in L^{\infty}(\Omega')$, then $u \in L^{q(\cdot)}(\Omega')$.

(c) $|\mathscr{D}_u| < +\infty$.

(d) $u \in L^{p(\cdot)}(\mathscr{D}_u) \cap L^{q(\cdot)}(\mathscr{D}_u^c)$.

(e) The infimum in (1.20) *is attained.*

(f) If $B \subset \Omega$, then, $|u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} \leq |u|_{L^{p(\cdot)}(B)+L^{q(\cdot)}(B)} + |u|_{L^{p(\cdot)}(\Omega\backslash B)+L^{q(\cdot)}(\Omega\backslash B)}$.

(g) We have

$$\begin{aligned}&\max\left\{\frac{1}{1+2\,|\mathscr{D}_u|^{\frac{1}{p(\zeta)}-\frac{1}{q(\zeta)}}}\,|u|_{L^{p(\cdot)}(\mathscr{D}_u)},\, c\,\min\left\{|u|_{L^{q(\cdot)}(\mathscr{D}_u^c)},\, |u|^{\frac{q(\zeta)}{p(\zeta)}}_{L^{q(\cdot)}(\mathscr{D}_u^c)}\right\}\right\}\\&\leq |u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} \leq |u|_{L^{p(\cdot)}(\mathscr{D}_u)} + |u|_{L^{q(\cdot)}(\mathscr{D}_u^c)},\end{aligned}$$

where, $\zeta \in \Omega$ and c is a small positive constant.

Proof. Let $v \in L^{p(\cdot)}(\Omega)$ and $w \in L^{q(\cdot)}(\Omega)$ such that $u = v + w$, then $v \in L^{p'(\cdot)}(\Omega')$ and $w \in L^{q'(\cdot)}(\Omega')$.

(a) To show that $u \in L^{p(\cdot)}(\Omega')$, it is enough to show that $w \in L^{p(\cdot)}(\Omega')$. And by Young's inequality, we get the results.

(b) To prove $u \in L^{q(\cdot)}(\Omega')$, it is sufficient to prove that $v \in L^{q(\cdot)}(\Omega')$. For that, we have

$$\begin{aligned}\int_{\Omega'} |v(x)|^{q(x)}\,dx &= \int_{\Omega'} |v(x)|^{q(x)-p(x)}|v(x)|^{p(x)}\,dx\\&\leq (1+|\sup v|)^{q^+-p^-}\int_{\Omega'}|v(x)|^{p(x)}\,dx < +\infty.\end{aligned}$$

Thus, $v \in L^{q(\cdot)}(\Omega')$. Hence, $u \in L^{q(\cdot)}(\Omega')$.

(c) We use the fact that $1 < |u| \leq |v| + |w|$ implies that $|v| \geq \frac{1}{2}$ or $|w| \geq \frac{1}{2}$ and for all $x \in \Omega$, we get

$$+\infty > \int_{\Omega} |u|^{p(x)} + |w|^{q(x)}\,dx \geq \left|\frac{1}{2}\right|^{p^+ + q^+} |\mathscr{D}_u|.$$

Hence, the result.

(d) From the assumptions $(a)-(c)$ we get the result.

(e) Let $u \in L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)$, we consider a minimizing sequence for u, namely $v_n \in L^{p(\cdot)}(\Omega)$ and $w_n \in L^{q(\cdot)}(\Omega)$ such that $u = v_n + w_n$ and

$$\lim_{n\to+\infty}\left(|v_n|_{L^{p(\cdot)}(\Omega)} + |w_n|_{L^{q(\cdot)}(\Omega)}\right) = |u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)}.$$

Then, by the reflexibility of $L^{p(\cdot)}(\Omega)$ and $L^{q(\cdot)}(\Omega)$, there exist $v_0 \in L^{p(\cdot)}(\Omega)$ and $w_0 \in L^{q(\cdot)}(\Omega)$ such that $v_n \rightharpoonup v_0$ in $L^{p(\cdot)}(\Omega)$ and $w_n \rightharpoonup w_0$ in $L^{q(\cdot)}(\Omega)$. By lower continuity, we have

$$\begin{aligned}|u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} &= \lim_{n\to+\infty}\left(|v_n|_{L^{p(\cdot)}(\Omega)} + |w_n|_{L^{q(\cdot)}(\Omega)}\right)\\ &\geq \lim_{n\to+\infty}\inf|v_n|_{L^{p(\cdot)}(\Omega)} + \lim_{n\to+\infty}\inf|w_n|_{L^{q(\cdot)}(\Omega)}\\ &= |v_0|_{L^{p(\cdot)}(\Omega)} + |w_0|_{L^{q(\cdot)}(\Omega)}.\end{aligned}$$

According to the Definition 1.3.2, we get

$$|u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} = |v_0|_{L^{p(\cdot)}(\Omega)} + |w_0|_{L^{q(\cdot)}(\Omega)}.$$

(f) See Proposition 2.2 in [18].

(g) From (d) and the Definition 1.3.2, we notice that

$$|u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} \leq |u|_{L^{p(\cdot)}(\mathscr{D}_u^c)} + |u|_{L^{q(\cdot)}(\mathscr{D}_u^c)}.$$

By (e), we obtain that

$$|u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} \geq |u|_{L^{p(\cdot)}(\mathscr{D}_u)+L^{q(\cdot)}(\mathscr{D}_u)},$$

and

$$|u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} \geq |u|_{L^{p(\cdot)}(\mathscr{D}_u^c)+L^{q(\cdot)}(\mathscr{D}_u^c)}.$$

By (1.20), we deduce that

$$|u|_{L^{p(\cdot)}(\mathscr{D}_u)} \leq 2|\mathscr{D}_u|^{\frac{1}{p(\zeta)}-\frac{1}{q(\zeta)}}|w|_{L^{q(\cdot)}(\mathscr{D}_u)} \;\; \forall\zeta \in \Omega.$$

It follows that

$$|u|_{L^{p(\cdot)}(\mathscr{D}_u)} \leq |v|_{L^{p(\cdot)}(\mathscr{D}_u)} + |w|_{L^{p(\cdot)}(\Omega)} \leq \left(1+2|\mathscr{D}_u|^{\frac{1}{p(\zeta)}-\frac{1}{q(\zeta)}}\right)|u|_{L^{p(\cdot)}(\mathscr{D}_u)+L^{q(\cdot)}(\mathscr{D}_u)}$$
$$\leq \left(1+2|\mathscr{D}_u|^{\frac{1}{p(\zeta)}-\frac{1}{q(\zeta)}}\right)|u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)},$$

without loss of generality, we may assume that u is nonnegative, such that

$$|u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} = |v|_{L^{p(\cdot)}(\Omega)} + |w|_{L^{q(\cdot)}(\Omega)}.$$

Evidently, $0 \leq w \leq 1$ on $\mathscr{D}_u^c$. We denote $\sigma_p = |w|_{L^{p(\cdot)}(\mathscr{D}_u^c)}$ and $\sigma_q = |w|_{L^{q(\cdot)}(\mathscr{D}_u^c)}$. According to the fact that $|w| \leq 1$ on $\mathscr{D}_u^c$, we get

$$1 = \int_{\mathscr{D}_u^c} \left|\frac{w}{\sigma_p}\right|^{p(x)} dx \geq \frac{\sigma_q^{q(\zeta)}}{\sigma_p^{p(\zeta)}} \int_{\mathscr{D}_u^c} \left|\frac{w}{\sigma_q}\right|^{q(x)} dx = \frac{\sigma_q^{q(\zeta)}}{\sigma_p^{p(\zeta)}} \ \forall \zeta \in \Omega.$$

Thus, $\sigma_p \geq \sigma_q^{\frac{q(\zeta)}{p(\zeta)}}$. Similarly,

$$|u|_{L^{p(\cdot)}(\mathscr{D}_u^c)+L^{q(\cdot)}(\mathscr{D}_u^c)} = |v|_{L^{p(\cdot)}(\mathscr{D}_u^c)} + |w|_{L^{q(\cdot)}(\mathscr{D}_u^c)} \geq |v|_{L^{q(\cdot)}(\mathscr{D}_u^c)}^{\frac{q(\zeta)}{p(\zeta)}} + |w|_{L^{q(\cdot)}(\mathscr{D}_u^c)}$$
$$\geq c \min\left\{|u|_{L^{q(\cdot)}(\mathscr{D}_u^c)}, |u|_{L^{q(\cdot)}(\mathscr{D}_u^c)}^{\frac{q(\zeta)}{p(\zeta)}}\right\}.$$

According to the previous results, we obtain the result.

□

Proposition 1.3.6. *[103] Assume that assumptions $(\mathscr{A}_1)$-(iv) are true. Then $\left(L^{p'(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega), |\cdot|_{L^{p'(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)}\right)$ is a reflexive Banach space.*

Now, we consider

$$X(\Omega) = \left\{u \in L^{\alpha(\cdot)}(\Omega) / \nabla u \in (L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega))^N\right\},$$

with the norm

$$||u||_\Omega = ||u||_{L^{\alpha(\cdot)}(\Omega)} + |\nabla u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)}.$$

Proposition 1.3.7. *[103] Under assumptions $(\mathscr{A}_1)$-(iv). $(X(\Omega), ||u||_\Omega)$ is a reflexive and a Banach space.*

Theorem 1.3.1. *[103] Suppose that hypotheses $(\mathscr{A}_1)$-(iv), $1 \ll p^*(\cdot)\frac{q'(\cdot)}{p'(\cdot)}$ hold, α satisfies $1 \ll \alpha(\cdot) \ll p^*(\cdot)\frac{N-1}{N}$ and $1 \ll \alpha(\cdot) \leq p^*\frac{q'(\cdot)}{p'(\cdot)}$. Then, the space $X(\Omega)$ is continuously embedded into $L^{p^*(\cdot)}(\Omega)$.*

Proposition 1.3.8. *Assume that hypotheses $(\mathscr{A}_1)$-(iv), $1 \ll p^*(\cdot)\frac{q'(\cdot)}{p'(\cdot)}$ hold, α satisfies $1 \ll \alpha(\cdot) \ll p^*(\cdot)\frac{N-1}{N}$ and $1 \ll \alpha(\cdot) \leq p^*\frac{q'(\cdot)}{p'(\cdot)}$. The following properties are true :*

(i) For any $u \in X(\Omega)$, $\psi_n u \to u$ in $X(\Omega)$.

(ii) For any $u \in X(\Omega)$, we have $u_\varepsilon = u * j_\varepsilon \to u$ in $X(\Omega)$ (where $j_\varepsilon(x) = \varepsilon^{-N} j\left(\frac{x}{\varepsilon}\right)$ and $j : \Omega \to \mathbb{R}^+$ is in $C_c^\infty(\Omega)$, a function inducing a probability measure.

(iii) For any $u \in X(\Omega)$, there exists a sequence $\{u_n\} \subset C_c^\infty(\Omega)$ such that $u_n \to u$ in $X(\Omega)$.

Proof. (i) See Theorem 3.12 in [103].

(ii) Using the mollifiers method, we get that $u_\varepsilon \to u$ in $L^{\alpha(\cdot)}(\Omega)$ as $\varepsilon \to 0$. Moreover, if $\nabla u = a + b$, with $a \in (L^{p(\cdot)}(\Omega))^N$ and $b \in (L^{q(\cdot)}(\Omega))^N$, we have $\nabla u_\varepsilon = \nabla u * j_\varepsilon = a * j_\varepsilon + b * j_\varepsilon$ with $a * j_\varepsilon \in (L^{p(\cdot)}(\Omega))^N$ and $b * j_\varepsilon \in (L^{q(\cdot)}(\Omega))^N$. Then

$$|\nabla u_\varepsilon - \nabla u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} \leq |a * j_\varepsilon - a|_{L^{p(\cdot)}(\Omega)} + |b * j_\varepsilon - b|_{L^{q(\cdot)}(\Omega)} \to 0.$$

Hence, $u_\varepsilon \longrightarrow u$ in $X(\Omega)$.

(iii) We can easily conclude the proof of this step, by using the above results (i) and (ii).

□

1.4 Sobolev-Orlicz Riemannian manifold with variable exponents

In this section, we begin with a brief description of the Lebesgue Riemannian manifolds and Sobolev Riemannian manifolds spaces with variable exponents. A detailed insight into the these spaces and a review of the bibliography can be found in [1, 2, 16, 63, 68, 70]

1.4.1 Definitions and Propositions

In the following, all the manifolds we consider are smooth, and we will use the following conditions on (M, g), depending on the context :

Definition 1.4.1. *Let (M, g) be a smooth Riemannain n-manifolds and let ∇ be the Levi-Civita connection. If u is a smooth function on M, then $\nabla^k u$ denotes the $k-th$ covariant derivative of u,* *and* $|\nabla^k u|$ *the norm of* $\nabla^k u$ *defined in local coordinates by*

$$|\nabla^k u|^2 = g^{i_1 j_1} \cdots g^{i_k j_k} (\nabla^k u)_{i_1 \cdots i_k} (\nabla^k u)_{j_1 \cdots j_k},$$

where Einstein's convention is used.

Remark 6. *A smooth manifold* M *of dimension* n *is a connected topological manifold* M *of dimension* n *together with a* $C^\infty-$*complete atlas.*

Example 2. *The following examples are classical examples of smooth manifolds :*

- *The Euclidean space* $\mathbb{R}^n$ *itself.*
- *The torus* T^n.
- *The unit sphere* S^n *of* $\mathbb{R}^{n+1}$.
- *The real projective space* $\mathbb{P}^n(\mathbb{R})$.

Definition 1.4.2. *To define variable Sobolev spaces, given a variable exponent* q *in* $\mathscr{P}(M)$ *(the set of all measurable functions* $p(\cdot): M \to (1,\infty)$ *) and a natural number k, introduce*

$$C_k^{q(\cdot)}(M) = \left\{ u \in C^\infty(M) \text{ such that } \forall j \; 0 \leq j \leq k \, |\nabla^j u| \in L^{q(\cdot)}(M) \right\}.$$

On $C_k^{q(\cdot)}(M)$ *define the norm*

$$||u||_{L_k^{q(\cdot)}(M)} = \sum_{j=0}^{k} ||\nabla^j u||_{L^{q(\cdot)}(M)}.$$

Definition 1.4.3. *The Sobolev spaces* $L_k^{q(\cdot)}(M)$ *is the completion of* $C_k^{q(\cdot)}(M)$ *with respect to the norm* $||u||_{L_k^{q(\cdot)}}$. *If* Ω *is a subset of* M, *then* $L_{k,0}^{q(\cdot)}(\Omega)$ *is the completion of* $C_k^{q(\cdot)}(M) \cap C_0(\Omega)$ *with respect to* $||.||_{L_k^{q(\cdot)}}$, *where* $C_0(\Omega)$ *denotes the vector space of continuous functions whose support is a compact subset of* Ω.

Definition 1.4.4. *Given* (M,g) *a smooth Riemannian manifold, and* $\gamma: [a,b] \longrightarrow M$ *is a curve of class* C^1. *The length of* γ *is*

$$l(\gamma) = \int_a^b \sqrt{g\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right)}\, dt,$$

and for a pair of points $x, y \in M$, *we define the distance* $d_g(x,y)$ *between x and y by*

$$d_g(x,y) = \inf \left\{ l(\gamma) : \gamma : [a,b] \to M \text{ such that } \gamma(a) = x \text{ and } \gamma(b) = y \right\}.$$

Definition 1.4.5. *A function $s : M \longrightarrow \mathbb{R}$ is log-Hölder continuous if there exists a constant c such that for every pair of points $\{x, y\}$ in M we have*

$$|\, s(x) - s(y)\,| \leq \frac{c}{log\left(e + \frac{1}{d_g(x,y)}\right)}.$$

We note by $\mathscr{P}^{log}(M)$ the set of log-Hölder continuous variable exponents. The relation between $\mathscr{P}^{log}(M)$ and $\mathscr{P}^{log}(\mathbb{R}^N)$ is the following :

Proposition 1.4.1. *[16, 63, 70] Let $q \in \mathscr{P}^{log}(M)$, and let $(B_R(q), \phi)$ be a chart such that*

$$\frac{1}{2}\delta_{ij} \leq g_{ij} \leq 2\,\delta_{ij}$$

as bilinear forms, where δ_{ij} is the delta Kronecker symbol. Then $qo\phi^{-1} \in \mathscr{P}^{log}(\phi(B_{\frac{R}{3}}(q)))$.

Proposition 1.4.2. *(**Hölder's inequality**) For all $u \in L^{q(\cdot)}(M)$ and $v \in L^{q'(\cdot)}(M)$ we have*

$$\int_M |\,u(x)\,v(x)\,|\,dv_g(x) \leq r_q\,||\,u\,||_{L^{q(\cdot)}(M)}\,.\,||\,v\,||_{L^{q'(\cdot)}(M)}.$$

Where r_q be a positive constant depend to q^- and q^+.

Proof. Obviously, we can suppose that $||u||_{L^{q(x)}(M)} \neq 0$ and $||v||_{L^{q'(x)}(M)} \neq 0$, we have

$$1 < q(x) < \infty,\ |u(x)| < \infty,\ |v(x)| < \infty \text{ a.e } x \in M.$$

By young inequality, we have

$$\frac{u(x)\,.\,v(x)}{||u(x)||_{L^{q(x)}(M)}\,.\,||v(x)||_{L^{q'(x)}(M)}} \leq \frac{1}{q(x)}\left(\frac{|u(x)|}{||u(x)||_{L^{q(x)}(M)}}\right)^{q(x)} + \frac{1}{q'(x)}\left(\frac{|v(x)|}{||v(x)||_{L^{q'(x)}(M)}}\right)^{q'(x)}$$

Integrating over M, we obtain

$$\begin{aligned}\int_M \frac{|\,u(x)\,.\,v(x)\,|}{||u(x)||_{L^{q(x)}(M)}\,.\,||v(x)||_{L^{q'(x)}(M)}}\,dv_g(x) &\leq \frac{1}{q^-}\int_M\left(\frac{|u(x)|}{||u(x)||_{L^{q(x)}(M)}}\right)^{q(x)} dv_g(x) \\ &\quad + \big(1 - \frac{1}{q^+}\big)\int_M\left(\frac{|v(x)|}{||v(x)||_{L^{q'(x)}(M)}}\right)^{q'(x)} dv_g(x) \\ &\leq 1 + \frac{1}{q^-} - \frac{1}{q^+},\end{aligned}$$

then, using the same technique as in the proof of Theorem 1.15 in [61], we get that

$$\int_M |u(x).v(x)|\, dv_g(x) \leq \big(1+\frac{1}{q^-}+\frac{1}{q^+}\big)\, ||u(x)||_{L^{q(x)}(M)}\,.\,||v(x)||_{L^{q'(x)}(M)}$$
$$\leq r_q ||u(x)||_{L^{q(x)}(M)}\,.\,||v(x)||_{L^{q'(x)}(M)},$$

Which complete the proof. □

Remark 7. *If a and b are two positive functions on M, then by Hölder's inequality and [60, 63] we have*

$$\int_{q^-<2} a^{\frac{q^-}{2}}\, b^{\frac{2q^- - q^- 2}{2}} \leq 2\,||\,\mathbb{1}_{q^-<2}\, a^{\frac{q^-}{2}}\,||_{L^{\frac{2}{q^-}}}\,.\,||\,\mathbb{1}_{q^-<2}\, b^{\frac{2q^- - q^- 2}{2}}\,||_{L^{\frac{2}{2-q^-}}}\,. \tag{1.21}$$

where $\mathbb{1}$ is the indicator function of M, moreover, since

$$||\,\mathbb{1}_{q^-<2}\, a^{\frac{2}{q^-}}\,||_{L^{\frac{2}{q^-}}} \leq \max\{\,\rho_1(a), \rho_1(a)^{\frac{q^-}{2}}\,\}$$

and

$$||\,\mathbb{1}_{q^-<2}\, b^{\frac{2q^- - q^- 2}{2}}\,||_{L^{\frac{2}{2-q^-}}} \leq \max\{\,\rho_{q(\cdot)}(b)^{\frac{2-q^-}{2}}, 1\,\},$$

we get,

$$\int_{q^-<2} a^{\frac{q^-}{2}}\,.\,b^{\frac{2q^- - q^- 2}{2}} \leq 2\,\max\{\,\rho_1(a), \rho_1(a)^{\frac{q^-}{2}}\,\}\,\max\{\,\rho_{q(\cdot)}(b)^{\frac{2-q^-}{2}}, 1\,\}. \tag{1.22}$$

Definition 1.4.6. *We say that the Riemannian n-manifold (M,g) has property $B_{vol}(\lambda, v)$ where λ is a constant, if its geometry is bounded in the following sense :*

- *The Ricci tensor of g noted by Rc (g) verifies, $Rc(g) \geq \lambda(n-1)g$ for some λ, where n is the dimension of M.*
- *There exists some $v > 0$ such that $|B_1(x)|_g \geq v\ \forall x \in M$, where $B_1(x)$ are the balls of radius 1 centered at some point x in terms of the volume of smaller concentric balls.*

Remark 8. *The first condition in the definition 1.4.6, is very necessary for the Sobolev estimates to be true.*

1.4.2 Embdeddings results

Remark 9. *If $M = \Omega \subseteq \mathbb{R}^N$ is a bounded open set, then the following inequality is related to the two exponents p, q (isotropic case)*

$$\frac{q}{p} < 1 + \frac{1}{N}.$$

This condition is essential, among others, for the embeddings of spaces to be satisfied.

Proposition 1.4.3. *[16, 70] Assume that the complete compact Riemannian n-manifold (M,g) has property $B_{vol}(\lambda,v)$ for some (λ,v). Then there exist positive constants $\delta_0=\delta_0(n,\lambda,v)$ and $A=A(n,\lambda,v)$, we have, if $R\leq\delta_0$, if $x\in M$ if $1\leq q\leq n$, and if $u\in L^q_{1,0}(B_R(x))$ the estimate*

$$||u||_{L^p}\leq A\,p\,||\nabla u||_{L^q},$$

where $\frac{1}{p}=\frac{1}{q}-\frac{1}{n}$.

We can extend the above proposition from the case when exponents p and q are constant, to the case when $p(\cdot)$ and $q(\cdot)$ are functions.

Proposition 1.4.4. *Assume that for some (λ,v) the complete compact Riemannian n-manifold (M,g) has property $B_{vol}(\lambda,v)$. Then there exist positive constants $\delta_0=\delta_0(n,\lambda,v)$ and $A=A(n,\lambda,v)$, we have, if $R\leq\delta_0$, if $x\in M$ if $1\leq q(\cdot)\leq n$, and if $u\in L^{q(\cdot)}_{1,0}(B_R(x))$ the estimate*

$$||u||_{L^{p(\cdot)}}\leq A\,p^-\,||\nabla u||_{L^{q(\cdot)}},$$

where $\frac{p(\cdot)}{q(\cdot)}<1+\frac{1}{n}$.

Proof. To demonstrate this Proposition, we use the same technique as proposition 1.4.3, for more detail see [16, 70]. □

In the following, we denote for all $u\in W_0^{1,q(x)}(M)$ that

$$\rho_{p(\cdot)}(u)=\int_M |u(x)|^{p(x)}\,dv_g(x)\ \text{and}\ \rho_{q(\cdot)}(u)=\int_M |u(x)|^{q(x)}\,dv_g(x).$$

With $dv_g=\sqrt{det(g_{ij})}\,dx$ is the Riemannian volume element on (M,g), where the g_{ij} are the components of the Riemannian metric g in the chart and dx is the Lebesgue volume element of $\mathbb{R}^N$.

Proposition 1.4.5. *[16, 63, 70] Assume that for some (λ,v) the complete compact Riemannian n-manifold (M,g) has property $B_{vol}(\lambda,v)$. Let $p\in\mathscr{P}(M)$ be uniformly continuous with $q^+<n$. Then $L_1^{q(\cdot)}(M)\hookrightarrow L^{p(\cdot)}(M), \forall q\in\mathscr{P}(M)$ such that $q\ll p\ll q*=\frac{nq}{n-q}$. In fact, for $||u||_{L_1^{q(\cdot)}}$ sufficiently small we have the estimate*

$$\rho_{p(\cdot)}(u)\leq G(\rho_{q(\cdot)}(u)+\rho_{q(\cdot)}(|\nabla u|)),$$

where the positive constant G depend on n, λ, v, q and p.

Proposition 1.4.6. *[68] Let $u \in L^{q(x)}(M)$, $\{u_k\} \subset L^{q(x)}(M)$, $k \in \mathbb{N}$, then we have*

(i) $||u||_{q(x)} < 1$ *(resp. = 1, > 1)* $\iff \rho_{q(x)}(u) < 1$ *(resp. = 1, > 1),*

(ii) $||u||_{q(x)} < 1 \Rightarrow ||u||_{q(x)}^{q^+} \leq \rho_{q(x)}(u) \leq ||u||_{q(x)}^{q^-}$,

(iii) $||u||_{q(x)} > 1 \Rightarrow ||u||_{q(x)}^{q^-} \leq \rho_{q(x)}(u) \leq ||u||_{q(x)}^{q^+}$,

(iv) $\lim_{k\to+\infty} ||u_k - u||_{q(x)} = 0 \iff \lim_{k\to+\infty} \rho_{q(x)}(u_k - u) = 0.$

To compare the functionals $||\cdot||_{q(\cdot)}$ and $\rho_{q(\cdot)}(\cdot)$, one has the relation

$$\min\{\rho_{q(\cdot)}(u)^{\frac{1}{q^-}}, \rho_{q(\cdot)}(u)^{\frac{1}{q^+}}\} \leq ||u||_{L^{q(\cdot)}} \leq \max\{\rho_{q(\cdot)}(u)^{\frac{1}{q^-}}, \rho_{q(\cdot)}(u)^{\frac{1}{q^+}}\}.$$

So, if $q^+ < n$, we have the embedding

$$L_{1,0}^{q(\cdot)}(B_R(x)) \hookrightarrow L^{p(\cdot)}(M),$$

where, $p(x) = \frac{nq(x)}{n-q(x)}$. In fact, there exists a positive constant $D = D(n, \lambda, v, q^+, q^-)$ such as for every u in $L_{1,0}^{q(\cdot)}(B_R(x))$, we have by Poincaré inequality and Proposition 1.4.6 that

$$\begin{aligned} ||u||_{L^{p(\cdot)}(M)} &\leq D\,||u||_{L_1^{q(\cdot)}(M)} = D(||u||_{L^{q(\cdot)}(M)} + ||\nabla u||_{L^{q(\cdot)}(M)}) \\ &\leq D(c+1)\,||\nabla u||_{L^{q(\cdot)}(M)}, \end{aligned}$$

where c is the Poincaré constant. Hence,

$$\begin{aligned} \rho_{p(\cdot)}(u) \leq ||u||_{L^{p(\cdot)}(M)}^{p^+} &\leq D^{p^+}(c+1)^{p^+}\,||\nabla u||_{L^{q(\cdot)}(M)}^{p^+} \\ &\leq D^{p^+}(c+1)^{p^+} \max\{\rho_{q(\cdot)}(|\nabla u|)^{\frac{p^+}{q^-}}, \rho_{q(\cdot)}(|\nabla u|)^{\frac{p^+}{q^+}}\} \\ &\leq D^{p^+}(c+1)^{p^+} \rho_{q(\cdot)}(|\nabla u|)^{\frac{p^+}{q^-}}. \end{aligned} \tag{1.23}$$

Definition 1.4.7. *The Sobolev space $W^{1,q(x)}(M)$ consists of such functions $u \in L^{q(x)}(M)$ for which $\nabla^k u \in L^{q(x)}(M)$ $k = 1, 2, \cdots, n$. The norm is defined by*

$$||\,u\,||_{W^{1,q(x)}(M)} = ||\,u\,||_{L^{q(x)}(M)} + \sum_{k=1}^{n} ||\,\nabla^k u\,||_{L^{q(x)}(M)}.$$

The space $W_0^{1,q(x)}(M)$ is defined as the closure of $C_c^\infty(M)$ in $W^{1,q(x)}(M)$, with $C_c^\infty(M)$ be the vector *space of smooth functions with compact support on M.*

Theorem 1.4.1. *Let M be a compact Riemannian manifold with a smooth boundary or without boundary and $q(x)$, $p(x) \in C(\overline{M}) \cap L^{\infty}(M)$. Assume that*

$$q(x) < N, \qquad p(x) < \frac{N\,q(x)}{N - q(x)} \text{ for } x \in \overline{M}.$$

Then,

$$W^{1,q(x)}(M) \hookrightarrow L^{p(x)}(M)$$

is a continuous and compact embedding.

Proof. This proof is based to an idea introduced in [61, 68]. Let $f : U(\subset M) \longrightarrow \mathbb{R}^N$ be an arbitrary local chart on M, and V be any open set in M, whose closure is compact and is contained in U. Choosing a finite subcovering $\{V_\alpha\}_{\alpha=1,\cdots,k}$ of M such that V_α is homeomorphic to the open unit ball $B_0(1)$ of $\mathbb{R}^N$ and for any α the components g^{α}_{ij} of g in (V_α, f_α) satisfy

$$\frac{1}{\varepsilon\,\delta_{ij}} \leq g^{\alpha}_{ij} < \varepsilon\,\delta_{ij}$$

as bilinear forms, where the constant $\varepsilon > 1$ is given. Let $\{\pi_\alpha\}_{\alpha=1,\cdots,k}$ be a smooth partition of unity subordinate to the finite covering $\{V_\alpha\}_{\alpha=1,\cdots,k}$. It is obvious that if $u \in W^{1,q(x)}(M)$, then $\pi_\alpha u \in W^{1,q(x)}(V_\alpha)$ and $(f_\alpha^{-1})^*(\pi_\alpha u) \in W^{1,q(f_\alpha^{-1}(x))}(B_0(1))$. According to proposition 1.2.4 and the Sobolev embeddings Theorem in [61, 63], we obtain the continuous and compact embedding

$$W^{1,q(x)}(V_\alpha) \hookrightarrow L^{p(x)}(V_\alpha) \qquad \text{for each } \alpha = 1, \cdots, k.$$

Since $u = \sum_{\alpha=1}^{k} \pi_\alpha u$, we can conclude that

$$W^{1,q(x)}(M) \subset L^{p(x)}(M),$$

and the embedding is continuous and compact. □

Proposition 1.4.7. *[16] If (M, g) is complete, then $W^{1,q(x)}(M) = W_0^{1,q(x)}(M)$.*

1.4.3 The weighted variable exponent Lebesgue space

The weighted variable exponent Lebesgue space $L^{q(x)}_{\mu(x)}(M)$ is defined as follows :

$$L^{q(x)}_{\mu(x)}(M) = \left\{ u : M \to \mathbb{R} \text{ is measurable such that, } \int_M \mu(x)|u(x)|^{q(x)}\, dv_g(x) < +\infty \right\},$$

with the norm

$$||u||_{q(x),\mu(x)} = \inf\left\{ \gamma > 0 : \int_M \mu(x) \left| \frac{u(x)}{\gamma} \right|^{q(x)} dv_g(x) \leq 1 \right\}.$$

Moreover, the weighted modular on $L^{q(x)}_{\mu(x)}(M)$ is the mapping $\rho_{q(\cdot),\mu(\cdot)} : L^{q(x)}_{\mu(x)}(M) \to \mathbb{R}$ defined like

$$\rho_{q(\cdot),\mu(\cdot)}(u) = \int_M \mu(x)|u(x)|^{q(x)}\, dv_g(x).$$

Example 3. *As a simple example of $\mu(x)$, we can take $\mu(x) = (1+|x|)^{\varepsilon(x)}$ with $\varepsilon(\cdot) \in C_+(\overline{M})$.*

Proposition 1.4.8. *Let u and $\{u_n\} \subset L^{q(x)}_{\mu(x)}(M)$, then we have the following results :*

(1) $||u||_{q(\cdot),\mu(\cdot)} < 1$ *(resp.* $= 1, > 1$*)* $\Longleftrightarrow \rho_{q(\cdot),\mu(\cdot)}(u) < 1$ *(resp.* $= 1, > 1$*).*

(2) $||u||_{q(\cdot),\mu(\cdot)} < 1 \Rightarrow ||u||^{q^+}_{q(\cdot),\mu(\cdot)} \leq \rho_{q(\cdot),\mu(\cdot)}(u) \leq ||u||^{q^-}_{q(\cdot),\mu(\cdot)}$.

(3) $||u||_{q(\cdot),\mu(\cdot)} > 1 \Rightarrow ||u||^{q^-}_{q(\cdot),\mu(\cdot)} \leq \rho_{q(\cdot),\mu(\cdot)}(u) \leq ||u||^{q^+}_{q(\cdot),\mu(\cdot)}$.

(4) $\lim\limits_{n\to+\infty} ||u_n||_{q(\cdot),\mu(x)} = 0 \Longleftrightarrow \lim\limits_{n\to+\infty} \rho_{q(\cdot),\mu(\cdot)}(u_n) = 0$.

(5) $\lim\limits_{n\to+\infty} ||u_n||_{q(\cdot),\mu(x)} = \infty \Longleftrightarrow \lim\limits_{n\to+\infty} \rho_{q(\cdot),\mu(\cdot)}(u_n) = \infty$.

Note that, the non-negative weighted function $\mu \in C(\overline{M})$ satisfy the following hypothesis : $\mu(\cdot) : \overline{M} \longrightarrow \mathbb{R}^+_*$ such that $\mu(\cdot) \in L^{\varepsilon(x)}(M)$ with

$$\frac{Np(x)}{Np(x) - q(x)(N - p(x))} < \varepsilon(x) < \frac{p(x)}{p(x) - q(x)} \text{ for all } x \in \overline{M}. \tag{1.24}$$

Indeed, since $\mu(\cdot) : \overline{M} \longrightarrow \mathbb{R}^+_*$, then, there exists $\mu_0 > 0$, and for all $x \in M$, we have that $\mu(x) > \mu_0$.

Theorem 1.4.2. *Let M be a compact Riemannian manifold with a smooth boundary or without boundary and $p(x), q(x) \in C(\overline{M}) \cap L^\infty(M)$. Assume that the assumption* (1.24) *is true. Then, the embedding*

$$W^{1,q(x)}(M) \hookrightarrow L^{q(x)}_{\mu(x)}(M),$$

is compact

Proof. Let $\theta(x) = \frac{\varepsilon(x)}{\varepsilon(x)-1} q(x) = \hat{\varepsilon}(x) q(x)$, where $\frac{1}{\varepsilon(x)} + \frac{1}{\hat{\varepsilon}(x)} = 1$. From (1.24), we deduce that $\theta(x) < p^*(x)$ for all $x \in \overline{M}$, which implies by Theorem 1.4.1, that $W^{1,q(x)}(M) \hookrightarrow L^{\theta(x)}(M)$. Hence, we have that $|u|^{q(x)} \in L^{\theta(x)}(M)$ for any $u \in W^{1,q(x)}(M)$. Now, using the Hölder inequality, we get

$$\rho_{q(\cdot),\mu(\cdot)}(u) \leq r_q \, . ||\mu(x)||_{\varepsilon(x)} || \, |u|^{q(x)} \, ||_{\hat{\varepsilon}(x)} < +\infty. \tag{1.25}$$

It follows that $u \in L^{q(x)}_{\mu(x)}(M)$, that is

$$W^{1,q(x)}(M) \hookrightarrow L^{q(x)}_{\mu(x)}(M).$$

Next, we prove that this embedding is compact. For that, we consider $\{u_n\} \subset W^{1,q(x)}(M)$ such that $u_n \rightharpoonup 0$ weakly in $W^{1,q(x)}(M)$ and since $W^{1,q(x)}(M) \hookrightarrow\hookrightarrow L^{\theta(x)}(M)$, we obtain that

$$u_n \longrightarrow 0 \text{ in } L^{\theta(x)}(M).$$

Then, it follows that $|| \, |u|^{q(x)} \, ||_{\hat{\varepsilon}(x)} \to 0$ as $n \to +\infty$. By Hölder inequality and (1.25), we have

$$\rho_{q(\cdot),\mu(\cdot)}(u_n) \longrightarrow 0.$$

From proposition 1.4.8, result (4), we deduce that

$$||u_n||_{q(\cdot),\mu(\cdot)} \longrightarrow 0 \text{ as } n \to +\infty. \tag{1.26}$$

Hence, the embedding $W^{1,q(x)}(M) \hookrightarrow L^{q(x)}_{\mu(x)}(M)$ is compact. □

2

On some nonlinear elliptic equations in anisotropic Sobolev-Orlicz space

Our objective in this chapter, is to study a certain class of anisotropic elliptic equations with the second term, which is a low-order term and non-polynomial growth ; described by an N-uplet of N-function satisfying the Δ_2-condition in the framework of anisotropic Orlicz spaces. We prove the existence and uniqueness of entropic solution for a source in the dual or in L^1. In the former we study our problem when the domain is bounded, and in the latter, when the domain is unbounded namely ; without assuming any condition on the behaviour of the solutions when $|x|$ tends towards infinity. Moreover, we are giving some examples of an anisotropic elliptic equation that verifies all our demonstrated results.

2.1 Existence of entropy solutions in a bounded domain

2.1.1 Introduction :

Let Ω be a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$. This section is concerned with the study of the existence of entropy solutions to an anisotropic elliptic non-linear equation, driven by low-order term and non-polynomial growth, described by n-uplet of N-functions satisfying the Δ_2-condition, in Orlicz anisotropic space $\mathring{W}_B^1(\Omega) = \overline{C^\infty(\Omega)}^{\mathring{W}_B^1(\Omega)}$, given by

$$(\mathscr{P}) \begin{cases} \mathscr{A}(x,u,\nabla u) + \mathscr{H}(x,u,\nabla u) = g(x) & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where, $\mathscr{A}(x,u,\nabla u) = \sum_{i=1}^{N} (\,\mathsf{a}_i(x,u,\nabla u)\,)_{x_i}$ is a Leray-Lions operator defined on $\mathring{W}_B^1(\Omega)$ into its dual, $B = (\,B_1,\cdots,B_N\,)$ are N-uplet Orlicz functions that satisfy Δ_2-condition, $\mathscr{H}(x,u,\nabla u) = \sum_{i=1}^{N} H_i(x,u,\nabla u)$ and for $i = 1,\cdots,N$, $H_i(x,u,\nabla u) : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ are Carathéodory functions do not satisfy any sign condition, and the growth described by the vector N-function B. As well as $g \in L^1(\Omega)$.

Problem $(\mathscr{P})$ and its versions have been intensively studied in several recent works. For the classical Orlics space, Benkirane and Elmahi in [24] have studied the following problem

$$\begin{cases} A(u) + g(x,u,\nabla u) = f(x) & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where A is a Leray–Lions operator, $g(x,s,\xi)$ is a Carathéodory function, that satisfy the following coercivity condition

$$|\,g(x,s,\xi)\,| \geq \nu\,|\,\xi\,|^p \text{ for } |\,s\,| \geq \gamma,$$

where ν and γ are two positive constants, and the data f is assumed to be in the dual $W^{-1,p'}(\Omega)$. The same problem was demonstrated by Elmahi and Meskine without assuming the above condition in [53]. After that, Porretta and Segura de Leon in [89] investigated the existence results of the same problem using the rearrangement techniques, and they suppose only some growth condition on $g(x,s,\xi)$.

It is noteworthy here to mention the relevant work of Benilan [21] who presented the idea of entropy solutions adjusted to Boltzmann conditions. For more results, we refer the reader to

[4, 5, 12, 34, 35, 55, 66, 69, 77, 91].

For the anisotropic Orlicz space, there are very few results dealing with this topic. For instance, in [48, 72] Korolev and Cianchy proved the embeddings of this space. Kozhevnikova in [74] established the existence of entropy solutions to the following problem :

$$(\mathscr{P}_1)\begin{cases}\sum_{i=1}^{N}(a_i(x,\nabla u))_{x_i}=a_0(x,u) & \text{in } \Omega,\\ \\ u(x)=\psi(x) & \text{on } \partial\Omega,\end{cases}$$

where Ω be an arbitrary domain in $\mathbb{R}^N, N\geq 2$, and the functions a, b are Carathéodory functions satisfying some suitable assumptions.

In [75] Kozhevnikova proved again the existence of entropy solution to the following second-order anisotropic quasilinear elliptic equation :

$$(\mathscr{P}_2)\begin{cases}\sum_{i=1}^{N}(a_i(x,u,\nabla u))_{x_i}-a_0(x,u,\nabla u)=0 & \text{in } \Omega,\\ \\ u=0 & \text{on } \partial\Omega,\end{cases}$$

where $a_i(x,s_0,s)$, $i=0,\cdots,N$are Carathéodory functions. For a deeper comprehension, we refer the reader to [38–41] and the references therein.

The input of this section is the very general framework, we therefore establish a more involved version of equations $(\mathscr{P}_1)$ and $(\mathscr{P}_2)$, given by $(\mathscr{P})$, which is managed by low-order term and non-polynomial growth ; described by an N-uplet of N-function satisfying the Δ_2-condition, the source f is merely integrable and $\Omega\subset\mathbb{R}^N,(N\geq 2)$ is a bounded domain with smooth boundary. Using An approximation procedure and some priori estimates, we proved the existence of entropy solutions to the problem $(\mathscr{P})$.

2.1.2 Assumptions on data and definition of solution

2.1.2.1 Assumptions on data

In this part, we assume they have non-negative measurable functions ϕ, $\varphi\in L^1(\Omega)$ and $\bar{\mathsf{a}}$, $\tilde{\mathsf{a}}$ are two positive constants such that

$$\sum_{i=1}^{N}|\,\mathsf{a}_i(x,s,\xi)\,|\leq\tilde{\mathsf{a}}\sum_{i=1}^{N}\bar{B}_i^{-1}B_i(|\,\xi\,|)+\varphi(x), \tag{2.1}$$

$$\sum_{i=1}^{N}\big(\mathsf{a}_i(x,s,\xi)-\mathsf{a}_i(x,s,\xi^{'})\big).(\xi_i-\xi_i^{'})>0, \tag{2.2}$$

$$\sum_{i=1}^{N}\mathsf{a}_i(x,s,\xi).\xi_i>\bar{\mathsf{a}}\sum_{i=1}^{N}B_i(|\xi|)-\phi(x), \tag{2.3}$$

and there exists $h\in L^1(\Omega)$ and $l:\mathbb{R}\longrightarrow\mathbb{R}^+$ a positive continuous function such that $l\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$.

$$\sum_{i=1}^{N}|H_i(x,s,\xi)|\leq l(s).\sum_{i=1}^{N}B_i(|\xi|)+h(x). \tag{2.4}$$

2.1.2.2 Definition of entropy solutions

Definition 2.1.1. *A measurable function $u:\Omega\longrightarrow\mathbb{R}$ is called an entropy solution of the problem $(\mathscr{P})$ if it satisfies the following conditions :*

1/ $u\in\mathscr{T}_0^{1,B}(\Omega)=\{u:\Omega\longrightarrow\mathbb{R}$ *measurable*, $T_\theta(u)\in\mathring{W}_B^1(\Omega)$ *for any* $\theta>0\}$.

2/ $H(x,u,\nabla u)\in L^1(\Omega)$.

3/ *For any* $\theta>0$,

$$\int_\Omega \mathsf{a}(x,u,\nabla u).\nabla T_\theta(u-\xi)\,dx+\int_\Omega H(x,u,\nabla u).T_\theta(u-\xi)\,dx$$
$$\leq\int_\Omega g(x).T_\theta(u-\xi)\,dx\ \ \forall\xi\in\mathring{W}_B^1(\Omega)\cap L^\infty(\Omega).$$

2.1.3 Main result

In this subsection, we will prove the existence of our problem $(\mathscr{P})$. For that, we suppose that g^m be a sequence of regular functions which strongly converge to g in $L^1(\Omega)$ and such that $||g^m||_{L^1}\leq c$ for some constant c and for each $m\in\mathbb{N}^*$.

$$\mathsf{a}^m(x,s,\xi)=(\mathsf{a}_1^m(x,s,\xi),\cdots,\mathsf{a}_N^m(x,s,\xi)),$$

where $\mathsf{a}_i^m(x,s,\xi)=\mathsf{a}_i(x,T_m(s),\xi)$ for $i=1,\cdots,N$.

$$H^m(x,s,\xi)=T_m(H(x,s,\xi)),$$

and for any $v\in\mathring{W}_B^1(\Omega)$, we consider the following approximate equations

$$(\mathscr{P}_m):\int_\Omega \mathsf{a}(x,T_m(u^m),\nabla u^m)\nabla v\,dx+\int_\Omega H^m(x,u^m,\nabla u^m)\,v\,dx=\int_\Omega g^m\,v\,dx.$$

Lemma 2.1.1. *Suppose that conditions* (2.1) - (2.4) *are satisfied, and let* $(u^m)_{m\in\mathbb{N}}$ *be sequence in* $\mathring{W}^1_B(\Omega)$ *such as*

(a) $u^m \rightharpoonup u$ *in* $\mathring{W}^1_B(\Omega)$.

(b) $a^m(x,u^m,\nabla u^m)$ *is bounded in* $L_{\bar{B}}(\Omega)$.

(c) $\displaystyle\sum_{i=1}^{N}\int_\Omega \Big[a_i^m(x,u^m,\nabla u^m)-a_i^m(x,u^m,\nabla u\chi_s)\Big].(\nabla u^m-\nabla u\,\chi_s)\,dx \longrightarrow 0$ *as* $m\to+\infty,\ s\to\infty$.

Where χ_s *is the characteristic function of* $\Omega^s=\{x\in\Omega:\,|\nabla u|\le s\}$. *Then,*

$$\nabla u^m \longrightarrow \nabla u \ a.e \ in\ \Omega, \tag{2.5}$$

and

$$B(|\nabla u^m|)\longrightarrow B(|\nabla u|)\ in\ L^1(\Omega). \tag{2.6}$$

Proof. Let $\vartheta>0$ fixed and $s>\vartheta$, then from (2.2) we have

$$\begin{aligned}
0 &\le \sum_{i=1}^{N}\int_{\Omega^\vartheta}\Big[a_i^m(x,u^m,\nabla u^m)-a_i^m(x,u^m,\nabla u)\Big].(\nabla u^m-\nabla u)\,dx\\
&=\sum_{i=1}^{N}\int_{\Omega^s}\Big[a_i^m(x,u^m,\nabla u^m)-a_i^m(x,u^m,\nabla u\,\chi_s)\Big].(\nabla u^m-\nabla u\,\chi_s)\,dx\\
&\le\sum_{i=1}^{N}\int_{\Omega}\Big[a_i^m(x,u^m,\nabla u^m)-a_i^m(x,u^m,\nabla u\,\chi_s)\Big].(\nabla u^m-\nabla u\,\chi_s)\,dx.
\end{aligned}$$

According to (c), we get

$$\lim_{m\to\infty}\sum_{i=1}^{N}\int_{\Omega^\vartheta}\Big[a_i^m(x,u^m,\nabla u^m)-a_i^m(x,u^m,\nabla u)\Big].(\nabla u^m-\nabla u)\,dx=0.$$

Proceeding as in [11], we obtain

$$\nabla u^m \longrightarrow \nabla u \text{ a.e in } \Omega.$$

On the other hand, we have

$$\begin{aligned}
\sum_{i=1}^{N}\int_\Omega a_i^m(x,u^m,\nabla u^m).\nabla u^m\,dx &= \sum_{i=1}^{N}\int_\Omega\Big[a_i^m(x,u^m,\nabla u^m)-a_i^m(x,u^m,\nabla u\,\chi_s)\Big].(\nabla u^m-\nabla u\,\chi_s)\,dx\\
&\quad+\sum_{i=1}^{N}\int_\Omega a_i^m(x,u^m,\nabla u\,\chi_s).(\nabla u^m-\nabla u\,\chi_s).dx\\
&\quad+\sum_{i=1}^{N}\int_\omega a_i^m(x,u^m,\nabla u^m).\nabla u\chi_s\,dx,
\end{aligned}$$

using (b) and (2.5), we obtain

$$\sum_{i=1}^{N} a_i^m(x,u^m,\nabla u^m) \rightharpoonup \sum_{i=1}^{N} a_i(x,u,\nabla u) \text{ weakly in } (L_{\bar{B}}(\Omega))^N.$$

Therefore

$$\sum_{i=1}^{N}\int_{\Omega} a_i^m(x,u^m,\nabla u^m)\,\nabla u\,\chi_s\,dx \longrightarrow \sum_{i=1}^{N}\int_{\omega} a_i(x,u,\nabla u)\,.\,\nabla u,$$

as $m\to\infty$, $s\to\infty$. So,

$$\sum_{i=1}^{N}\int_{\Omega}\Big[a_i^m(x,u^m,\nabla u^m)-a_i^m(x,u^m,\nabla u\,\chi_s)\Big]\,.\,(\nabla u^m-\nabla u\,\chi_s)\,dx \longrightarrow 0,$$

and

$$\sum_{i=1}^{N}\int_{\Omega} a_i^m(x,u^m,\nabla u\,\chi_s)\,.\,(\nabla u^m-\nabla u\,\chi_s)\,.\,dx \longrightarrow 0.$$

Thus,

$$\lim_{m\to\infty}\sum_{i=1}^{N}\int_{\Omega} a_i^m(x,u^m,\nabla u^m)\,.\,\nabla u^m\,dx = \sum_{i=1}^{N}\int_{\Omega} a_i(x,u,\nabla u)\,.\,\nabla u\,dx,$$

from (2.3) and vitali's Theorem, we get

$$\bar{a}\sum_{i=1}^{N}\int_{\Omega} B_i(|\nabla u^m|)\,dx-\int_{\Omega}\phi(x)\,dx \geq \bar{a}\sum_{i=1}^{N}\int_{\Omega} B_i(|\nabla u|)\,dx-\int_{\Omega}\phi(x)\,dx.$$

Consequently, by Lemma 2.6 in [67], we get

$$B(|\nabla u^m|) \longrightarrow B(|\nabla u|) \text{ in } \mathring{W}_B^1(\Omega).$$

Thanks to Lemma 1 in [72], we have

$$B(|\nabla u^m|) \longrightarrow B(|\nabla u|) \text{ in } L^1(\Omega).$$

□

Lemma 2.1.2. *Let an N-functions $\bar{B}(\theta)$ satisfy the Δ_2-condition and u^m, u are two functions of $L_B(\Omega)$ with $m=1,\cdots,\infty$, such as*

$$||\,u^m\,||_B \leq c \qquad m\geq 1.$$

$$u^m \longrightarrow u \textit{ almost everywhere in } \Omega,\ m \longrightarrow \infty.$$

Then,

$$u^m \rightharpoonup u \textit{ weakly in } L_B(\Omega) \textit{ as } m \to \infty.$$

Proof. The proof of Lemma 2.1.2 is similar to the proof of Lemma 1.3, Ch. I in [79]. □

Remark 10. *we can extend the previous Lemmas 2.1.1 and 2.1.2 to an unbounded domain, using the same technique.*

Theorem 2.1.1. *Assume that conditions* (2.1) - (2.4) *hold true, then there exists at least one weak solution of the approximate problem* $(\mathscr{P}_m)$.

Proof. Let

$$\begin{aligned} S^m : \mathring{W}_B^1(\Omega) &\longrightarrow (\mathring{W}_B^1(\Omega))' \\ v \quad &\longmapsto \langle S^m(u), v\rangle = \int_\Omega \sum_{i=1}^N \left(\mathtt{a}_i^m(x,u,\nabla u)\,.\,\nabla v + H_i^m(x,u,\nabla u)\,.\,v \right) dx \\ &\qquad - \int_\Omega g^m(x)\,.\,v\,dx \end{aligned}$$

- <u>S is bounded</u>

We denote $L_{\bar{B}}(\Omega) = \prod_{i=1}^N L_{\bar{B}_i}(\Omega)$ with the norm

$$||v||_{L_{\bar{B}}(\Omega)} = \sum_{i=1}^N ||v_i||_{\bar{B}_i,\Omega} \qquad v = (v_1,\cdots,v_N) \in L_{\bar{B}}(\Omega).$$

Where $\bar{B}_i(t)$ are N-functions satisfying the Δ_2-conditions.

Sobolev-space $\mathring{W}_B^1(\Omega)$ is the completions of the space $C_0^\infty(\Omega)$.

$$\mathtt{a}^m(x,s,\xi) = \big(\mathtt{a}_1^m(x,s,\xi),\cdots,\mathtt{a}_N^m(x,s,\xi)\big)$$

and

$$H^m(x,s,\xi) = \big(H_1^m(x,s,\xi),\cdots,H_N^m(x,s,\xi)\big).$$

For $u \in \mathring{W}_B^1(\Omega)$, according to (1.10) and (2.1) we get

$$\begin{aligned} ||\,\mathsf{a}^m(x,u,\nabla u)\,||_{L_{\bar{B}}(\Omega)} &= \sum_{i=1}^{N} ||\,\mathsf{a}_i^m(x,u,\nabla u)\,||_{L_{\bar{B}_i}(\Omega)} \\ &\leq \sum_{i=1}^{N} \int_\Omega \bar{B}_i(\,\mathsf{a}_i^m(x,u,\nabla u)\,)\,dx + N \\ &\leq \tilde{\mathsf{a}}(\Omega)\,.\,||B(u)\,||_{1,\Omega} + ||\,\varphi||_{1,\Omega} + N. \end{aligned} \tag{2.7}$$

Further, for $\mathsf{a}^m(x,u,\nabla u) \in L_{\bar{B}_i}(\Omega),\ v \in \mathring{W}_B^1(\Omega)$ using Hölder's inequality we have

$$\begin{aligned} |\,\langle S^m(u),\,v\rangle_\Omega\,| \leq 2\,||\,\mathsf{a}^m(x,u,\nabla u)\,||_{L_{\bar{B}}(\Omega)}\,.\,||\,v\,||_{\mathring{W}_B^1(\Omega)} \\ + 2\,||\,H^m(x,u,\nabla u)\,||_{L_B(\Omega)}\,.\,||\,v\,||_{\mathring{W}_B^1(\Omega)} + c_0\,.\,||\,v\,||_{\mathring{W}_B^1(\Omega)}. \end{aligned} \tag{2.8}$$

Thus, S^m is bounded.

- <u>S^m is coercive</u>

For $u \in \mathring{W}_B^1(\Omega)$

$$\begin{aligned} \langle S^m(u),u\rangle_\Omega =& \sum_{i=1}^{N} \int_\Omega \mathsf{a}_i^m(x,u,\nabla u)\,.\,\frac{\partial u}{\partial x_i}\,dx + \sum_{i=1}^{N} \int_\Omega H_i^m(x,u,\nabla u)\,.\,u\,dx \\ & - \int_\Omega g^m(x)\,.\,u\,dx. \end{aligned}$$

Then,

$$\begin{aligned} \frac{\langle S^m(u),\,u\rangle_\Omega}{||\,u\,||_{\mathring{W}_B^1(\Omega)}} \geq& \frac{1}{||\,u\,||_{\mathring{W}_B^1(\Omega)}}\,.\,\Bigg[\,\bar{\mathsf{a}} \sum_{i=1}^{N} \int_\Omega B_i\Big(\Big|\,\frac{\partial u}{\partial x_i}\Big|\Big)\,dx - c_1 - c_0 \\ & - l(u)\,.\,\sum_{i=1}^{N} \int_\Omega B_i\Big(\Big|\,\frac{\partial u}{\partial x_i}\Big|\Big)\,dx - \int_\Omega g(x)\,dx\Bigg] \\ \geq& \frac{1}{||\,u\,||_{\mathring{W}_B^1(\Omega)}}\,.\,\Bigg[\,(\,\bar{\mathsf{a}}(\Omega) - c_2\,)\,.\,\sum_{i=1}^{N} \int_\Omega B_i\Big(\Big|\,\frac{\partial u}{\partial x_i}\Big|\Big)\,dx - c_0 - c_1 - c_3\Bigg] \end{aligned}$$

According to (2.1), we have for all $\theta > 0$, $\exists\,\alpha_0 > 0$ such that

$$b_i(\,|\,u_{x_i}\,|\,) > \theta\,b_i\bigg(\frac{|\,u_{x_i}\,|}{||\,u_{x_i}\,||_{B_i,\Omega}}\bigg), \qquad i = 1,\cdots,N.$$

We take $||\,u_{x_i}^\alpha\,||_{B_i,\Omega} > \eta_0 \quad i = 1,\cdots,N.$

Suppose that $||\, u^{\alpha}_{x_i} \,||_{\mathring{W}^1_B(\Omega)} \longrightarrow 0$ as $\alpha \to \infty$. We can assume that

$$||\, u^{\alpha}_{x_1} \,||_{B_1,\Omega} + \cdots + ||\, u^{\alpha}_{x_N} \,||_{B_N,\Omega} \geq N\,\eta_0.$$

According to (1.10) for $c > 1$, we have

$$|\, u^{\alpha} \,|\, b(\,|\, u^{\alpha} \,|\,) < c\, B(u^{\alpha})$$

then, by (1.9) we obtain

$$\begin{aligned}
\frac{\langle S^m(u^{\alpha}),\, u^{\alpha} \rangle_{\Omega}}{||\, u^{\alpha} \,||_{\mathring{W}^1_B(\Omega)}} &\geq \frac{\bar{\mathsf{a}}(\Omega) - c_2}{N\,\eta_0} \cdot \sum_{i=1}^{N} \int_{\Omega} B_i\Big(\Big|\, \frac{\partial u}{\partial x_i} \,\Big|\Big)\, dx - \frac{c_4}{N\,\eta_0} \\
&\geq \frac{\bar{\mathsf{a}}(\Omega) - c_2}{N\,\eta_0} \cdot \sum_{i=1}^{N} \int_{\Omega} |\, u^{\alpha}_{x_i} \,|\, b(\,|\, u^{\alpha}_{x_i} \,|\,)\, dx - \frac{c_4}{N\,\eta_0} \\
&\geq \frac{(\,\bar{\mathsf{a}}(\Omega) - c_2\,)\,.\,\theta}{c N\, ||\, u^{\alpha}_{x_i} \,||_{B_i}} \cdot \sum_{i=1}^{N} \int_{\Omega} |\, u^{\alpha}_{x_i} \,|\, b_i\Big(\frac{|\, u^{\alpha}_{x_i} \,|}{||\, u^{j}_{x_i} \,||_{B_i,\Omega}} \Big)\, dx - \frac{c_4}{N\,\eta_0} \\
&\geq \frac{(\,\bar{\mathsf{a}}(\Omega) - c_2\,)\,.\,\theta}{c N} \cdot \sum_{i=1}^{N} \int_{\Omega} B_i\Big(\frac{|\, u^{\alpha}_{x_i} \,|}{||\, u^{\alpha}_{x_i} \,||_{B_i,\Omega}} \Big)\, dx - \frac{c_4}{N\,\eta_0} \\
&\geq \frac{(\,\bar{\mathsf{a}}(\Omega) - c_2\,)\,.\,\theta}{c N} - \frac{c_4}{N\,\eta_0}.
\end{aligned}$$

Which shows that S^m is coercive, because θ is arbitrary.

- <u>S^m is a pseudo-monotone</u>

Consider a sequence $\{\, u^m \,\}_{m=1}^{\infty}$ in the space $\mathring{W}^1_B(\Omega)$ such that

$$u^m \rightharpoonup u \text{ weakly in } \mathring{W}^1_B(\Omega) \qquad m \to \infty. \tag{2.9}$$

$$\lim_{m\to\infty} \sup \langle\, S(u^m),\, u^m - u \,\rangle \leq 0 \tag{2.10}$$

we demonstrate that

$$S^m(u^m) \rightharpoonup S^m(u) \text{ weakly in } \big(\,\mathring{W}^1_B(\Omega)\,\big)',\ m \to \infty. \tag{2.11}$$

$$\langle\, S^m(u^m),\, u^m - u \,\rangle \longrightarrow 0,\ m \to \infty. \tag{2.12}$$

Since $B(t)$ satisfy the Δ_2-condition, then by (1.10) we have

$$\int_{\Omega} B(t)\; dt \leq c_0\, ||\, t \,||_{B,\Omega}. \tag{2.13}$$

According to (2.9) we get

$$||u^m||_{\mathring{W}_B^1(\Omega)} \leq c_1 \qquad m \geq 1, \tag{2.14}$$

and

$$||B(\nabla u^m)||_1 \leq c_2 \qquad m \geq 1. \tag{2.15}$$

Combining to subsection 2.1.2 and (2.14) we obtain

$$||\mathsf{a}^m(x,u,\nabla u)||_{\bar{B}} = \sum_{i=1}^N ||\mathsf{a}_i^m(x,u^m,\nabla u^m)||_{\bar{B}_i} \leq c_3 \; m \in \mathbb{N}^*. \tag{2.16}$$

And for $m \in \mathbb{N}^*$, $|H^m(x,u,\nabla u)| = |T_m(H(x,u,\nabla u)| \leq m$. Then, by (2.4) and (2.14) we have

$$||H^m(x,u,\nabla u)||_B = \sum_{i=1}^N ||H_i^m(x,u^m,\nabla u^m)||_{B_i} \leq c_4 \; m \in \mathbb{N}^*.$$

According to Lemma 1.1.2, we have

$$\mathring{W}_B^1(\Omega) \hookrightarrow L_{B_i}(\Omega) \text{ for } i = 1, \cdots, N.$$

We set

$$\begin{aligned} S^m(x) = &\sum_{i=1}^N \left[\mathsf{a}_i^m(x,u^m,\nabla u^m) - \mathsf{a}_i^m(x,u,\nabla u)\right](u^m-u)_{x_i} \\ &+ \sum_{i=1}^N \left[H_i^m(x,u^m,\nabla u^m) - H_i^m(x,u,\nabla u)\right](u^m-u), \; m \in \mathbb{N}^*. \end{aligned}$$

then

$$\langle S(u^m) - S(u), u^m - u\rangle = \int_\Omega S^m(x)\,dx \qquad m \in \mathbb{N}^*.$$

By (2.9) and (2.10), we get

$$\lim_{m\to\infty} \sup \int_\Omega S^m(x)\,dx \leq 0.$$

So,

$$\begin{aligned} S^m(x) = &\sum_{i=1}^N \left[\mathsf{a}_i^m(x,u^m,\nabla u^m) - \mathsf{a}_i^m(x,u^m,\nabla u)\right](u^m-u)_{x_i} + \sum_{i=1}^N \left[\mathsf{a}_i^m(x,u^m,\nabla u) - \mathsf{a}_i^m(x,u,\nabla u)\right](u^m-u)_{x_i} \\ &+ \sum_{i=1}^N \left[H_i^m(x,u^m,\nabla u^m) - H_i^m(x,u,\nabla u)\right](u^m-u) \\ = &\; S_1^m(x) + S_2^m(x) + S_3^m(x) \qquad m \in \mathbb{N}^*. \end{aligned} \tag{2.17}$$

We demonstrate that

$$S_1^m(x) \longrightarrow 0 \text{ almost everywhere in } \Omega \quad m \to \infty. \tag{2.18}$$

$$S_2^m(x) \longrightarrow 0 \text{ almost everywhere in } \Omega \quad m \to \infty. \tag{2.19}$$

$$S_3^m(x) \longrightarrow 0 \text{ almost everywhere in } \Omega \quad m \to \infty. \tag{2.20}$$

$$\begin{aligned} S_1^m(x) &= \sum_{i=1}^{N} \left[\mathtt{a}_i^m(x,u^m,\nabla u^m) - \mathtt{a}_i^m(x,u^m,\nabla u) \right] (u^m-u)_{x_i} \\ &= \sum_{i=1}^{N} \mathtt{a}_i^m(x,u^m,\nabla u^m) \,.\, u_{x_i}^m - \sum_{i=1}^{N} \mathtt{a}_i^m(x,u^m,\nabla u^m) \,.\, u_{x_i} - \sum_{i=1}^{N} \mathtt{a}_i^m(x,u,\nabla u) \,.\, u_{x_i}^m + \sum_{i=1}^{N} \mathtt{a}_i^m(x,u,\nabla u) \,.\, u_{x_i} \end{aligned}$$

using (1.1), (2.3), (2.15) and (2.16) we obtain

$$S_1^m(x) \geq c(m) \longrightarrow 0 \text{ as } m \to \infty.$$

Hence, we deduce the convergence (2.18).

As in [76], let $S_i(u) = \mathtt{a}_i(x,u,\nabla v)$ $i = 1, \cdots, N$ be Nemytsky operators for $v \in \mathring{W}_B^1(\Omega)$ fixed and $x \in \Omega$, continuous in $L_{\bar{B}_i}(\Omega)$.

Thus, according to (1.11) and (2.26), we have

$$S_2^m(x) \longrightarrow 0 \text{ almost everywhere in } \Omega \quad m \to \infty.$$

Using the inequality (1.11) we obtain

$$\begin{aligned} S_3^m(x) &\leq 2\sum_{i=1}^{N} ||H_i^m(x,u^m,\nabla u^m) - H_i^m(x,u,\nabla u)||_{B_i,\Omega(R)} \,.\, ||u^m-u||_{\mathring{W}_B^1(\Omega)} \\ &\leq 2c(m) \,.\, ||u^m-u||_{\mathring{W}_B^1(\Omega)}. \end{aligned}$$

Hence, combining to (2.26), we have

$$S_3^m(x) \longrightarrow 0 \text{ almost everywhere in } \Omega \quad m \to \infty.$$

Consequently, applying (2.18), (2.19), (2.20) and the selective convergences, we deduce that

$$S^m(x) \longrightarrow 0 \text{ almost everywhere in } \Omega \quad m \to \infty. \tag{2.21}$$

Let $\Omega' \subset \Omega$, $\operatorname{meas}\Omega' = \operatorname{meas}\Omega$, and the conditions (2.26), (2.21) hold, and (2.1) - (2.4) are satisfied. We show the convergence

$$u^m_{x_i}(x) \longrightarrow u_{x_i}(x) \text{ everywhere in } \Omega \text{ for } i = 1, \cdots, N\,,\ m \to \infty. \tag{2.22}$$

By the absurd, we assume that we do not have convergence at the point $x^* \in \Omega'$.
Let $u^m = u^m_{x_i}(x^*)$, $u = u_{x_i}(x^*)$, $i = 1, \cdots, N$, and $\hat{\mathsf{a}} = \varphi_1(x^*)$, $\bar{\mathsf{a}} = \varphi(x^*)$. We consider that the sequence $\sum_{i=1}^{N} B_i(u^m)$ $m = 1, \cdots, \infty$ is unbounded.
Let $\varepsilon \in \left(0, \frac{\bar{\mathsf{a}}}{1+\hat{\mathsf{a}}}\right)$ is fixed, according to (1.2), (1.4) and the conditions (2.1), (2.3), we have

$$\begin{aligned}
S^m(x^*) &= \sum_{i=1}^{N} \Big(\mathsf{a}^m_i(x^*, u^m, \nabla u^m) - \mathsf{a}^m_i(x^*, u, \nabla u) \Big) \nabla(u^m - u) \\
&\quad + \sum_{i=1}^{N} \Big(H^m_i(x^*, u^m, \nabla u^m) - H^m_i(x^*, u, \nabla u) \Big) (u^m - u) \\
&= \sum_{i=1}^{N} \mathsf{a}^m_i(x^*, u^m, \nabla u^m) \nabla u^m - \sum_{i=1}^{N} \mathsf{a}^m_i(x^*, u^m, \nabla u^m) \nabla u \\
&\quad - \sum_{i=1}^{N} \mathsf{a}^m_i(x^*, u, \nabla u) \nabla u^m + \sum_{i=1}^{N} \mathsf{a}^m_i(x^*, u, \nabla u) \nabla u \\
&\quad + \sum_{i=1}^{N} H^m_i(x^*, u^m, \nabla u^m)\, u^j - \sum_{i=1}^{N} H^m_i(x^*, u^m, \nabla u^m)\, u \\
&\quad - \sum_{i=1}^{N} H^m_i(x^*, u, \nabla u)\, u^m + \sum_{i=1}^{N} H^m_i(x^*, u, \nabla u)\, u.
\end{aligned}$$

Using the generalized Young inequality and (2.14), we get

$$\begin{aligned}
S^m(x^*) &\geq \sum_{i=1}^{N} \mathsf{a}^m_i(x^*, u, \nabla u) \,.\, \nabla u + \sum_{i=1}^{N} \mathsf{a}^m_i(x^*, u^m, \nabla u^m) \,.\, \nabla u^m \\
&\quad - \varepsilon \sum_{i=1}^{N} \bar{B}_i(\mathsf{a}^m_i(x^*, u^m, \nabla u^m)) - c_1(\varepsilon) \sum_{i=1}^{N} B_i(\nabla u) \\
&\quad - \varepsilon \sum_{i=1}^{N} \bar{B}_i(\mathsf{a}^m_i(x^*, u, \nabla u)) - c_2(\varepsilon) \sum_{i=1}^{N} B_i(\nabla u^m) \\
&\quad + \sum_{i=1}^{N} H^m_i(x^*, u^m, \nabla u^m) \,.\, \nabla u^m + \sum_{i=1}^{N} H^m_i(x^*, u, \nabla u) \,.\, \nabla u \\
&\quad - \sum_{i=1}^{N} H^m_i(x^*, u^m, \nabla u^m) \,.\, \nabla u - \sum_{i=1}^{N} H^m_i(x^*, u, \nabla u) \,.\, \nabla u^m,
\end{aligned}$$

then,

$$\begin{aligned}
S^m(x^*) \geq \bar{a} \sum_{i=1}^N B_i(\nabla u) - \psi(x^*) + \sum_{i=1}^N B_i(\nabla u^m) - \psi(x^*) - \varepsilon\, \hat{a} \sum_{i=1}^N B_i(\nabla u^m) \\
- \varepsilon\, \varphi(x^*) - c_1(\varepsilon) \sum_{i=1}^N B_i(\nabla u) - \varepsilon\, \hat{a} \sum_{i=1}^N B_i(\nabla u) - \varepsilon\, \varphi(x^*) \\
- c_2 \sum_{i=1}^N B_i(\nabla u^m) - 4h(x^*) - c_3\, l(u) \sum_{i=1}^N B_i(\nabla u) - c_4\, l(u^m) \sum_{i=1}^N B_i(\nabla u^m) \\
\geq \left[\bar{a} - c_1(\varepsilon) - \varepsilon\, \hat{a} - c_3\, l(u)\right] \sum_{i=1}^N B_i(\nabla u) + \left[\bar{a} - \varepsilon\, \hat{a} c_2 - c_4\, l(u^m)\right] \sum_{i=1}^N B_i(\nabla u^m) - c_5(\varepsilon).
\end{aligned}$$

We deduced that the sequence $S^m(x^*)$ is not bounded, which is absurd as far as (2.21) is concerned. Thus, the sequences $u^m_{x_i}, i = 1, \cdots, N, m \to \infty$ are bounded.
Let $u^* = (u^*_1, u^*_2, \cdots, u^*_N)$ the limits of subsequence $u^m = (u^m_1, \cdots, u^m_N)$ with $m \to \infty$. Then, taking into account (2.26), we obtain

$$u^m_{x_i} \longrightarrow u^*_{x_i} \qquad , \; i = 1, \cdots, N. \tag{2.23}$$

Therefore, from (2.21), (2.23) and the fact that $a^m_i(x^*, u, \nabla u)$ are continuous in u (because they are Carathéodory functions), we obtain

$$\sum_{i=1}^N \left(a^m_i(x^*, u^m, \nabla u^m) - a^m_i(x^*, u, \nabla u)\right) . (u^m_{x_i} - u_{x_i}) = 0,$$

and from (2.2) we have, $u^*_{x_i} = u_{x_i}$. This contradicts the fact that there is no convergence at the point x^*.
According to (2.26), (2.23) and the fact that $a^m_i(x^*, u, \nabla u)$ are continuous in u, so for $m \to \infty$ we get

$$a^m_i(x, u^m, \nabla u^m) \longrightarrow a^m_i(x, u, \nabla u), \; i = 1, \cdots, N \text{ almost everywhere in } \Omega.$$

By Lemma 2.1.2 we get the weak convergences

$$a^m_i(x, u^m, \nabla u^m) \rightharpoonup a^m_i(x, u, \nabla u) \text{ in } L_{\bar{B}_i(\Omega)}, i = 1, \cdots, N. \tag{2.24}$$

Then, the weak convergence (2.11) follows from (2.24).

Moreover, to complete the proof, we note that (2.12) is implied from (2.9) and (2.21) :

$$\langle S^m(u^m), u^m - u\rangle = \langle S^m(u^m) - S^m(u), u^m - u\rangle + \langle S^m(u), u^m - u\rangle \to 0,\, m \to \infty.$$

□

Theorem 2.1.2. *Let* Ω *be a bounded domain of* $\mathbb{R}^N$. *Under assumptions* (2.1) - (2.4), *there exists at least one entropy solution of the problem* $(\mathscr{P})$ *on the sense of definition 2.1.1.*

Proof. We divide our proof in six steps.

Step 1 : **A priori estimate of** $\{u^m\}$**.**

In this step we are concerning to show the following proposition :

Proposition 2.1.1. *Suppose that the assumptions* (2.1) - (2.4) *hold, and let* $(u^m)_{m\in\mathbb{N}}$ *be solutions of the approximate problem* $(\mathscr{P}_m)$. *Then, for any* $\theta > 0$, *there exists a constant* $\theta\,.c_1 + c_2$ *(not depending on m), such that*

$$\int_\Omega B(|\nabla T_\theta(u^m)|) \le \theta\,.c_1 + c_2$$

Proof. Taking $v = \exp(G(u^m))\,.T_\theta(u^m)$, as a test function with $G(s) = \int_0^s \frac{l(t)}{\bar{a}}\,dt$ and $\bar{a}$ is the coercivity constant, we obtain

$$\begin{aligned}
&\sum_{i=1}^N \int_\Omega \mathrm{a}_i^m(x,u^m,\nabla u^m)\,.\nabla(\exp(G(u^m))\,.T_\theta(u^m))\,dx \\
&\quad + \sum_{i=1}^N \int_\Omega H_i^m(x,u^m,\nabla u^m)\,.\exp(G(u^m))\,.T_\theta(u^m)\,dx \\
&= \int_\Omega g^m\,.\exp(G(u^m))\,.T_\theta(u^m)\,dx.
\end{aligned}$$

Then,

$$\begin{aligned}
&\sum_{i=1}^N \int_\Omega \mathrm{a}_i^m(x,u^m,\nabla u^m)\exp(G(u^m))\,\nabla T_\theta(u^m))\,dx \\
&+ \sum_{i=1}^N \int_\Omega \mathrm{a}_i^m(x,u^m,\nabla u^m).\nabla u^m.\frac{l(u^m)}{\bar{a}}.\exp(G(u^m))T_\theta(u^m)dx \\
&= \sum_{i=1}^N \int_\Omega |H_i^m(x,u^m,\nabla u^m)|\,.\exp(G(u^m))\,.T_\theta(u^m)\,dx + \int_\Omega g^m\,.\exp(G(u^m))\,.T_\theta(u^m)\,dx \\
&\le \sum_{i=1}^N \int_\Omega \big[h(x) + l(u^m)\,.B_i(\nabla u^m)\big]\,.\exp(G(u^m))\,.T_\theta(u^m)\,dx + \int_\Omega g^m\,.\exp(G(u^m))\,.T_\theta(u^m)\,dx \\
&\le \sum_{i=1}^N \int_\Omega l(u^m)\,.B_i(\nabla u^m).\exp(G(u^m))\,.T_\theta(u^m)\,dx + \int_\Omega \big(g^m + h(x)\big)\,.\exp(G(u^m))\,.T_\theta(u^m)\,dx,
\end{aligned}$$

so,

$$\sum_{i=1}^{N}\int_{\{\Omega:|u^m|<\theta\}} \mathsf{a}_i^m(x,u^m,\nabla u^m)\,.\,\nabla u^m\,.\,\exp(G(u^m))\,dx$$
$$\leq \int_{\Omega}\big[g^m(x)+h(x)+\phi(x)\frac{l(u^m)}{\bar{\mathsf{a}}}\big]\,.\,\exp(G(u^m))\,T_\theta(u^m)\,dx,$$

by (2.3), we get

$$\bar{\mathsf{a}}\sum_{i=1}^{N}\int_{\{\Omega:|u^m|\leq\theta\}} B_i(\nabla u^m)\exp(G(u^m))\,dx$$
$$\leq \int_{\{\Omega:|u^m|\leq\theta\}}\phi(x)\exp(G(u^m))\,dx$$
$$+\int_{\Omega}\big[g^m(x)+h(x)+\phi(x)\frac{l(u_m)}{\bar{\mathsf{a}}}\big]\,.\,\exp(G(u^m))\,T_\theta(u^m)\,dx,$$

since ϕ, h and $g^m \in L^1(\Omega)$, and the fact that $\exp(G(\pm\infty)) \leq \exp\left(\frac{||l||_{L^1(\mathbb{R})}}{\bar{\mathsf{a}}}\right)$, we deduce that,

$$\int_{\{\Omega:|u^m|<\theta\}} B(\nabla T_\theta(u^m))\,dx \leq \theta\,.\,c_1+c_2 \quad \theta>0.$$

Finally

$$\int_{\Omega} B(\nabla T_\theta(u^m))\,dx \leq \theta\,.\,c_1+c_2 \quad \theta>0.$$

□

Step 2 : Almost everywhere convergence of $\{u^m\}$.

Lemma 2.1.3. *For all u^m measurable function on Ω, we have*

$$meas\{x\in\Omega,\,|u^m|>\theta\}\longrightarrow 0.$$

Proof. According to Lemma 1.1.1 and Lemma 1.1.2, we have

$$\begin{aligned}||T_\theta(u^m)||_{B^*} &\leq A\,.\,||\nabla T_\theta(u^m)||_B \\ &\leq A\,.\,\varepsilon(\theta)\int_{\Omega} B(\nabla T_\theta(u^m)\,dx \\ &\leq c\,.\,\theta\,.\,\varepsilon(\theta), \qquad \text{for } \theta>1, \end{aligned} \tag{2.25}$$

with $\varepsilon(\theta)\longrightarrow 0$ as $\theta\longrightarrow\infty$.

Form (2.25) we have

$$B^*\left(\frac{\theta}{||T_\theta(u^m)||_{B^*}}\right)\text{meas}\{x\in\Omega:\,|u^m|\geq\theta\}\leq\int_\Omega B^*\left(\frac{T_\theta(u^m)}{||T_\theta(u^m)||_{B^*}}\right)dx$$
$$\leq\int_\Omega B^*\left(\frac{\theta}{||T_\theta(u^m)||_{B^*}}\right)dx,$$

by (2.25) again, we obtain

$$B^*\left(\frac{\theta}{||T_\theta(u^m)||_{B^*}}\right)\longrightarrow\infty \text{ as } \theta\longrightarrow\infty.$$

Hence,

$$\text{meas}\{x\in\Omega:|u^m|\geq\theta\}\longrightarrow 0 \text{ as } \theta\longrightarrow\infty \text{ for all } m\in\mathbb{N}.$$

□

We have now to prove the almost everywhere convergence of $\{u^m\}$

$$u^m\longrightarrow u \text{ a.e in } \Omega. \tag{2.26}$$

Let $K(\theta)=\sup_{m\in\mathbb{N}}\text{meas}\{x\in\Omega:\,|u^m|>\theta\}\longrightarrow 0$ as $\theta\longrightarrow\infty$.
Form proposition 2.1.1 and Lemma 2.1.3, the sequence $\{T_\theta(u^m)\}$ is bounded in $\mathring{W}_B^1(\Omega)$. Then, there exists a subsequence of $\{T_\theta(u^m)\}$, denoted again by $\{T_\theta(u^m)\}$ and a function v_θ in $\mathring{W}_B^1(\Omega)$ such that for $P\ll B$

$$T_\theta(u^m)\longrightarrow v_\theta \text{ in } L_P(\Omega) \text{ as } m\longrightarrow\infty.$$

Consequently, we assume that $\{T_\theta(u^m)\}$ is a Cauchy sequence in Ω, since $\{T_\theta(u^m)\}$ is bounded in $\mathring{W}_B^1(\Omega)$. Then, there exists $\lambda>0$ and a constant c_0 such as

$$\int_\Omega B\left(\frac{|T_\theta(u^m)|}{\lambda}\right)dx\leq c_0.$$

On the other hand, for any $\delta>0$, we have

$$\text{meas}\{|u^m-u^n|>\delta\}\leq\text{meas}\{|u^m|>\theta\}+\text{meas}\{|u^n|>\theta\}$$
$$+\text{meas}\{|T_\theta(u^m)-T_\theta(u^n)|>\delta\}.$$

Let $\varepsilon>0$, using (2.25), and the fact that $\{T_\theta(u^m)\}$ is a Cauchy sequence, there exists some constant

$\theta(\varepsilon)$ such as

$$\text{meas}\{\,|u^m - u^n| > \delta\,\} \leq \varepsilon, \;\; \text{for all } m, n \geq c(\theta(\varepsilon), \delta).$$

This shows that $\{\,u^m\,\}$ is a Cauchy sequence in Ω. Thus, we prove that there is $u : \Omega \longrightarrow \mathbb{R}$ measurable such that $u^m \longrightarrow u$ a.e in Ω. This implies the (2.26).

Step 3 : Weak convergence of the gradient.

Since $\mathring{W}_B^1(\Omega)$ reflexive, then there exists a subsequence such that

$$T_\theta(u^m) \rightharpoonup v \text{ weakly in } \mathring{W}_B^1(\Omega),\; m \to \infty.$$

And since,

$$\mathring{W}_B^1(\Omega) \hookrightarrow L_B(\Omega),$$

we have

$$\nabla T_\theta(u^m) \rightharpoonup \nabla v \text{ in } L_B(\Omega) \text{ as } m \to \infty,$$

Beside, $T_\theta(u^m) \longrightarrow T_\theta(u)$ a.e in Ω, gives $T_\theta(u^m) \longrightarrow T_\theta(u)$ strongly in $L_B(\Omega)$. Then, we obtain for any fixed $\theta > 0$

$$\nabla T_\theta(u^m) \longrightarrow \nabla T_\theta(u) \text{ a.e in } \Omega.$$

Applying Lemma 2.1.2, we have the following weak convergence

$$\nabla T_\theta(u^m) \rightharpoonup \nabla T_\theta(u) \text{ in } L_B(\Omega) \text{ as } m \to \infty.$$

Step 4 : Strong convergence of the gradient.

For $\alpha > \theta > 0$, we introduce the following function defined as

$$h_\alpha(s) = \begin{cases} 1 & \text{if } \; |s| \leq \alpha\,, \\ 1 - |s-\alpha| & \text{if } \; \alpha \leq |s| \leq \alpha+1\,, \\ 0 & \text{if } \; s \geq \alpha+1\,. \end{cases}$$

and we show that the following assertions are true :

Assertion 1 :

$$\lim_{\alpha\to\infty} \lim_{m\to\infty} \sum_{i=1}^{N} \int_{\{\,\alpha\leq|u^m|\leq\alpha+1\,\}} \mathsf{a}_i^m(x, u^m, \nabla u^m)\,.\,\nabla u^m\, dx = 0. \tag{2.27}$$

Assertion 2 :

$$\nabla u^m \longrightarrow \nabla u \text{ a.e in } \Omega. \tag{2.28}$$

Proof. We take $v = \exp(G(u^m))\,T_{1,\alpha}(u^m) = \exp(G(u^m))\,T_1(u^m - T_\alpha(u^m))$ as a test function in the problem $(\mathscr{P}_m)$, we obtain

$$\begin{aligned}\sum_{i=1}^{N}\int_\Omega \mathsf{a}_i^m(x,u^m,\nabla u^m)\,.\,\nabla\Big(\exp(G(u^m))\,.\,T_1(u^m - T_\alpha(u^m))\Big)\,dx \\ \leq \sum_{i=1}^{N}\int_\Omega |\,H_i^m(x,u^m,\nabla u^m)\,|\,.\,\exp(G(u^m))\,.\,T_1(u^m - T_\alpha(u^m))\,dx \\ + \int_\Omega g^m(x)\,.\,\exp(G(u^m))\,.\,T_1(u^m - T_\alpha(u^m))\,dx,\end{aligned}$$

according to (2.3) and (2.4) we deduce that

$$\begin{aligned}\sum_{i=1}^{N}\int_{\{\alpha<|u^m|<\alpha+1\}} \mathsf{a}_i^m(x,u^m,\nabla u^m)\,.\,\nabla u^m\,.\,\exp(G(u^m))\,dx \\ \leq \int_\Omega \left[g^m(x)+h(x)+\phi(x)\,.\,\frac{l(u^m)}{\bar{\mathsf{a}}}\right].\,\exp(G(u^m))\,.\,T_1(u^m - T_\alpha(u^m))\,dx,\end{aligned}$$

since $\phi \in L^1(\Omega)$, $h \in L^1(\Omega)$, $g^m \in (L^1(\Omega))^N$, and the fact that $\exp(G(\pm)) \leq \exp\left(\frac{||l||_{L^1(\mathbb{R})}}{\bar{\mathsf{a}}}\right)$, we deduce from vitali's Theorem that

$$\lim_{\alpha\to\infty}\lim_{m\to\infty}\int_\Omega \left[g^m(x)+h(x)+\phi(x)\,.\,\frac{l(u^m)}{\bar{\mathsf{a}}}\right].\,\exp(G(u^m))\,.\,T_1(u^m - T_\alpha(u^m))\,dx = 0.$$

Hence,

$$\lim_{\alpha\to\infty}\lim_{m\to\infty}\int_{\{\alpha<|u^m|<\alpha+1\}} \mathsf{a}_i^m(x,u^m,\nabla u^m)\,.\,\nabla u^m\,dx = 0.$$

And to show that assertion 2 is true, we take

$$v = \exp(G(u^m))\,(T_\theta(u^m) - T_\theta(u))\,h_\alpha(u^m),$$

as a test function in the problem $(\mathscr{P}_m)$. We have

$$\sum_{i=1}^{N}\int_{\Omega}\mathsf{a}_i^m(x,u^m,\nabla u^m)\,.\,\nabla\big(\exp(G(u^m))\,.\,(T_\theta(u^m)-T_\theta(u))\,.\,h_\alpha(u^m)\big)\,dx$$
$$+\sum_{i=1}^{N}\int_{\Omega}H_i^m(x,u^m,\nabla u^m)\,.\,\exp(G(u^m))\,.\,(T_\theta(u^m)-T_\theta(u))\,.\,h_\alpha(u^m)\,dx$$
$$\leq\int_{\Omega}g^m(x)\,.\,\exp(G(u^m))\,.\,(T_\theta(u^m)-T_\theta(u))\,.\,h_\alpha(u^m)\,dx,$$

which implies

$$\sum_{i=1}^{N}\int_{\Omega}\mathsf{a}_i^m(x,u^m,\nabla u^m)\,.\,\nabla u^m\,.\,\frac{l(u^m)}{\bar{\mathsf{a}}}\,.\,\exp(G(u^m))\,.\,(T_\theta(u^m)-T_\theta(u))\,.\,h_\alpha(u^m)\,dx$$
$$+\sum_{i=1}^{N}\mathsf{a}_i^m(x,u^m,\nabla u^m)\,.\,\exp(G(u^m))\,.\,(\nabla T_\theta(u^m)-\nabla T_\theta(u))\,.\,h_\alpha(u^m)\,dx$$
$$+\sum_{i=1}^{N}\int_{\Omega}\mathsf{a}_i^m(x,u^m,\nabla u^m)\,.\,\exp(G(u^m))\,.\,(T_\theta(u^m)-T_\theta(u))\,.\,\nabla h_\alpha(u^m)\,dx$$
$$\leq\sum_{i=1}^{N}\int_{\Omega}|H_i^m(x,u^m,\nabla u^m)|\,.\,\exp(G(u^m))\,.\,(T_\theta(u^m)-T_\theta(u))\,.\,h_\alpha(u^m)\,dx$$
$$+\int_{\Omega}g^m(x)\,.\,\exp(G(u^m))\,.\,(T_\theta(u^m)-T_\theta(u))\,.\,h_\alpha(u^m)\,dx,$$

thanks to (2.3) and (2.4), we obtain

$$\sum_{i=1}^{N}\int_{\Omega}\mathsf{a}_i^m(x,u^m,\nabla u^m)\,.\,\exp(G(u^m))\,.\,(\nabla T_\theta(u^m)-\nabla T_\theta(u))\,.\,h_\alpha(u^m)\,dx$$
$$+\sum_{i=1}^{N}\int_{\{\Omega:\,\alpha\leq|u^m|\leq\alpha+1\}}\mathsf{a}_i^m(x,u^m,\nabla u^m)\,.\,\nabla u^m\,.\,\exp(G(u^m))\,.\,(T_\theta(u^m)-T_\theta(u))\,dx$$
$$\leq\int_{\Omega}\left[g^m(x)+h(x)+\phi(x)\,.\,\frac{l(u^m)}{\bar{\mathsf{a}}}\right].\,\exp(G(u^m))\,.\,(T_\theta(u^m)-T_\theta(u))\,.\,h_\alpha(u^m)\,dx,$$

sine $h_\alpha\geq 0$, and $u^m\,(T_\theta(u^m)-T_\theta(u))\geq 0$ we have

$$\sum_{i=1}^{N}\int_{\{\Omega:\,|u^m|\leq\theta\}}\mathsf{a}_i(x,T_\theta(u^m),\nabla T_\theta(u^m))\exp(G(u^m))\,.\,(\nabla T_\theta(u^m)-\nabla T_\theta(u))\,dx$$
$$+\int_{\{\Omega:\,\alpha\leq|u^m|\leq\alpha+1\}}\mathsf{a}_i^m(x,u^m,\nabla u^m)\,\nabla u^m\exp(G(u^m))\,(T_\theta(u^m)-T_\theta(u))\,dx$$
$$\leq\int_{\Omega}\left[g^m(x)+h(x)+\phi(x)\,.\,\frac{l(u^m)}{\bar{\mathsf{a}}}\right].\,\exp(G(u^m))\,.\,(T_\theta(u^m)-T_\theta(u))\,dx$$
$$+\sum_{i=1}^{N}\int_{\{\Omega:\,\theta\leq|u^m|\leq\alpha+1\}}\mathsf{a}_i(x,T_{\alpha+1}(u^m),\nabla T_{\alpha+1}(u^m))\,.\,\exp(G(u^m))\,.\,|\nabla T_\theta(u)|\,dx$$
$$+\sum_{i=1}^{N}\int_{\{\Omega:\alpha\leq|u^m|\leq\alpha+1\}}\mathsf{a}_i^m(x,u^m,\nabla u^m)\,.\,\nabla u^m\,.\,\exp(G(u^m))\,.\,|T_\theta(u^m)-T_\theta(u)|\,dx.$$

The first term in the right hand side goes to zero as m tend to ∞, since $T_\theta(u^m) \rightharpoonup T_\theta(u)$ weakly in $\mathring{W}^1_B(\Omega)$.

Since $\mathsf{a}_i^m(x, T_{\alpha+1}(u^m), \nabla T_{\alpha+1}(u^m))$ is bounded in $L_{\bar{B}}(\Omega)$, there exists $\tilde{\mathsf{a}}^m \in L_{\bar{B}}(\Omega)$ such as

$$|\, \mathsf{a}_i^m(x, T_{\alpha+1}(u^m), \nabla T_{\alpha+1}(u^m))\,| \rightharpoonup \tilde{\mathsf{a}}^m \text{ in } L_{\bar{B}}(\Omega). \tag{2.29}$$

Thus, the second term of the right hand side goes also to zero.

According with (2.27) and the fact that $T_\theta(u^m) \longrightarrow T_\theta(u)$ strongly in $\mathring{W}^1_{B,loc}(\Omega)$ we deduce that

$$\sum_{i=1}^N \int_{\{\Omega: |u^m| \le \theta\}} \mathsf{a}_i(x, T_\theta(u^m), \nabla T_\theta(u^m)) \,.\, \exp(G(u^m)) \,.\, |\, \nabla T_\theta(u^m) - \nabla T_\theta(u)\,|\, dx$$
$$\le \varepsilon(\alpha, m).$$

Then,

$$\begin{aligned}
&\sum_{i=1}^N \int_\Omega \Big[\mathsf{a}_i(x, T_\theta(u^m), \nabla T_\theta(u^m)) - \mathsf{a}_i(x, T_\theta(u^m), \nabla T_\theta(u)) \Big] \,.\, (\, \nabla T_\theta(u^m) - T_\theta(u)\,)\, dx \\
&\le - \sum_{i=1}^N \int_\Omega \mathsf{a}_i(x, T_\theta(u^m), \nabla T_\theta(u)) \,.\, \exp(G(u^m)) \,.\, |\, \nabla T_\theta(u^m) - \nabla T_\theta(u)\,|\, dx \\
&\quad - \sum_{i=1}^N \int_{\{\Omega: |u^m| \le \theta\}} \mathsf{a}_i(x, T_\theta(u^m), \nabla T_\theta(u^m)) \,.\, \exp(G(u^m)) \,.\, \nabla T_\theta(u)\, dx + \varepsilon(\alpha, m).
\end{aligned} \tag{2.30}$$

According to the Lebesgue dominated convergence Theorem, we have $T_\theta(u^m) \longrightarrow T_\theta(u)$ in $\mathring{W}^1_{B,loc}(\Omega)$ and $\nabla T_\theta(u^m) \rightharpoonup \nabla T_\theta(u)$ in $\mathring{W}^1_B(\Omega)$, then the terms on the right hand side of (2.30) go to zero as m and α tend to infinity.

Which implies

$$\sum_{i=1}^N \int_\Omega \Big[\mathsf{a}_i(x, T_\theta(u^m), \nabla T_\theta(u^m)) - \mathsf{a}_i(x, T_\theta(u^m), \nabla T_\theta(u)) \Big] \times (\, \nabla T_\theta(u^m) - T_\theta(u)\,)\, dx \longrightarrow 0. \tag{2.31}$$

Thanks to Lemma 2.1.1, we have for $\theta \ge 1$,

$$\nabla T_\theta(u^m) \longrightarrow \nabla T_\theta(u) \text{ a.e in } \Omega, \tag{2.32}$$

Then,

$$\nabla u^m \longrightarrow \nabla u \text{ a.e in } \Omega.$$

□

Step 5 : Equi-integrability of $H^m(x,u^m,\nabla u^m)$.

Let $v=\exp(2G(|u^m|)).T_1(u^m-T_R(u^m))$ as a test function in the problem $(\mathscr{P}_m)$, we obtain

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega \mathsf{a}_i^m(x,u^m,\nabla u^m).\nabla\big(\exp(2G(|u^m|)).T_1(u^m-T_R(u^m))\big)\,dx\\&\quad+\sum_{i=1}^{N}\int_\Omega H_i^m(x,u^m,\nabla u^m).\exp(2G(|u^m|)).T_1(u^m-T_R(u^m))\,dx\\&\le\int_\Omega g^m(x).\exp(2G(|u^m|)).T_1(u^m-T_R(u^m))\,dx,\end{aligned}$$

which implies that

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega \mathsf{a}_i^m(x,u^m,\nabla u^m).\nabla u^m.\frac{l(u^m)}{\bar{\mathsf{a}}}.\exp(2G(|u^m|)).T_1(u^m-T_R(u^m))\,dx\\&\quad+\sum_{i=1}^{N}\int_{\{\Omega:R\le|u^m|\le R+1\}}\mathsf{a}_i^m(x,u^m,\nabla u^m).\nabla u^m.\exp(2G(|u^m|))\,dx\\&\le\sum_{i=1}^{N}\int_\Omega |H_i^m(x,u^m,\nabla u^m)|.\exp(2G(|u^m|)).T_1(u^m-T_R(u^m))\,dx\\&\quad+\int_\Omega g^m(x).\exp(2G(|u^m|)).T_1(u^m-T_R(u^m))\,dx,\end{aligned}$$

by (2.3) and (2.4), we obtain

$$\begin{aligned}&\bar{\mathsf{a}}\sum_{i=1}^{N}\int_{\{\Omega:R\le|u^m|\le R+1\}}B_i(|\nabla u^m|).\exp(2G(|u^m|)\,dx\\&\le\int_\Omega\left[g^m(x)+h(x)+\phi(x).\frac{l(u^m)}{\bar{\mathsf{a}}}\right].\exp(2G(|u^m|)).T_1(u^m-T_R(u^m))\,dx\\&\quad+\int_{\{\Omega:R\le|u^m|\le R+1\}}\phi(x).\exp(2G(|u^m|)\,dx.\end{aligned}$$

Since $\exp(G(\pm\infty))\le\exp\left(2\,\frac{||l||_{L^1(\mathbb{R})}}{\bar{\mathsf{a}}}\right)$, $g^m\in(L^1(\Omega))^N$, ϕ and $h\in L^1(\Omega)$. Then, $\forall\varepsilon>0$, $\exists R(\varepsilon)>0$ such as

$$\sum_{i=1}^{N}\int_{\{\Omega:|u^m|>R+1\}}B(|\nabla u^m|)\,dx\le\frac{\varepsilon}{2}\qquad\forall R>R(\varepsilon).$$

Let $\mathring{V}(\Omega)$ be an arbitrary bounded subset for Ω, then, for any measurable set $E\subset\mathring{V}(\Omega)$ we have

$$\begin{aligned}\sum_{i=1}^{N}\int_E B_i(|\nabla u^m|)\,dx\le&\sum_{i=1}^{N}\int_E B_i(|\nabla T_R(u^m)|)\,dx\\&+\sum_{i=1}^{N}\int_{\{|u^m|>R+1\}}B_i(|\nabla u^m|)\,dx,\end{aligned}\tag{2.33}$$

we conclude that $\forall E \subset \mathring{V}(\Omega)$ with $\text{meas}\,(E) < \beta(\varepsilon)$ and $T_R(u^m) \longrightarrow T_R(u)$ in $\mathring{W}^1_B(\Omega)$

$$\sum_{i=1}^{N} \int_E B_i(|\nabla T_R(u^m)|)\, dx \leq \frac{\varepsilon}{2}. \tag{2.34}$$

Finally, according to (2.33) and (2.34), we obtain

$$\sum_{i=1}^{N} \int_E B_i(|\nabla u^m|)\, dx \leq \varepsilon \qquad \forall E \subset \mathring{V}(\Omega) \text{ such as meas}\,(E) < \beta(\varepsilon).$$

Which gives the results.

Step 6 : Passing to the limit.

Let $\xi \in \mathring{W}^1_B(\Omega) \cap L^\infty(\Omega)$, using the following test function $v = T_\theta(u^m - \xi)$ in the problem $(\mathscr{P}_m)$, with the fact that $|u^m| - ||\xi||_\infty < |u^m - \xi| \leq \alpha$ and $\{|u^m - \xi| \leq \alpha\} \subset \{|u^m| \leq \alpha + ||\xi||_\infty\}$, we obtain

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega \mathsf{a}_i(x, T_m(u^m), \nabla u^m) \nabla T_\theta(u^m - \xi)\, dx + \sum_{i=1}^{N} \int_\Omega H_i^m(x, u^m, \nabla u^m) \,.\, T_\theta(u^m - \xi)\, dx \\ &\quad \leq \int_\Omega g^m(x) \,.\, T_\theta(u^m - \xi)\, dx, \end{aligned} \tag{2.35}$$

which implies that

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega \mathsf{a}_i(x, T_m(u^m), \nabla u^m) \,.\, T_\theta(u^m - \xi)\, dx \\ &= \sum_{i=1}^{N} \int_\Omega \mathsf{a}_i(x, T_{\alpha+||\xi||_\infty}(u^m), \nabla T_{\alpha+||\xi||_\infty}(u^m)) \times T_{\alpha+||\xi||_\infty}(u^m - \xi) \,.\, \chi_{\{|u^m - \xi| < \alpha\}} dx \\ &= \sum_{i=1}^{N} \int_\Omega \Big[\mathsf{a}_i(x, T_{\alpha+||\xi||_\infty}(u^m), \nabla T_{\alpha+||\xi||_\infty}(u^m)) - \mathsf{a}_i(x, T_{\alpha+||\xi||_\infty}(u^m), \nabla \xi)\Big] \nabla T_{\alpha+||\xi||_\infty}(u^m - \xi) \,.\, \chi_{\{|u^m - \xi| < \alpha\}}\, dx \\ &\quad + \sum_{i=1}^{N} \int_\Omega \mathsf{a}_i(x, T_{\alpha+||\xi||_\infty}(u^m), \nabla \xi) \,.\, \nabla T_{\alpha+||\xi||_\infty}(u^m - \xi) \,.\, \chi_{\{|u^m - \xi| < \alpha\}}\, dx. \end{aligned} \tag{2.36}$$

By Fatou's Lemma, we have

$$\begin{aligned} &\liminf_{m\to\infty} \sum_{i=1}^{N} \int_\Omega \mathsf{a}_i(x, T_m(u^m), \nabla u^m) \,.\, \nabla T_\theta(u^m - \xi)\, dx \\ &\geq \sum_{i=1}^{N} \int_\Omega \Big[\mathsf{a}_i(x, T_{\alpha+||\xi||_\infty}(u^m), \nabla T_{\alpha+||\xi||_\infty}(u^m)) - \mathsf{a}_i(x, T_{\alpha+||\xi||_\infty}(u^m), \nabla \xi)\Big] \\ &\quad \times \nabla T_{\alpha+||\xi||_\infty}(u^m - \xi) \,.\, \chi_{\{|u^m - \xi| < \alpha\}}\, dx \\ &\quad + \lim_{m\to\infty} \sum_{i=1}^{N} \int_\Omega \mathsf{a}_i(x, T_{\alpha+||\xi||_\infty}(u^m), \nabla \xi) \times \nabla T_{\alpha+||\xi||_\infty}(u^m - \xi) \,.\, \chi_{\{|u^m - \xi| < \alpha\}}\, dx. \end{aligned} \tag{2.37}$$

The second term on the right hand side of the previous inequality is equal to

$$\int_{\Omega} \mathsf{a}_i(x, T_{\alpha+||\xi||_\infty}(u), \nabla\xi) \,.\, \nabla T_{\alpha+||\xi||_\infty}(u-\xi) \,.\, \chi_{\{|u-\xi|<\alpha\}} \; dx.$$

Then, since $T_\theta(u^m-\xi) \rightharpoonup T_\theta(u-\xi)$ weakly in $\mathring{W}^1_B(\Omega)$, and by (2.28), (2.32) we have

$$\sum_{i=1}^{N} \int_{\Omega} H_i^m(x,u^m,\nabla u^m) \,.\, T_\theta(u^m-\xi)\; dx \longrightarrow \sum_{i=1}^{N} \int_{\Omega} H_i(x,u,\nabla u) \,.\, T_\theta(u-\xi)\; dx, \tag{2.38}$$

and

$$\int_{\Omega} g^m(x) \,.\, T_\theta(u^m-\xi)\; dx \longrightarrow \int_{\Omega} g(x) \,.\, T_\theta(u-\xi)\; dx. \tag{2.39}$$

Combining (2.35) - (2.39) and passing to the limit as $m \longrightarrow \infty$, we have the condition 3 in definition 2.1.1.

□

Example 4. *Let Ω be a bounded domain of $\mathbb{R}^N$, $(N \geq 2)$. By Theorems 2.1.1 and 2.1.2 it exists an entropy solutions based on the Definition 2.1.1 of the following anisotropic problem $(\mathscr{P}_3)$:*

$$(\mathscr{P}_3)\begin{cases} \tilde{\mathsf{a}} \sum_{i=1}^{N} \bar{B}_i^{-1} B_i(|\nabla u|) + l(u) \,.\, \sum_{i=1}^{N} B_i(|\nabla u|) = g(x) & in \;\; \Omega \,, \\ u = 0 & on \;\; \partial\Omega \,, \end{cases}$$

with $\tilde{\mathsf{a}}$ is a positive constant, $l : \mathbb{R} \longrightarrow \mathbb{R}^+$ a positive continuous functions such as $l \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $g \in L^1(\Omega)$ and

$$B(t) \,=\, |t|^{p-2}\,\theta, \; with \; t \in \mathbb{R} \; and \; p \,>\, 1,$$

satisfying the Δ_2-condition.

Example 5. *The same problem $(\mathscr{P}_3)$ but*

$$B(t) = \frac{|t|^{p-2}\,\theta}{ln(1+|t|)},$$

with $t \neq 0$ and $B(0) = 0$ satisfying the Δ_2-condition.

2.2 Existence and Uniqueness of entropy solution in unbounded domain

2.2.1 Introduction

In this section, we focused on the study of existence and uniqueness solution to anisotropic elliptic non-linear equation, driven by low-order term and non-polynomial growth; described by n-uplet of N-function satisfying the Δ_2-condition, in Sobolev-Orlicz anisotropic space $\mathring{W}_B^1(\omega) = \overline{C^\infty(\omega)}^{\mathring{W}_B^1(\omega)}$.

To be more precise, ω is an unbounded domain of $\mathbb{R}^N$, $N \geq 2$, we study the following equation :

$$(\mathscr{P}) \begin{cases} A(u) + \sum_{i=1}^{N} b_i(x,u,\nabla u) = f(x) & \text{in } \omega \, , \\ \\ u = 0 & \text{on } \partial\omega \, . \end{cases}$$

Where, $A(u) = \sum_{i=1}^{N} (a_i(x,u,\nabla u))_{x_i}$ is a Leray-Lions operator defined from $\mathring{W}_B^1(\omega)$ into its dual, $B(\theta) = (B_1(\theta), \cdots, B_N(\theta))$ are N-uplet Orlicz functions that satisfy the Δ_2-condition, and for $i = 1, \cdots, N$, $b_i(x,u,\nabla u) : \omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ the Carathéodory functions that do not satisfy any sign condition and the growth described by the vector N-function $B(\theta)$.

In the recent studies, specifically in the case when ω is an unbounded domain; namely, without assuming any condition on the behaviour of the solutions when $|x| \to +\infty$. The existing result has been established by Brézis [43] for the semi-linear equation :

$$-\Delta u + |u|^{p_0-2} u = f(x),$$

where $x \in \mathbb{R}^N$, $p_0 > 2$, $f \in L_{1,loc}(\mathbb{R}^N)$. Karlson and Bendahmane in [22] solved the problem $(\mathscr{P})$ in the classic case such as $b(x,u,\nabla u) = \text{div}(g(u))$, with $g(u)$ has a growth like $|u|^{q-1}$ where $q \in (1, p_0 - 1)$. For more result, we refer to [28, 50].

Our goal, in this section, is to show the existence and uniqueness of entropy solution for the equations $(\mathscr{P})$; governed with growth and described by an N-uplet of N-functions satisfying the Δ_2-condition. The functions $b_i(x,u,\nabla u)$ do not satisfy any sign condition and the source f is merely integrable, within the fulfilling of anisotropic Orlicz spaces. An approximation procedure and some a priori estimates are used to solve the problem, the challenges that we had were due to behaviour of solution near infinity.

2.2.2 Existence results

In this subsection, we assume they have non-negative measurable functions ϕ, $\varphi \in L^1(\omega)$ and $\bar{a}$, $\tilde{a}$ are two positive constants such that

$$\sum_{i=1}^{N} |\, a_i(x,s,\xi)\,| \leq \tilde{a} \sum_{i=1}^{N} \bar{B}_i^{-1} B_i(|\,\xi\,|) + \varphi(x), \tag{2.40}$$

$$\sum_{i=1}^{N} \big(a_i(x,s,\xi) - a_i(x,s,\xi^{'}) \big) \,.\, (\xi_i - \xi_i^{'}) > 0, \tag{2.41}$$

$$\sum_{i=1}^{N} a_i(x,s,\xi) \,.\, \xi_i > \bar{a} \sum_{i=1}^{N} B_i(|\,\xi\,|) - \phi(x), \tag{2.42}$$

and there exists $h \in L^1(\Omega)$ and $l : \mathbb{R} \longrightarrow \mathbb{R}^+$ a positive continuous functions such that $l \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

$$\sum_{i=1}^{N} |\, b_i(x,s,\xi)\,| \leq l(s) \,.\, \sum_{i=1}^{N} B_i(|\,\xi\,|) + h(x). \tag{2.43}$$

Definition 2.2.1. *A measurable function $u : \omega \longrightarrow \mathbb{R}$ is called an entropy solution of the problem $(\mathscr{P})$ if it satisfies the following conditions :*

1/ $u \in \mathscr{T}_0^{1,B}(\omega) = \{\, u : \omega \longrightarrow \mathbb{R}$ *measurable*, $T_k(u) \in \mathring{W}_B^1(\omega)$ *for any* $k > 0\}$.

2/ $b(x,u,\nabla u) \in L^1(\omega)$.

3/ *For any* $k > 0$,

$$\int_\omega a(x,u,\nabla u) \,.\, \nabla T_k(u-\xi)\, dx + \int_\omega b(x,u,\nabla u) \,.\, T_k(u-\xi)\, dx$$
$$\leq \int_\omega f(x) \,.\, T_k(u-\xi)\, dx \qquad \forall\, \xi \in \mathring{W}_B^1(\omega) \cap L^\infty(\omega).$$

Theorem 2.2.1. *Let ω be an unbounded domain of $\mathbb{R}^N$. Under assumptions* (2.40) - (2.43)*, there exists a least one entropy solution of the problem $(\mathscr{P})$ on the sense of definition 2.2.1.*

Proof. Let $\omega(m) = \{\, x \in \omega : |\,x\,| \leq m\,\}$ and $f^m(x) = \frac{f(x)}{1+\frac{1}{m}\,|\,f(x)\,|} \cdot \chi_{\omega(m)}$.
We have $f^m \longrightarrow f$ in $L^1(\omega)$, $m \to \infty$, $|\, f^m(x)\,| \leq |\, f(x)\,|$ and $|f^m| \leq m\chi_{\omega(m)}$.

$$a^m(x,s,\xi) = (\, a_1^m(x,s,\xi), \cdots, a_N^m(x,s,\xi)\,)$$

where $a_i^m(x,s,\xi) = a_i(x, T_m(s), \xi)$ for $i = 1, \cdots, N$.

$$b^m(x,s,\xi) = T_m(\, b(x,s,\xi)\,) \,.\, \chi_{\omega(m)}$$

and for any $v \in \mathring{W}_B^1(\omega)$, we consider the following approximate equations

$$(\mathscr{P}_m): \int_\omega a(x, T_m(u^m), \nabla u^m)\, \nabla v\, dx + \int_\omega b^m(x, u^m, \nabla u^m)\, v\, dx = \int_\omega f^m\, v\, dx.$$

For the proof. See Appendix 2.2.4.

We divide our proof in six steps.

Step 1 : **A priori estimate of** $\{u^m\}$**.**

Proposition 2.2.1. *Suppose that the assumptions* (2.40) - (2.43) *hold true, and let* $(u^m)_{m\in\mathbb{N}}$ *be a solution of the approximate problem* $(\mathscr{P}_m)$*. Then, for all* $k > 0$*, there exists a constant* $c.k$ *(not depending on m), such that*

$$\int_\omega B(|\nabla T_k(u^m)|) \le c.k.$$

Proof. Taking $v = \exp(G(u^m)).T_k(u^m)$, as a test function with $G(s) = \int_0^s \frac{l(t)}{\bar{a}}\, dt$ and $\bar{a}$ is the coercivity constant, we obtain

$$\sum_{i=1}^N \int_\omega a_i^m(x, u^m, \nabla u^m).\nabla(\exp(G(u^m)).T_k(u^m))\, dx + \sum_{i=1}^N \int_\omega b_i^m(x, u^m, \nabla u^m).\exp(G(u^m)).T_k(u^m)\, dx$$
$$\le \int_\omega f^m.\exp(G(u^m)).T_k(u^m)\, dx.$$

Then,

$$\begin{aligned}
&\sum_{i=1}^N \int_\omega a_i^m(x, u^m, \nabla u^m) \exp(G(u^m))\, \nabla T_k(u^m))\, dx \\
&\quad + \sum_{i=1}^N \int_\omega a_i^m(x, u^m, \nabla u^m).\nabla u^m.\frac{l(u^m)}{\bar{a}}.\exp(G(u^m))T_k(u^m)dx \\
&\le \sum_{i=1}^N \int_\omega |b_i^m(x, u^m, \nabla u^m)|.\exp(G(u^m)).T_k(u^m)\, dx + \int_\omega f^m.\exp(G(u^m)).T_k(u^m)\, dx \\
&\le \sum_{i=1}^N \int_\omega \big[h(x) + l(u^m).B_i(\nabla u^m)\big].\exp(G(u^m)).T_k(u^m)\, dx + \int_\omega f^m.\exp(G(u^m)).T_k(u^m)\, dx \\
&\le \sum_{i=1}^N \int_\omega l(u^m).B_i(\nabla u^m).\exp(G(u^m)).T_k(u^m)\, dx + \int_\omega \big(f^m + h(x)\big).\exp(G(u^m)).T_k(u^m)\, dx,
\end{aligned}$$

so,

$$\begin{aligned}
&\sum_{i=1}^N \int_{\{\omega:|u^m|<k\}} a_i^m(x, u^m, \nabla u^m).\nabla u^m.\exp(G(u^m))\, dx \\
&\quad \le \int_\omega \big[f^m(x) + h(x) + \phi(x)\frac{l(u^m)}{\bar{a}}\big].\exp(G(u^m))\, T_k(u^m)\, dx,
\end{aligned}$$

by (2.42), we get

$$
\begin{aligned}
&\bar{a}\sum_{i=1}^{N}\int_{\{\omega:|u^m|\le k\}} B_i(\nabla u^m)\exp(G(u^m))\,dx\\
&\le \int_{\{\omega:|u^m|\le k\}} \phi(x)\exp(G(u^m))\,dx\\
&\quad+\int_{\omega}\big[\,f^m(x)+h(x)+\phi(x)\frac{l(u_m)}{\bar{a}}\,\big].\exp(G(u^m))\,T_k(u^m)\,dx,
\end{aligned}
$$

since ϕ, h and $f^m \in L^1(\omega)$, and the fact that $\exp(G(\pm\infty)) \le \exp\left(\frac{||l||_{L^1(\mathbb{R})}}{\bar{a}}\right)$, we deduce that,

$$\int_{\{\omega:|u^m|<k\}} B(\nabla T_k(u^m))\,dx \le k.c,\quad k>0.$$

Finally

$$\int_{\omega} B(\nabla T_k(u^m))\,dx \le k.c,\quad k>0.$$

□

Step 2 : Almost everywhere convergence of $\{u^m\}$.

Lemma 2.2.1. *For all u^m measurable function on ω, we have*

$$meas\{x\in\omega, |u^m|>k\}\longrightarrow 0.$$

Proof. According to Lemma 1.1.1 and Lemma 1.1.2, we have

$$
\begin{aligned}
||T_k(u^m)||_{B^*} &\le A.||\nabla T_k(u^m)||_B\\
&\le A.\varepsilon(k)\int_{\omega} B(\nabla T_k(u^m)\,dx\\
&\le c.k.\varepsilon(k), \qquad \text{for } k>1,
\end{aligned}
\tag{2.44}
$$

with $\varepsilon(k)\longrightarrow 0$ as $k\longrightarrow\infty$.

Form (2.44) we have,

$$
\begin{aligned}
B^*\left(\frac{k}{||T_k(u^m)||_{B^*}}\right)\text{meas}\{x\in\omega: |u^m|\ge k\} &\le \int_{\omega} B^*\left(\frac{T_k(u^m)}{||T_k(u^m)||_{B^*}}\right)dx\\
&\le \int_{\omega} B^*\left(\frac{k}{||T_k(u^m)||_{B^*}}\right)dx,
\end{aligned}
$$

by (2.44) again, we obtain

$$B^*\left(\frac{k}{||\,T_k(u^m)\,||_{B^*}}\right) \longrightarrow \infty \text{ as } k \longrightarrow \infty.$$

Hence,

$$\text{meas}\{\, x \in \omega : |\, u^m \,| \geq k \,\} \longrightarrow 0 \text{ as } k \longrightarrow \infty \text{ for all } m \in \mathbb{N}.$$

□

Lemma 2.2.2. *For all u^m measurable function on ω, such that*

$$T_k(u^m) \in \mathring{W}^1_B(\omega) \qquad \forall k \geq 1.$$

We have,

$$meas\{\, \omega : B(\nabla u^m) \geq r \,\} \longrightarrow 0 \; as \; r \longrightarrow \infty.$$

Proof.

$$\begin{aligned}\text{meas}\{\, x \in \omega : B(\nabla u^m) \geq 0 \,\} = &\text{meas}\{\, \{\, x \in \omega : |\, u^m \,| \geq k, \; B(\nabla u^m) \geq r \,\} \\ &\cup \{\, x \in \omega : |\, u^m \,| < k, \; B(\nabla u^m) \geq r \,\}\,\},\end{aligned}$$

if we denote

$$g(r,k) = \text{meas}\{\, x \in \omega : |\, u^m \,| \geq k, \, B(\nabla u^m) \geq r \,\},$$

we have

$$\text{meas}\{\, x \in \omega : |\, u^m \,| < k, \; B(\nabla u^m) \geq r \,\} = g(r,0) - g(r,k).$$

Then,

$$\int_{\{x \in \omega : |\, u^m \,| < k\}} B(\nabla u^m)\, dx = \int_0^\infty \big(g(r,0) - g(r,k)\big)\, dr \leq c\,.\,k, \tag{2.45}$$

with $r \longrightarrow g(r,k)$ is a decreasing map. Then,

$$\begin{aligned} g(r,0) &\leq \frac{1}{r}\int_0^r g(r,0)\, dr \\ &\leq \frac{1}{r}\int_0^r \big(g(r,0) - g(r,k)\big)\, dr + \frac{1}{r}\int_0^r g(r,k)\, dr \\ &\leq \frac{1}{r}\int_0^r \big(g(r,0) - g(r,k)\big)\, dr + g(0,k), \end{aligned} \tag{2.46}$$

combining (2.45) and (2.46), we obtain

$$g(r,0) \leq \frac{c\,.\,k}{r} + g(0,k),$$

by Lemma 1.1.1,

$$\lim_{k \to \infty} g(0,k) = 0.$$

Thus

$$g(r,0) \longrightarrow 0 \text{ as } r \longrightarrow \infty.$$

□

We have now to prove the almost everywhere convergence of $\{ u^m \}$.

$$u^m \longrightarrow u \text{ a.e in } \Omega. \tag{2.47}$$

Let $g(k) = \sup_{m \in \mathbb{N}} \operatorname{meas}\{x \in \omega : |u^m| > k\} \longrightarrow 0$ as $k \longrightarrow \infty$.
Since ω is unbounded domain in $\mathbb{R}^N$, we define η_R as

$$\eta_R(r) = \begin{cases} 1 & \text{if } \; r < R\;, \\ R+1-r & \text{if } \; R \leq r < R+1\;, \\ 0 & \text{if } \; r \geq R+1\;. \end{cases}$$

For R, $k > 0$, we have by (1.6)

$$\begin{aligned} \int_\omega B(\nabla \eta_R(|x|)\,.\,T_k(u^m))\,dx &\leq c \int_{\{x \in \omega : |u^m| < k\}} B(\nabla u^m)\,dx + c\int_\omega B(T_k(u^m)\,.\,\nabla \eta_R(|x|)\,dx \\ &\leq c(k,R), \end{aligned}$$

which implies that the sequence $\{ \eta_R(|x|)\,T_K(u^m) \}$ is bounded in $\mathring{W}^1_B(\omega(R+1))$ and by embedding Theorem, for $P \ll B$ we have

$$\mathring{W}^1_B(\omega(R+1)) \hookrightarrow L_P(\omega(R+1)),$$

and since $\eta_R = 1$ in $\omega(R)$, we have

$$\eta_R T_k(u^m) \longrightarrow v_k \text{ in } L_P(\omega(R+1)) \text{ as } m \longrightarrow \infty.$$

For $k \geq 1$,

$$T_k(u^m) \longrightarrow v_k \text{ in } L_P(\omega(R+1)) \text{ as } m \longrightarrow \infty,$$

by diagonal process, we prove that there is $u : \omega \longrightarrow \mathbb{R}$ measurable such that $u^m \longrightarrow u$ a.e in ω. This implies the (2.47).

Lemma 2.2.3. *Let an N-functions $\bar{B}(t)$ satisfy the Δ_2-condition and u^m, $m = 1, \cdots, \infty$, and u be two functions of $L_B(\omega)$ such as*

$$||u^m||_B \leq c \qquad m = 1, 2, \cdots.$$

$$u^m \longrightarrow u \text{ almost everywhere in } \omega,\ m \longrightarrow \infty.$$

Then,

$$u^m \rightharpoonup u \text{ weakly in } L_B(\omega) \text{ as } m \to \infty.$$

Proof. See Lemma 1.3 in [79]. □

Step 3 : Weak convergence of the gradient.

Since $\mathring{W}^1_B(\omega)$ reflexive, then, there exists a subsequence

$$T_k(u^m) \rightharpoonup v \text{ weakly in } \mathring{W}^1_B(\omega),\ m \to \infty.$$

And since,

$$\mathring{W}^1_B(\omega) \hookrightarrow L_B(\omega),$$

we have

$$\nabla T_k(u^m) \rightharpoonup \nabla v \text{ in } L_B(\omega) \text{ as } m \to \infty,$$

since

$$u^m \longrightarrow u \text{ a.e in } \omega \text{ as } m \to \infty,$$

we get

$$\nabla u^m \longrightarrow \nabla u \text{ a.e in } \omega \text{ as } m \to \infty.$$

Then, we obtain for any fixed $k > 0$,

$$\nabla T_k(u^m) \longrightarrow \nabla T_k(u) \text{ a.e in } \omega.$$

Applying Lemma 2.2.3, we have the following weak convergence

$$\nabla T_k(u^m) \rightharpoonup \nabla T_k(u) \text{ in } L_B(\omega) \text{ as } m \to \infty,$$

for more detail see page 11 in [26].

Step 4 : Strong convergence of the gradient.

For $j > k > 0$, we introduce the following function defined as

$$h_j(s) = \begin{cases} 1 & \text{if } \ |s| \le j \ , \\ 1 - |s-j| & \text{if } \ j \le |s| \le j+1 \ , \\ 0 & \text{if } \ s \ge j+1 \ . \end{cases}$$

and we show that the following assertions are true :

Assertion 1 :

$$\lim_{j\to\infty} \lim_{m\to\infty} \sum_{i=1}^{N} \int_{\{ j \le |u^m| \le j+1 \}} a_i^m(x, u^m, \nabla u^m) \,.\, \nabla u^m \,.\, \eta_R(|x|)\, dx = 0. \tag{2.48}$$

Assertion 2 :

$$\nabla u^m \longrightarrow \nabla u \text{ a.e in } \omega(m). \tag{2.49}$$

Proof. We take $v = \exp(G(u^m))\, T_{1,j}(u^m)\, \eta_R(|x|) = \exp(G(u^m))\, T_1(u^m - T_j(u^m))\, \eta_R(|x|)$ as a test function in the problem $(\mathscr{P}_m)$, we obtain

$$\begin{aligned}\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u^m,\nabla u^m)\,.\,\nabla\Big(\exp(G(u^m))\,.\,T_1(u^m-T_j(u^m))\,.\,\eta_R(|x|)\Big)\,dx\\ \leq\sum_{i=1}^{N}\int_{\omega}|b_i^m(x,u^m,\nabla u^m)|\,.\,\exp(G(u^m))\,.\,T_1(u^m-T_j(u^m))\,.\,\eta_R(|x|)\,dx\\ +\int_{\omega} f^m(x)\,.\,\exp(G(u^m))\,.\,T_1(u^m-T_j(u^m))\,.\,\eta_R(|x|)\,dx,\end{aligned}$$

according to (2.42) and (2.43) we deduce that

$$\begin{aligned}\sum_{i=1}^{N}\int_{\{j<|u^m|<j+1\}} a_i^m(x,u^m,\nabla u^m)\,.\,\nabla u^m\,.\,\exp(G(u^m))\,.\,\eta_R(|x|)\,dx\\ \leq\int_{\omega}\left[f^m(x)+h(x)+\phi(x)\,.\,\frac{l(u^m)}{\bar{a}}\right].\exp(G(u^m)).T_1(u^m-T_j(u^m))\,.\,\eta_R(|x|)\,dx,\end{aligned}$$

since $\phi\in L^1(\omega)$, $h\in L^1(\omega)$, $f^m\in(L^1(\omega))^N$, and the fact that $\exp(G(\pm\infty))\leq\exp\left(\frac{\|l\|_{L^1(\mathbb{R})}}{\bar{a}}\right)$, we deduce from vitali's Theorem that

$$\lim_{j\to\infty}\lim_{m\to\infty}\int_{\omega}\left[f^m(x)+h(x)+\phi(x)\,.\,\frac{l(u^m)}{\bar{a}}\right].\exp(G(u^m))\,.\,T_1(u^m-T_j(u^m))\,.\,\eta_R(|x|)\,dx=0.$$

Hence,

$$\lim_{j\to\infty}\lim_{m\to\infty}\int_{\{j<|u^m|<j+1\}} a_i^m(x,u^m,\nabla u^m)\,.\,\nabla u^m\,.\,\eta_R(|x|)\,dx=0.$$

And to show that assertion 2 is true, we take

$$v=\exp(G(u^m))\,(T_k(u^m)-T_k(u))\,h_j(u^m)\,\eta_R(|x|),$$

as a test function in the problem $(\mathscr{P}_m)$. We have

$$\begin{aligned}\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u^m,\nabla u^m)\,.\,\nabla\big(\exp(G(u^m))\,.\,(T_k(u^m)-T_k(u))\,.\,h_j(u^m)\,.\,\eta_R(|x|)\big)\,dx\\ +\sum_{i=1}^{N}\int_{\omega} b_i^m(x,u^m,\nabla u^m)\,.\,\exp(G(u^m))\,.\,(T_k(u^m)-T_k(u))\,.\,h_j(u^m)\,.\,\eta_R(|x|)\,dx\\ \leq\int_{\omega} f^m(x)\,.\,\exp(G(u^m))\,.\,(T_k(u^m)-T_k(u))\,.\,h_j(u^m)\,.\,\eta_R(|x|)\,dx,\end{aligned}$$

which implies

$$\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u^m,\nabla u^m)\,.\,\nabla u^m\,.\,\frac{l(u^m)}{\bar{a}}\,.\,\exp(G(u^m))\,.\,(T_k(u^m)-T_k(u))\,.\,h_j(u^m)\times\eta_R(|x|)\,dx$$
$$+\sum_{i=1}^{N} a_i^m(x,u^m,\nabla u^m)\,.\,\exp(G(u^m))\,.\,(\nabla T_k(u^m)-\nabla T_k(u))\,.\,h_j(u^m)\,.\,\eta_R(|x|)\,dx$$
$$+\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u^m,\nabla u^m)\,.\,\exp(G(u^m))\,.\,(T_k(u^m)-T_k(u))\,.\,\nabla h_j(u^m)\,.\,\eta_R(|x|)\,dx$$
$$+\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u^m,\nabla u^m)\,.\,\exp(G(u^m))\,.\,(T_k(u^m)-T_k(u))\,.\,h_j(u^m)\,.\,\nabla\eta_R(|x|)\,dx$$
$$\leq\sum_{i=1}^{N}\int_{\omega}|b_i^m(x,u^m,\nabla u^m)|\,.\,\exp(G(u^m))\,.\,(T_k(u^m)-T_k(u))\,.\,h_j(u^m)\,.\,\eta_R(|x|)\,dx$$
$$+\int_{\omega} f^m(x)\,.\,\exp(G(u^m))\,.\,(T_k(u^m)-T_k(u))\,.\,h_j(u^m)\,.\,\eta_R(|x|)\,dx,$$

thanks to (2.42) and (2.43), we obtain

$$\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u^m,\nabla u^m)\,.\,\exp(G(u^m))\,.\,(\nabla T_k(u^m)-\nabla T_k(u))\,.\,h_j(u^m)\,.\,\eta_R(|x|)\,dx$$
$$+\sum_{i=1}^{N}\int_{\{\omega:\, j\leq|u^m|\leq j+1\}} a_i^m(x,u^m,\nabla u^m)\,.\,\nabla u^m\,.\,\exp(G(u^m))\times(T_k(u^m)-T_k(u))\,.\,\eta_R(|x|)\,dx$$
$$+\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u^m,\nabla u^m)\,.\,\exp(G(u^m))\,.\,(T_k(u^m)-T_k(u))\,.\,h_j(u^m)\times\nabla\eta_R(|x|)\,dx$$
$$\leq\int_{\omega}\left[f^m(x)+h(x)+\phi(x)\,.\,\frac{l(u^m)}{\bar{a}}\right].\,\exp(G(u^m))\,.\,(T_k(u^m)-T_k(u))\,.\,h_j(u^m)\times\eta_R(|x|)\,dx$$

sine $h_j\geq 0$, $\eta_R(|x|)\geq 0$ and $u^m\,(T_k(u^m)-T_k(u))\geq 0$ we have

$$\sum_{i=1}^{N}\int_{\{\omega:\,|u^m|\leq k\}} a_i(x,T_k(u^m),\nabla T_k(u^m))\exp(G(u^m))\,.\,(\nabla T_k(u^m)-\nabla T_k(u))\times\eta_R(|x|)\,dx$$
$$+\int_{\{\omega:\, j\leq|u^m|\leq j+1\}} a_i^m(x,u^m,\nabla u^m)\,\nabla u^m\,\exp(G(u^m))\,(T_k(u^m)-T_k(u))\,\eta_R(|x|)\,dx$$
$$+\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u^m,\nabla u^m)\,.\,\exp(G(u^m))\,.\,(T_k(u^m)-T_k(u))\,.\,\nabla\eta_R(|x|)\,dx$$
$$\leq\int_{\omega}\left[f^m(x)+h(x)+\phi(x)\,.\,\frac{l(u^m)}{\bar{a}}\right].\,\exp(G(u^m))\,.\,(T_k(u^m)-T_k(u))\,.\,\eta_R(|x|)\,dx$$
$$+\sum_{i=1}^{N}\int_{\{\omega:\, k\leq|u^m|\leq j+1\}} a_i(x,T_{j+1}(u^m),\nabla T_{j+1}(u^m))\,.\,\exp(G(u^m))\,.\,|\nabla T_k(u)|\times\eta_R(|x|)\,dx$$
$$+\sum_{i=1}^{N}\int_{\{\omega:\, j\leq|u^m|\leq j+1\}} a_i^m(x,u^m,\nabla u^m)\,.\,\nabla u^m\,.\,\exp(G(u^m))\,.\,|T_k(u^m)-T_k(u)|\times\eta_R(|x|)\,dx.$$

The first term in the right hand side goes to zero as m tend to ∞, since $T_k(u^m)\rightharpoonup T_k(u)$ weakly in

$\mathring{W}_B^1(\omega(m))$.

Since $a_i^m(x, T_{j+1}(u^m), \nabla T_{j+1}(u^m))$ is bounded in $L_{\bar{B}}(\omega(m))$, there exists $\tilde{a}^m \in L_{\bar{B}}(\omega(m))$ such as

$$|\, a_i^m(x, T_{j+1}(u^m), \nabla T_{j+1}(u^m))\, | \rightharpoonup \tilde{a}^m \text{ in } L_{\bar{B}}(\omega(m)). \tag{2.50}$$

Thus, the second term of the right hand side goes also to zero.

Since $T_k(u^m) \longrightarrow T_K(u)$ strongly in $\mathring{W}_{B,loc}^1(\omega(m))$. The third term of the left hand side increased by a quantity that tends to zero as m tend to zero, and according to (2.48) we deduce that

$$\sum_{i=1}^N \int_{\{\omega:|u^m|\le k\}} a_i(x, T_k(u^m), \nabla T_k(u^m)) \,.\, \exp(G(u^m)) \,.\, |\,\nabla T_k(u^m) - \nabla T_k(u)\,| \times \eta_R(|x|)\, dx$$
$$\le \varepsilon(j, m).$$

Then,

$$\sum_{i=1}^N \int_\omega \Big[a_i(x, T_k(u^m), \nabla T_k(u^m)) - a_i(x, T_k(u^m), \nabla T_k(u)) \Big] \,.\, (\,\nabla T_k(u^m) - T_K(u)\,) \times \eta_R(|x|)\, dx$$
$$\le -\sum_{i=1}^N \int_\omega a_i(x, T_k(u^m), \nabla T_k(u)) \,.\, \exp(G(u^m)) \,.\, |\,\nabla T_k(u^m) - \nabla T_k(u)\,| \times \eta_R(|x|)\, dx$$
$$- \sum_{i=1}^N \int_{\{\omega:|u^m|\le k\}} a_i(x, T_k(u^m), \nabla T_k(u^m)) \,.\, \exp(G(u^m)) \,.\, \nabla T_k(u) \,.\, \eta_R(|x|)\, dx + \varepsilon(j, m). \tag{2.51}$$

According to Lebesgue dominated convergence Theorem, we have $T_k(u^m) \longrightarrow T_k(u)$ in $\mathring{W}_{B,loc}^1(\omega)$ and $\nabla T_k(u^m) \rightharpoonup \nabla T_k(u)$ in $\mathring{W}_B^1(\omega)$, then the terms on the right had side of (2.51) go to zero as m and j tend to infinity.

Which implies that

$$\sum_{i=1}^N \int_\omega \Big[a_i(x, T_k(u^m), \nabla T_k(u^m)) - a_i(x, T_k(u^m), \nabla T_k(u)) \Big] \times (\,\nabla T_k(u^m) - T_K(u)\,)\, dx \longrightarrow 0. \tag{2.52}$$

Thanks to Lemma 2.1.1, we have for $k \ge 1$,

$$\nabla T_k(u^m) \longrightarrow \nabla T_k(u) \text{ a.e in } \omega(m), \tag{2.53}$$

and by diagonal process, we prove that

$$\nabla u^m \longrightarrow \nabla u \text{ a.e in } \Omega(m).$$

□

Step 5 : Equi-integrability of $b^m(x,u^m,\nabla u^m)$.

Let $v=\exp(2G(|u^m|)).T_1(u^m-T_R(u^m)).\eta_R(|x|)$ as a test function in the problem $(\mathscr{P}_m)$, we obtain

$$\begin{aligned}&\sum_{i=1}^N\int_\omega a_i^m(x,u^m,\nabla u^m).\nabla\big(\exp(2G(|u^m|)).T_1(u^m-T_R(u^m)).\eta_R(|x|)\big)\,dx\\&\quad+\sum_{i=1}^N\int_\omega b_i^m(x,u^m,\nabla u^m).\exp(2G(|u^m|)).T_1(u^m-T_R(u^m)).\eta_R(|x|)\,dx\\&\leq\int_\omega f^m(x).\exp(2G(|u^m|)).T_1(u^m-T_R(u^m)).\eta_R(|x|)\,dx,\end{aligned}$$

which implies that

$$\begin{aligned}&\sum_{i=1}^N\int_\omega a_i^m(x,u^m,\nabla u^m).\nabla u^m.\frac{l(u^m)}{\bar a}.\exp(2G(|u^m|)).T_1(u^m-T_R(u^m))\times\eta_R(|x|)\,dx\\&+\sum_{i=1}^N\int_{\{\omega:R\leq|u^m|\leq R+1\}}a_i^m(x,u^m,\nabla u^m).\nabla u^m.\exp(2G(|u^m|)).\eta_R(|x|)\,dx\\&+\sum_{i=1}^N\int_\omega a_i^m(x,u^m,\nabla u^m).\exp(2G(|u^m|).T_1(u^m-T_R(u^m)).\nabla\eta_R(|x|)\,dx\\&\leq\sum_{i=1}^N\int_\omega|b_i^m(x,u^m,\nabla u^m)|.\exp(2G(|u^m|)).T_1(u^m-T_R(u^m)).\eta_R(|x|)\,dx\\&\quad+\int_\omega f^m(x).\exp(2G(|u^m|)).T_1(u^m-T_R(u^m)).\eta_R(|x|)\,dx,\end{aligned}$$

by (2.42) and (2.43), we obtain

$$\begin{aligned}&\bar a\sum_{i=1}^N\int_{\{\omega:R\leq|u^m|\leq R+1\}}B_i(|\nabla u^m|).\exp(2G(|u^m|).\eta_R(|x|)\,dx\\&\quad+\sum_{i=1}^N\int_\omega a_i^m(x,u^m,\nabla u^m).\exp(2G(|u^m|).T_1(u^m-T_R(u^m)).\nabla\eta_R(|x|)\,dx\\&\leq\int_\omega\left[f^m(x)+h(x)+\phi(x).\frac{l(u^m)}{\bar a}\right].\exp(2G(|u^m|)).T_1(u^m-T_R(u^m))\\&\quad\times\eta_R(|x|)\,dx+\int_{\{\omega:R\leq|u^m|\leq R+1\}}\phi(x).\exp(2G(|u^m|).\eta_R(|x|)\,dx.\end{aligned}$$

Since $\eta_R(|x|)\geq 0$, $\exp(G(\pm\infty))\leq\exp\left(2\frac{||l||_{L^1(\mathbb{R})}}{\bar a}\right)$, $f^m\in(L^1(\omega))^N$, ϕ and $h\in L^1(\omega)$. Then, $\forall\varepsilon>0$, $\exists R(\varepsilon)>0$ such as

$$\sum_{i=1}^N\int_{\{\omega:|u^m|>R+1\}}B(|\nabla u^m|)\,dx\leq\frac{\varepsilon}{2},\qquad\forall R>R(\varepsilon).$$

Let $\mathring{V}(\omega(m))$ be an arbitrary bounded subset for ω. Then, for any measurable set $E \subset \mathring{V}(\omega(m))$ we have

$$\sum_{i=1}^{N}\int_E B_i(|\nabla u^m|)\,dx \leq \sum_{i=1}^{N}\int_E B_i(|\nabla T_R(u^m)|)\,dx + \sum_{i=1}^{N}\int_{\{|u^m|>R+1\}} B_i(|\nabla u^m|)\,dx \tag{2.54}$$

we conclude that $\forall E \subset \mathring{V}(\omega(m))$ with $\text{meas}\,(E) < \beta(\varepsilon)$ and $T_R(u^m) \longrightarrow T_R(u)$ in $\mathring{W}^1_B(\omega)$

$$\sum_{i=1}^{N}\int_E B_i(|\nabla T_R(u^m)|)\,dx \leq \frac{\varepsilon}{2}. \tag{2.55}$$

Finally, according to (2.54) and (2.55), we obtain

$$\sum_{i=1}^{N}\int_E B_i(|\nabla u^m|)\,dx \leq \varepsilon \qquad \forall E \subset \mathring{V}(\omega(m)) \text{ such as } \text{meas}\,(E) < \beta(\varepsilon).$$

Which gives the results.

Step 6 : Passing to the limit.

Let $\xi \in \mathring{W}^1_B(\omega) \cap L^\infty(\omega)$, using the following test function $v = \vartheta_k\, T_k(u^m - \xi)$ in the problem $(\mathscr{P}_m)$ with

$$\vartheta_k = \begin{cases} 1 & \text{for } \ \omega(m)\ , \\ 0 & \text{for } \ \omega(m+1)\backslash\omega(m)\ , \end{cases}$$

and $|u^m| - ||\xi||_\infty < |u^m - \xi| \leq j$. Then, $\{|u^m - \xi| \leq j\} \subset \{|u^m| \leq j + ||\xi||_\infty\}$ we obtain

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\omega a_i(x, T_m(u^m), \nabla u^m)\,.\,\vartheta_k\,\nabla T_k(u^m - \xi)\,dx \\
&\quad + \sum_{i=1}^{N}\int_\omega a_i(x, T_m(u^m), \nabla u^m)\,.\,T_k(u^m - \xi)\,\nabla\vartheta_k\,dx \\
&\quad + \sum_{i=1}^{N}\int_\omega b_i^m(x, u^m, \nabla u^m)\,.\,\vartheta_k\,T_k(u^m - \xi)\,dx \\
&\leq \int_\omega f^m(x)\,.\,\vartheta_k\,T_k(u^m - \xi)\,dx,
\end{aligned} \tag{2.56}$$

which implies that

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\omega(m)} a_i(x,T_m(u^m),\nabla u^m)\,.\,T_k(u^m-\xi)\;dx\\
&=\sum_{i=1}^{N}\int_{\omega(m)} a_i(x,T_{j+\|\xi\|_\infty}(u^m),\nabla T_{j+\|\xi\|_\infty}(u^m))\,.\,T_{j+\|\xi\|_\infty}(u^m-\xi)\,.\,\chi_{\{|u^m-\xi|<j\}}dx\\
&=\sum_{i=1}^{N}\int_{\omega(m)} \Big[a_i(x,T_{j+\|\xi\|_\infty}(u^m),\nabla T_{j+\|\xi\|_\infty}(u^m))-a_i(x,T_{j+\|\xi\|_\infty}(u^m),\nabla\xi)\Big]\\
&\qquad\qquad\times\nabla T_{j+\|\xi\|_\infty}(u^m-\xi)\,.\,\chi_{\{|u^m-\xi|<j\}}\;dx\\
&\quad+\sum_{i=1}^{N}\int_{\omega(m)} a_i(x,T_{j+\|\xi\|_\infty}(u^m),\nabla\xi)\,.\,\nabla T_{j+\|\xi\|_\infty}(u^m-\xi)\,.\,\chi_{\{|u^m-\xi|<j\}}\;dx.
\end{aligned} \tag{2.57}$$

By Fatou's Lemma, we have

$$\begin{aligned}
&\liminf_{m\to\infty}\sum_{i=1}^{N}\int_{\omega(m)} a_i(x,T_m(u^m),\nabla u^m)\,.\,\nabla T_k(u^m-\xi)\;dx\\
&\geq\sum_{i=1}^{N}\int_{\omega(m)} \Big[a_i(x,T_{j+\|\xi\|_\infty}(u^m),\nabla T_{j+\|\xi\|_\infty}(u^m))-a_i(x,T_{j+\|\xi\|_\infty}(u^m),\nabla\xi)\Big]\\
&\qquad\qquad\times\nabla T_{j+\|\xi\|_\infty}(u^m-\xi)\,.\,\chi_{\{|u^m-\xi|<j\}}\;dx\\
&\quad+\lim_{m\to\infty}\sum_{i=1}^{N}\int_{\omega(m)} a_i(x,T_{j+\|\xi\|_\infty}(u^m),\nabla\xi).\nabla T_{j+\|\xi\|_\infty}(u^m-\xi)\,.\,\chi_{\{|u^m-\xi|<j\}}\;dx.
\end{aligned} \tag{2.58}$$

The second term on the right hand side of the previous inequality is equal to

$$\int_{\omega(m)} a_i(x,T_{j+\|\xi\|_\infty}(u),\nabla\xi)\,.\,\nabla T_{j+\|\xi\|_\infty}(u-\xi)\,.\,\chi_{\{|u-\xi|<j\}}\;dx.$$

Then, since $T_k(u^m-\xi)\rightharpoonup T_k(u-\xi)$ weakly in $\mathring{W}^1_B(\omega)$, and by (2.49), (2.53) we have

$$\sum_{i=1}^{N}\int_{\omega} b_i^m(x,u^m,\nabla u^m)\,.\,\vartheta_k\,T_k(u^m-\xi)\;dx\longrightarrow\sum_{i=1}^{N}\int_{\omega} b_i(x,u,\nabla u)\,.\,\vartheta_k\,T_k(u-\xi)\;dx, \tag{2.59}$$

and

$$\int_{\omega} f^m(x)\,.\,\vartheta_k\,T_k(u^m-\xi)\;dx\longrightarrow\int_{\omega} f(x)\,.\,\vartheta_k\,T_k(u-\xi)\;dx. \tag{2.60}$$

Combining (2.56) - (2.60) and passing to the limit as $m\longrightarrow\infty$, we have the condition 3 in definition 2.2.1. □

2.2.3 Uniqueness result in unbounded domain

In this subsection, we demonstrate the Theorem of uniqueness solution to the problem $(\mathscr{P})$ in an unbounded domain; using the fact given in [3, 32, 33] such as $b_i(x,u,\nabla u)$ are a contraction Lipschitz continuous functions.

Theorem 2.2.2. *Under assumptions* (2.40) - (2.43)*, and* $b_i(x,u,\nabla u) : \omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ *for* $i = 1,\cdots,N$ *contraction Lipschitz continuous functions do not satisfy any sign condition, and*

$$\sum_{i=1}^{N} \left[a_i(x,\xi,\nabla\xi) - a_i(x,\xi',\nabla\xi') \right] . (\nabla\xi - \nabla\xi') > 0. \tag{2.61}$$

The problem $(\mathscr{P})$ *has a unique solution.*

Proof. Let u^1 and u^2 be two solutions of problem $(\mathscr{P})$ with $u^1 \neq u^2$ then,

$$\sum_{i=1}^{N} \int_\omega a_i(x,u^1,\nabla u^1) . \nabla v \, dx + \sum_{i=1}^{N} \int_\omega b_i(x,u^1,\nabla u^1) . v \, dx = \int_\omega f(x) . v \, dx,$$

and

$$\sum_{i=1}^{N} \int_\omega a_i(x,u^2,\nabla u^2) . \nabla v \, dx + \sum_{i=1}^{N} \int_\omega b_i(x,u^2,\nabla u^2) . v \, dx = \int_\omega f(x) . v \, dx,$$

we subtract the previous inequality, we get

$$\begin{aligned} &\sum_{i=1}^{N} \int_\omega \left[a_i(x,u^1,\nabla u^1) - a_i(x,u^2,\nabla u^2) \right] . \nabla v \, dx \\ &+ \sum_{i=1}^{N} \int_\omega \left[b_i(x,u^1,\nabla u^1) - b_i(x,u^2,\nabla u^2) \right] . v \, dx = 0, \end{aligned}$$

we take $v = \eta(x) . (u^1 - u^2)(x)$ with

$$\eta(x) = \begin{cases} 0 & \text{if } \ x \geq k \ , \\ k - \frac{|x|^2}{k} & \text{if } \ |x| < k \ , \\ 0 & \text{if } \ x \leq -k \ . \end{cases}$$

Combine to (2.61), we obtain

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\omega}\left[a_i(x,u^1,\nabla u^1)-a_i(x,u^2,\nabla u^2)\right].(u^1-u^2).\nabla\eta(x)\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\omega}\left[b_i(x,u^1,\nabla u^1)-b_i(x,u^2,\nabla u^2)\right].(u^1-u^2).\eta(x)\,dx\\
&\leq 0,
\end{aligned}$$

according to (1.2) and the fact that $b_i(x,u,\nabla u)$ contraction Lipschitz functions for $i=1,\cdots,N$, we get

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\omega}\bar{B}_i\big(a_i(x,u^1,\nabla u^1)-a_i(x,u^2,\nabla u^2)\big)\,dx+\sum_{i=1}^{N}\int_{\omega}B_i(u^1-u^2)\,\nabla\eta(x))\,dx\\
&\leq\sum_{i=1}^{N}\int_{\omega}\bar{B}_i\big(\,a_i(x,u^1,\nabla u^1)-a_i(x,u^2,\nabla u^2)\big)\,dx+2\sum_{i=1}^{N}\int_{\omega}B_i(u^1-u^2)\,dx\\
&\leq\alpha\sum_{i=1}^{N}\int_{\omega}B_i(u^1-u^2)\,dx+\alpha\sum_{i=1}^{N}\int_{\omega}\bar{B}_i(\eta(x).(u^1-u^2))\,dx,
\end{aligned}\tag{2.62}$$

then

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\omega}\bar{B}_i\big(\,a_i(x,u^1,\nabla u^1)-a_i(x,u^2,\nabla u^2)\big)\,dx\\
&\leq(\alpha-2)\sum_{i=1}^{N}\int_{\omega}B_i(u^1-u^2)\,dx+\alpha\sum_{i=1}^{N}\int_{\omega}\bar{B}_i(\eta(x).(u^1-u^2))\,dx.
\end{aligned}\tag{2.63}$$

Since,

$$\begin{aligned}
\sum_{i=1}^{N}\int_{\omega}\bar{B}_i(\eta(x).(u^1-u^2))\,dx&\leq\sum_{i=1}^{N}\int_{\omega\cap\{|x|\leq k\}}\bar{B}_i\bigg(\bigg(k-\frac{|x|^2}{k}\bigg).(u^1-u^2)\bigg)\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\omega\cap\{|x|>k\}}\bar{B}_i(\,\eta(x).(u^1-u^2))\,dx\\
&\longrightarrow 0 \text{ as } k\longrightarrow 0,
\end{aligned}$$

and since the N-functions $\bar{B}_i$ verified the same conditions and properties of the B_i then, according to (1.6) and (2.40), we obtain

$$\begin{aligned}
\sum_{i=1}^{N}\int_{\omega}\bar{B}_i\big(\,a_i(x,u^1,\nabla u^1)-a_i(x,u^2,\nabla u^2)\big)\,dx&\leq\tilde{a}c\sum_{i=1}^{N}\int_{\omega}B_i(\nabla(u^2-u^2))\,dx\\
&\leq\tilde{a}c\,||\,B(u^1-u^2)\,||_{1,\omega}.
\end{aligned}$$

Combine to (2.62) and (2.63), we deduce that

$$0 \leq (\tilde{a}c+2-\alpha)\,||\,B(u^1-u^2)\,||_{1,\omega} \leq 0.$$

Thus

$$||\,B(u^1-u^2)\,||_{1,\omega} = 0.$$

Hence, $u^1 = u^2$ a.e in ω. □

2.2.4 appendix

Let

$$\begin{aligned} A^m : \mathring{W}^1_B(\omega) &\longrightarrow (\mathring{W}^1_B(\omega))' \\ v \quad &\longmapsto \langle A^m(u), v\rangle = \int_\omega \sum_{i=1}^{N} \left(a_i^m(x,u,\nabla u)\,.\,\frac{\partial v}{\partial x_i} + b_i^m(x,u,\nabla u)\,.\,v \right) dx - \int_\omega f^m(x)\,.\,v\,dx, \end{aligned}$$

and let denote $L_{\bar{B}}(\omega) = \prod_{k=1}^{N} L_{\bar{B}_i}(\omega)$ with the norm

$$||\,v\,||_{L_{\bar{B}}(\omega)} = \sum_{i=1}^{N} ||\,v_i\,||_{\bar{B}_i,\omega} \qquad v = (v_1,\cdots,v_N) \in L_{\bar{B}}(\omega).$$

Where $\bar{B}_i(t)$ are N-functions satisfying the Δ_2-conditions.

Sobolev-space $\mathring{W}^1_B(\omega)$ is the completions of the space $C_0^\infty(\omega)$.

$$a^m(x,s,\xi) = \big(a_1^m(x,s,\xi),\cdots,a_N^m(x,s,\xi)\big),$$

and

$$b^m(x,s,\xi) = \big(b_1^m(x,s,\xi),\cdots,b_N^m(x,s,\xi)\big).$$

Let's show that operator A is bounded, so for $u \in \mathring{W}^1_B(\omega)$, according to (1.9) and (2.40) we get

$$\begin{aligned} ||\,a^m(x,u,\nabla u)\,||_{L_{\bar{B}}(\omega)} &= \sum_{i=1}^{N} ||\,a_i^m(x,u,\nabla u)\,||_{L_{\bar{B}_i}(\omega)} \\ &\leq \sum_{i=1}^{N} \int_\omega \bar{B}_i(\,a_i^m(x,u,\nabla u)\,)\,dx + N \\ &\leq \tilde{a}(\omega)\,.\,||\,B(u)\,||_{1,\omega} + ||\,\varphi||_{1,\omega} + N. \end{aligned} \tag{2.64}$$

Further, for $a^m(x,u,\nabla u) \in L_{\bar{B}_i}(\omega),\ v \in \mathring{W}^1_B(\omega)$ using Hölder's inequality we have

$$|\langle A^m(u), v\rangle_\omega| \leq 2\,||a^m(x,u,\nabla u)||_{L_{\bar{B}}(\omega)}\,.\,||v||_{\mathring{W}^1_B(\omega)} + 2\,||b^m(x,u,\nabla u)||_{L_B(\omega)}\,.\,||v||_{\mathring{W}^1_B(\omega)} + c_0\,.\,||v||_{\mathring{W}^1_B(\omega)}. \tag{2.65}$$

Thus, A^m is bounded.

And that A^m is coercive, so for $u \in \mathring{W}^1_B(\omega)$

$$\langle A^m(u), u\rangle_\omega = \sum_{i=1}^{N}\int_\omega a_i^m(x,u,\nabla u)\,.\,\frac{\partial u}{\partial x_i}\,dx + \sum_{i=1}^{N}\int_\omega b_i^m(x,u,\nabla u)\,.\,u\,dx - \int_\omega f^m(x)\,.\,u\,dx.$$

Then,

$$\begin{aligned}\frac{\langle A^m(u), u\rangle_\omega}{||u||_{\mathring{W}^1_B(\omega)}} &\geq \frac{1}{||u||_{\mathring{W}^1_B(\omega)}}\,.\,\Bigg[\bar{a}\sum_{i=1}^{N}\int_\omega B_i\bigg(\bigg|\frac{\partial u}{\partial x_i}\bigg|\bigg)\,dx - c_1 - c_0 \\ &\qquad - l(u)\,.\,\sum_{i=1}^{N}\int_\omega B_i\bigg(\bigg|\frac{\partial u}{\partial x_i}\bigg|\bigg)\,dx - \int_\omega h(x)\,dx\Bigg] \\ &\geq \frac{1}{||u||_{\mathring{W}^1_B(\omega)}}\,.\,\Bigg[(\bar{a}(\omega) - c_2)\,.\,\sum_{i=1}^{N}\int_\omega B_i\bigg(\bigg|\frac{\partial u}{\partial x_i}\bigg|\bigg)\,dx - c_0 - c_1 - c_3\Bigg].\end{aligned}$$

According to (2.40), we have for all $k > 0$, $\exists\,\alpha_0 > 0$ such that

$$b_i(|u_{x_i}|) > k\,b_i\bigg(\frac{|u_{x_i}|}{||u_{x_i}||_{B_i,\omega}}\bigg), \qquad i = 1,\cdots,N.$$

We take $||u_{x_i}||_{B_i,\omega} > \alpha_0 \quad i = 1,\cdots,N.$

Suppose that $||u_{x_i}||_{\mathring{W}^1_B(\omega)} \longrightarrow 0$ as $j \to \infty$. We can assume that

$$||u^j_{x_1}||_{B_1,\omega} + \cdots + ||u^j_{x_N}||_{B_N,\omega} \geq N\,\alpha_0.$$

According to (1.9) for $c > 1$, we have

$$|u^j|\,b(|u^j|) < c\,B(u^j),$$

then, by (1.8) we obtain

$$\begin{aligned}
\frac{\langle A^m(u^j), u^j\rangle_\omega}{||\,u^j\,||_{\mathring{W}^1_B(\omega)}} &\geq \frac{\bar{a}(\omega)-c_2}{N\,\alpha_0}\cdot\sum_{i=1}^{N}\int_\omega B_i\Big(\Big|\,\frac{\partial u}{\partial x_i}\Big|\Big)\,dx - \frac{c_4}{N\,\alpha_0}\\
&\geq \frac{\bar{a}(\omega)-c_2}{N\,\alpha_0}\cdot\sum_{i=1}^{N}\int_\omega |\,u^j_{x_i}\,|\,b(\,|\,u^j_{x_i}\,|\,)\,dx - \frac{c_4}{N\,\alpha_0}\\
&\geq \frac{(\,\bar{a}(\omega)-c_2\,).k}{cN\,||\,u^j_{x_i}\,||_{B_i}}\cdot\sum_{i=1}^{N}\int_\omega |\,u^j_{x_i}\,|\,b_i\Big(\frac{|\,u^j_{x_i}\,|}{||\,u^j_{x_i}\,||_{B_i,\omega}}\Big)\,dx - \frac{c_4}{N\,\alpha_0}\\
&\geq \frac{(\,\bar{a}(\omega)-c_2\,).k}{cN}\cdot\sum_{i=1}^{N}\int_\omega B_i\Big(\frac{|\,u^j_{x_i}\,|}{||\,u^j_{x_i}\,||_{B_i,\omega}}\Big)\,dx - \frac{c_4}{N\,\alpha_0}\\
&\geq \frac{(\,\bar{a}(\omega)-c_2\,).k}{cN} - \frac{c_4}{N\,\alpha_0}.
\end{aligned}$$

Which shows that A^m is coercive, because k is arbitrary.

And for A^m pseudo-monotonic, we consider a sequence $\{\,u^m\,\}_{m=1}^{\infty}$ in the space $\mathring{W}^1_B(\omega)$ such that

$$u^m \rightharpoonup u \text{ weakly in } \mathring{W}^1_B(\omega) \qquad m\to\infty. \tag{2.66}$$

$$\limsup_{m\to\infty}\langle A^m(u^m),\, u^m-u\rangle \leq 0 \tag{2.67}$$

we demonstrate that

$$A^m(u^m) \rightharpoonup A^m(u) \text{ weakly in } (\,\mathring{W}^1_B(\omega)\,)',\; m\to\infty. \tag{2.68}$$

$$\langle A^m(u^m),\, u^m-u\rangle \longrightarrow 0,\; m\to\infty. \tag{2.69}$$

Since $B(\theta)$ satisfy the Δ_2-condition, then by (1.9) we have

$$\int_\omega B(\theta)\,dx \leq c_0\,||\,\theta\,||_{B,\omega}. \tag{2.70}$$

According to (2.66) we get

$$||\,u^m\,||_{\mathring{W}^1_B(\omega)} \leq c_1, \qquad m=1,2,\cdots \tag{2.71}$$

and

$$||\,B(\nabla u^m)\,||_1 \leq c_2, \qquad m=1,2,\cdots. \tag{2.72}$$

Combining to (2.64) and (2.71) we obtain

$$||a^m(x,u,\nabla u)||_{\bar{B}} = \sum_{i=1}^{N} ||a_i^m(x,u^m,\nabla u^m)||_{\bar{B}_i} \leq c_3 \; m = 1,2,\cdots. \tag{2.73}$$

And for $m \in \mathbb{N}^*$, $|b^m(x,u,\nabla u)| = |T_m(b(x,u,\nabla u)| \leq m$. Then, by (2.43) and (2.71) we have

$$||b^m(x,u,\nabla u)||_B = \sum_{i=1}^{N} ||b_i^m(x,u^m,\nabla u^m)||_{B_i} \leq c_4, \;\; m = 1,2,\cdots.$$

According again to proof of Lemma 2.2.2 and Lemma 1.1.2, we have

$$\mathring{W}_B^1(\omega(R+1)) \hookrightarrow L_{B_i}(\omega(R+1)) \text{ for } R > 0 \text{ and } i = 1,\cdots,N.$$

We set

$$\begin{aligned} A^m(x) = &\sum_{i=1}^{N} \left[a_i^m(x,u^m,\nabla u^m) - a_i^m(x,u,\nabla u) \right] (u^m - u)_{x_i} \\ &+ \sum_{i=1}^{N} \left[b_i^m(x,u^m,\nabla u^m) - b_i^m(x,u,\nabla u) \right] (u^m - u), \; m = 1,\cdots. \end{aligned}$$

Then

$$\langle A^m(u^m) - A^m(u), u^m - u \rangle = \int_\omega A^m(x)\, dx \qquad m = 1,\cdots.$$

By (2.66) and (2.67), we obtain

$$\lim_{m\to\infty} \sup \int_\omega A^m(x)\, dx \leq 0.$$

So,

$$\begin{aligned} A^m(x) &= \sum_{i=1}^{N} \left[a_i^m(x,u^m,\nabla u^m) - a_i^m(x,u^m,\nabla u) \right] (u^m - u)_{x_i} \\ &\quad + \sum_{i=1}^{N} \left[a_i^m(x,u^m,\nabla u) - a_i^m(x,u,\nabla u) \right] (u^m - u)_{x_i} \\ &\quad + \sum_{i=1}^{N} \left[b_i^m(x,u^m,\nabla u^m) - b_i^m(x,u,\nabla u) \right] (u^m - u) \\ &= A_1^m(x) + A_2^m(x) + A_3^m(x) \qquad m = 1,\cdots. \end{aligned} \tag{2.74}$$

We prove that

$$A_1^m(x) \longrightarrow 0 \text{ almost everywhere in } \Omega \qquad m \to \infty. \tag{2.75}$$

$$A_2^m(x) \longrightarrow 0 \text{ almost everywhere in } \Omega \quad m \to \infty. \tag{2.76}$$

$$A_3^m(x) \longrightarrow 0 \text{ almost everywhere in } \Omega \quad m \to \infty. \tag{2.77}$$

$$\begin{aligned} A^m(x) &= \sum_{i=1}^{N} \left[a_i^m(x,u^m,\nabla u^m) - a_i^m(x,u^m,\nabla u) \right] (u^m-u)_{x_i} \\ &= \sum_{i=1}^{N} a_i^m(x,u^m,\nabla u^m)\,.\,u_{x_i}^m - \sum_{i=1}^{N} a_i^m(x,u^m,\nabla u^m)\,.\,u_{x_i} \\ &\quad - \sum_{i=1}^{N} a_i^m(x,u,\nabla u)\,.\,u_{x_i}^m + \sum_{i=1}^{N} a_i^m(x,u,\nabla u)\,.\,u_{x_i}, \end{aligned}$$

applying (1.1), (2.42), (2.72) and (2.73) we obtain

$$A_1^m(x) \geq c(m) \longrightarrow 0 \text{ as } m \to \infty.$$

Hence, using the diagonal process, we conclude the convergence (2.75).

As in [76], let $A_i^m(u) = a_i^m(x,u,\nabla v)$ $i = 1,\cdots,N$ be Nemytsky operators for $v \in \mathring{W}_B^1(\omega)$ fixed and $x \in \omega(R)$, continuous in $L_{\bar{B}_i}(\omega(R))$ for any $R > 0$.

Thus, according to (1.10), (2.47) and the diagonal process, we have for any $R > 0$,

$$A_2^m(x) \longrightarrow 0 \text{ almost everywhere in } \omega \quad m \to \infty.$$

Applying the inequality (1.10) we obtain

$$\begin{aligned} A_3^m(x) &\leq 2 \sum_{i=1}^{N} ||\, b_i^m(x,u^m,\nabla u^m) - b_i^m(x,u,\nabla u)\, ||_{B_i,\omega(R)}\,.\, ||\, u^m - u\, ||_{\mathring{W}_B^1(\omega)} \\ &\leq 2c(m)\,.\, ||\, u^m - u\, ||_{\mathring{W}_B^1(\omega)}. \end{aligned}$$

Hence, combining to (2.47) and the diagonal process, we have for any $R > 0$

$$A_3^m(x) \longrightarrow 0 \text{ almost everywhere in } \omega \quad m \to \infty.$$

Consequently, by (2.75), (2.76), (2.77) and the selective convergences we deduce that

$$A^m(x) \longrightarrow 0 \text{ almost everywhere in } \omega \quad m \to \infty. \tag{2.78}$$

Let $\omega' \subset \omega,$ meas $\omega' =$ meas ω, and the conditions (2.47), (2.78) are true, and (2.40) - (2.43) are satisfied.

We prove the convergence

$$u^m_{x_i}(x) \longrightarrow u_{x_i}(x) \text{ everywhere in } \omega \text{ for } i = 1, \cdots, N\ ,\ m \to \infty. \tag{2.79}$$

By the absurd, suppose we do not have convergence at the point $x^* \in \omega'$.

Let $u^m = u^m_{x_i}(x^*)$, $u = u_{x_i}(x^*)$, $i = 1, \cdots, N$, and $\hat{a} = \varphi_1(x^*)$, $\bar{a} = \varphi(x^*)$. Suppose that the sequence $\sum_{i=1}^{N} B_i(u^m)$ $m = 1, \cdots, \infty$ is unbounded.

Let $\varepsilon \in \left(0, \frac{\bar{a}}{1+\hat{a}}\right)$ is fixed, according to (1.2), (1.4) and the conditions (2.40), (2.42), we get

$$\begin{aligned}
A^m(x^*) &= \sum_{i=1}^{N} \left(a^m_i(x^*, u^m, \nabla u^m) - a^m_i(x^*, u, \nabla u) \right) \nabla(u^m - u) \\
&\quad + \sum_{i=1}^{N} \left(b^m_i(x^*, u^m, \nabla u^m) - b^m_i(x^*, u, \nabla u) \right) (u^m - u) \\
&= \sum_{i=1}^{N} a^m_i(x^*, u^m, \nabla u^m) \nabla u^m - \sum_{i=1}^{N} a^m_i(x^*, u^m, \nabla u^m) \nabla u \\
&\quad - \sum_{i=1}^{N} a^m_i(x^*, u, \nabla u) \nabla u^m + \sum_{i=1}^{N} a^m_i(x^*, u, \nabla u) \nabla u \\
&\quad + \sum_{i=1}^{N} b^m_i(x^*, u^m, \nabla u^m) u^j - \sum_{i=1}^{N} b^m_i(x^*, u^m, \nabla u^m) u - \sum_{i=1}^{N} b^m_i(x^*, u, \nabla u) u^m + \sum_{i=1}^{N} b^m_i(x^*, u, \nabla u) u.
\end{aligned}$$

Applying the generalized Young inequality and (2.71), we obtain

$$\begin{aligned}
A^m(x^*) &\geq \sum_{i=1}^{N} a^m_i(x^*, u, \nabla u) . \nabla u + \sum_{i=1}^{N} a^m_i(x^*, u^m, \nabla u^m) . \nabla u^m - \varepsilon \sum_{i=1}^{N} \bar{B}_i(a^m_i(x^*, u^m, \nabla u^m)) \\
&\quad - c_1(\varepsilon) \sum_{i=1}^{N} B_i(\nabla u) - \varepsilon \sum_{i=1}^{N} \bar{B}_i(a^m_i(x^*, u, \nabla u)) - c_2(\varepsilon) \sum_{i=1}^{N} B_i(\nabla u^m) \\
&\quad + \sum_{i=1}^{N} b^m_i(x^*, u^m, \nabla u^m) . \nabla u^m + \sum_{i=1}^{N} b^m_i(x^*, u, \nabla u) . \nabla u - \sum_{i=1}^{N} b^m_i(x^*, u^m, \nabla u^m) . \nabla u \\
&\quad - \sum_{i=1}^{N} b^m_i(x^*, u, \nabla u) . \nabla u^m \\
&\geq \bar{a} \sum_{i=1}^{N} B_i(\nabla u) - \psi(x^*) + \sum_{i=1}^{N} B_i(\nabla u^m) - \psi(x^*) - \varepsilon\, \hat{a} \sum_{i=1}^{N} B_i(\nabla u^m) - \varepsilon\, \varphi(x^*) \\
&\quad - c_1(\varepsilon) \sum_{i=1}^{N} B_i(\nabla u) - \varepsilon\, \hat{a} \sum_{i=1}^{N} B_i(\nabla u) - \varepsilon\, \varphi(x^*) - c_2 \sum_{i=1}^{N} B_i(\nabla u^m) - 4h(x^*) \\
&\quad - c_3\, l(u) \sum_{i=1}^{N} B_i(\nabla u) - c_4\, l(u^m) \sum_{i=1}^{N} B_i(\nabla u^m).
\end{aligned}$$

So

$$A^j(x^*) \geq \left[\bar{a} - c_1(\varepsilon) - \varepsilon\,\hat{a} - c_3\,l(u)\right] \sum_{i=1}^{N} B_i(\nabla u) + \left[\bar{a} - \varepsilon\,\hat{a}\,c_2 - c_4\,l(u^m)\right] \sum_{i=1}^{N} B_i(\nabla u^m) - c_5(\varepsilon).$$

So we deduce that the sequence $A^m(x^*)$ is not bounded, which is absurd as far as what is in (2.78). As a consequence, the sequences $u^m_{x_i}, i = 1, \cdots, N,\ m \to \infty$ are bounded.

Let $u^* = (u^*_1, u^*_2, \cdots, u^*_N)$ the limits of subsequence $u^m = (u^m_1, \cdots, u^m_N)$ with $m \to \infty$. Then, taking into account (2.47), we obtain

$$u^m_{x_i} \longrightarrow u^*_{x_i} \qquad ,\ i = 1, \cdots, N. \tag{2.80}$$

As a result, from (2.78), (2.80) and the fact that $a^m_i(x^*, u, \nabla u)$ are continuous in u (because they are Carathéodory functions), we have

$$\sum_{i=1}^{N} \left(a^m_i(x^*, u^m, \nabla u^m) - a^m_i(x^*, u, \nabla u)\right) . (u^m_{x_i} - u_{x_i}) = 0,$$

and from (2.41) we have, $u^*_{x_i} = u_{x_i}$. This contradicts the fact that there is no convergence at the point x^*.

And referring to (2.47), (2.80) and the fact that $a^m_i(x^*, u, \nabla u)$ are continuous u, so for $m \to \infty$ we get

$$a^m_i(x, u^m, \nabla u^m) \longrightarrow a^m_i(x, u, \nabla u),\ i = 1, \cdots, N \text{ almost everywhere in } \omega.$$

Using Lemma 2.2.3 we find the weak convergences

$$a^m_i(x, u^m, \nabla u^m) \rightharpoonup a^m_i(x, u, \nabla u) \text{ in } L_{\bar{B}_i(\omega)}, i = 1, \cdots, N. \tag{2.81}$$

The weak convergence (2.68) follows from (2.81).

Furthermore, to complete the proof, we note that (2.69) is implied from (2.66) and (2.78) :

$$\begin{aligned}\langle A^m(u^m), u^m - u\rangle =& \langle A^m(u^m) - A^m(u), u^m - u\rangle \\ &+ \langle A^m(u), u^m - u\rangle \to 0, m \to \infty.\end{aligned}$$

We're ending this section by a suitable example, that checks all the above conditions and propositions,

Example 6. *Let ω be an unbounded domain of $\mathbb{R}^N$, $(N \geq 2)$. By Theorems 2.2.1 and 2.2.2 it exists a unique entropy solution based on the definition 2.2.1 of the following anisotropic problem $(\mathscr{P}_1)$:*

$$(\mathscr{P}_1)\begin{cases} \tilde{a}\displaystyle\sum_{i=1}^{N}\bar{B}_i^{-1}B_i(|\nabla u|)+l(u)\,.\sum_{i=1}^{N}B_i(|\nabla u|)=f(x) & in \quad \omega\,, \\ u=0 & on \quad \partial\omega\,, \end{cases}$$

with $\tilde{a}$ is a positive constant, $l:\mathbb{R}\longrightarrow\mathbb{R}^+$ a positive continuous functions such as $l\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$, $f\in L^1(\omega)$ and

$$B(z) = |z|^b\,(\,|ln|z||+1\,),\;\; b>1.$$

satisfying the Δ_2-condition.

3

The Existence and Uniqueness of an Entropy Solution to Unilateral Orlicz Anisotropic Equations in General Domain

In this chapter, we study a certain class of unilateral elliptical operators whose nonlinearity is given by a vector of N-functions in the framework of anisotropic Orlicz spaces. In the first part, we prove the existence of entropic solutions for our problem in the bounded domain. In the second part, we show the existence and uniqueness solution to the same problem but with an unbounded domain namely ; without assuming any condition on the behavior of the solutions when $|x|$ tends towards infinity. Moreover, we are giving some examples of an anisotropic elliptic equation that verifies all our demonstrated results in each section.

3.1 Existence of entropy solutions in a bounded domain

3.1.1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$. The aim of this section is the study of boundary value problems for a class of nonlinear anisotropic elliptic equations. More specifically, we consider the unilateral elliptical operators whose nonlinearity is given by a vector of N-functions like

$$(\mathscr{P}) \begin{cases} \mathfrak{A}(u) + \sum_{i=1}^{N} \mathfrak{b}_i(x,u,\nabla u) = f & \text{in } \Omega\,, \\ u \geq \zeta & \text{a.e in } \Omega\,, \end{cases}$$

where, $\mathfrak{A}(u) = \sum_{i=1}^{N} (\sigma_i(x,u,\nabla u))_{x_i}$ is a Leray–Lions operator defined in $\mathring{W}_M^1(\Omega)$ (defined as the adherence space $C_0^\infty(\Omega)$) into its dual (see assumptions (3.1), (3.2), (3.3) in Section 3.1.2 below); $M(t) = (M_1(t), \cdots, M_N(t))$ are N-uplet Orlicz functions that satisfy Δ_2-condition; the obstacle ζ is a measurable function that belongs to $L^\infty(\Omega) \cap \mathring{W}_M^1(\Omega)$; and for the $i = 1, \cdots, N$, $\mathfrak{b}_i(x,s,\xi)$: $\Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ are Carathéodory functions (measurable with respect to x in Ω for every $(s,\, \xi)$ in $\mathbb{R} \times \mathbb{R}^N$, and continuous with respect to $(s,\, \xi)$ in $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω) that does not satisfy any sign condition and the growth which is described by the vector N-function $(M_1(t), \cdots, M_N(t))$ (see assumption (3.4)). As well as $f \in L^1(\Omega)$.

For several years great effort has been devoted to the study of nonlinear elliptic equations with an operator which was described by polynomial growth. For example, in the classical Sobolev space, Boccardo and Gallouêt in [35], proved the existence of a weak solution of $(\mathscr{P})$ in the case $\phi \equiv g \equiv 0$. Bénilan in [21] presented the idea of entropy solutions which were adjusted to the Boltzmann condition. For a deeper comprehension of these types of equations in this field, we refer the reader to [5, 30, 34, 35, 67, 69, 77] and references therein.

Next, in the Orlicz space, Benkirane and Bennouna in [25] demonstrated the existence of entropy solutions to the following nonlinear elliptic problem :

$$-\mathrm{div}\, a(x,u,\nabla u) + \mathrm{div}(\phi(u)) = f,$$

where $\phi \in (C^0(\mathbb{R}))^N$ and $f \in L^1(\Omega)$. For more results, we refer the reader to [4, 11, 12, 38–41, 46, 47, 52, 79, 84, 91] and references therein.

And in the anisotropic Sobolev-Orlicz space, there are few results dealing with this topic. We will mention recent papers, and we are starting by the pertinent works of Korolev and Cianchi [48, 72] who proved the embeddings of this space. For more results dealing this field, we refer to [13, 20, 26, 28, 74, 75] and the references therein.

This kind of operator arises in a quite natural way in many different contexts, such as the study of fluid filtration in porous media, constrained heating, elasticity, electro-rheological fluids , optimal control, financial mathematics and other domains, see [19, 23, 45, 96] and the references therein.

As far we know, no previous research has investigated the existence of entropy solutions to unilateral problem $(\mathscr{P})$ with the second term as an operator with growth described by an n-uplet of N-functions satisfying the Δ_2-condition, within the fulfilling of anisotropic Sobolev-Orlicz space with bounded domain, the function $\mathfrak{b}_i(x,u,\nabla u)$ does not satisfy any sign condition and the source f is merely integrable. Hence, motived by the aforementioned papers, our main work is to obtain the existence Theorem for unilateral problems corresponding to $(\mathscr{P})$ via an approximation procedure and some priori estimates.

3.1.2 Assumptions on data and definition of solution

Statement of the problem : Suppose they have non-negative measurable functions ϕ, $\varphi \in L^1(\Omega)$ and positive constants $\bar{a}$ and $\tilde{a}$ such as :

$$\sum_{i=1}^{N}\big(\sigma_i(x,s,\xi)-\sigma_i(x,s,\xi')\big).(\xi_i-\xi_i')>0, \tag{3.1}$$

$$\sum_{i=1}^{N}\sigma_i(x,s,\xi).\xi_i \geq \bar{a}\sum_{i=1}^{N}M_i(|\xi_i|)-\phi(x), \tag{3.2}$$

$$\sum_{i=1}^{N}|\sigma_i(x,s,\xi)| \leq \tilde{a}\sum_{i=1}^{N}\bar{M}_i^{-1}M_i(|\xi|)+\varphi(x), \tag{3.3}$$

and

$$\sum_{i=1}^{N}|\mathfrak{b}_i(x,s,\xi)| \leq h(x)+l(s).\sum_{i=1}^{N}M_i(|\xi|), \tag{3.4}$$

with $\overline{M}(t)$ the complementary of $M(t)$, $h(x)\in L^1(\Omega)$ and $l:\mathbb{R}\longrightarrow\mathbb{R}^+$ a positive continuous function such as : $l\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$.

Definition of entropy solutions

Definition 3.1.1. *A measurable function u is said to be an entropy solution for the problem* $(\mathscr{P})$*, if*

$u \in \mathring{W}^1_M(\Omega)$ such that $u \geq \zeta$ a.e. in Ω and

$$\sum_{i=1}^{N} \int_\Omega \sigma_i(x,u,\nabla u)\,.\,\nabla(u-v)\,dx + \sum_{i=1}^{N} \int_\Omega \mathfrak{b}_i(x,u,\nabla u)\,.\,(u-v)\,dx \leq \int_\Omega f(x)\,.\,(u-v)\,dx \qquad \forall v \in K_\zeta \cap L^\infty(\Omega),$$

where, $K_\zeta = \{\, u \in \mathring{W}^1_M(\Omega)$ such as $u \geq \zeta$ a.e. in $\Omega\,\}$.

3.1.3 Main result

In this part, we will show the existence of our problem $(\mathscr{P})$. There exist $f^m \in C_0^\infty(\Omega)$ such that $f^m \longrightarrow f$ in $L^1(\Omega)$, $m \to \infty$, $|\,f^m(x)\,| \leq |\,f(x)\,|$ and for $i = 1, \cdots, N$,
$\sigma_i^m(x,u_m,\nabla u_m) : (\,\mathring{W}^1_M(\Omega)\,)^N \longrightarrow (\,\mathring{W}^{-1}_{\overline{M}}(\Omega)\,)^N$ being Carathéodory functions with

$$\sigma_i^m(x,u,\nabla u) = \sigma_i(x,T_m(u),\nabla u),$$

and $\mathfrak{b}_i^m(x,u_m,\nabla u_m) : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ again being Carathéodory functions not satisfying any sign condition, with

$$\mathfrak{b}^m(x,u,\nabla u) = \frac{\mathfrak{b}(x,u,\nabla u)}{1+\frac{1}{m}\,|\,\mathfrak{b}(x,u,\nabla u)\,|},$$

and

$$|\,\mathfrak{b}^m(x,u,\nabla u)\,| = |\,\mathfrak{b}(x,T_m(u),\nabla u)\,| \leq m \text{ for all } m \in \mathbb{N}^*, \tag{3.5}$$

Consider the penalized equations :

$$(\mathscr{P}_m) : \sum_{i=1}^{N} \int_\Omega \sigma_i^m(x,u_m,\nabla u_m)\,.\,\nabla(u_m-v)\,dx + \sum_{i=1}^{N} \int_\Omega \mathfrak{b}_i^m(x,u_m,\nabla u_m)\,.\,(u_m-v)\,dx + \int_\Omega m\,.\,T_m(u_m-\zeta)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,(u_m-v)\,dx = \int_\Omega f^m(x)\,.\,(u_m-v)\,dx \;\; \forall v \in \mathring{W}^1_M(\Omega),$$

with $sg_m(s) = \frac{T_m(s)}{m}$, for $m \in \mathbb{N}^*$, and we define the truncation at height m, $T_m(u) : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$T_m(u) = \begin{cases} u & \text{if } |\,u\,| \leq m\,, \\ m & \text{if } |\,u\,| > m\,. \end{cases}$$

Theorem 3.1.1. *Let's assume that conditions* (3.1)–(3.4) *and* (1.18) *hold true, then there exists at least one solution of the approximate problem $(\mathscr{P}_m)$.*

Proof. See appendix 3.1.4. □

Now, we will show some results in the form of propositions that will be useful for the demonstration of existence Theorem 3.1.2, see below.

Proposition 3.1.1. *(see [28]) Suppose that conditions* (3.1) - (3.4) *are satisfied, and let* $(u_m)_{m\in\mathbb{N}}$ *be a sequence in* $\mathring{W}^1_B(\Omega)$ *such as*

$(a)\quad u_m \rightharpoonup u \ \ in \ \ \mathring{W}^1_M(\Omega),$

$(b)\quad \sigma^m(x,u_m,\nabla u_m)$ *is bounded in* $L_{\overline{M}}(\Omega)$,

$(c)\quad \int_\Omega \big(\sigma^m(x,u_m,\nabla u_m) - \sigma^m(x,u_m,\nabla u\,\chi_K)\big)\,.\nabla(u_m - u\,\chi_K)\,dx \to 0$ *as* $K\to+\infty$ *(* χ_K *the characteristic function of* $\Omega_K = \{x\in\Omega;\,|\nabla u|\le K\}$ *).*

Then :

$$M(|\nabla u_m|) \longrightarrow M(|\nabla u|) \ \ in \ \ L^1(\Omega).$$

Proposition 3.1.2. *(see [26]) Let's assume that conditions* (3.1) - (3.4) *and* (1.18) *hold true, then the generalized solution of the problems* $(\mathscr{P}_m)$ *satisfies the following estimate :*

$$\int_\Omega M(|\nabla T_K(u_m)|) \le c = c(K)\,, \quad K>0.$$

Proposition 3.1.3. *(see [26]) Suppose that the conditions* (3.1) - (3.4) *and* (1.18) *are satisfied, and let* $(u_m)_{m\in\mathbb{N}}$ *be a solution of the problem* $(\mathscr{P}_m)$*, then there exists a measurable function* u *such as* $\forall K>0$*, we have for all subsequence noted again* u_n*,*

$(a)\quad u_m \longrightarrow u \ \ a.e \ \ in \ \ \Omega,$

$(b)\quad T_K(u_m) \rightharpoonup T_K(u) \ \ weakly \ \ in \ \ \mathring{W}^1_M(\Omega),$

$(c)\quad T_K(u_m) \longrightarrow T_K(u) \ \ strongly \ \ in \ \ \mathring{W}^{-1}_{\overline{M}}(\Omega).$

Proposition 3.1.4. *Suppose that the conditions* (3.1) - (3.4) *and* (1.18) *are satisfied, and let* $(u_m)_{m\in\mathbb{N}}$ *be a solution of the problem* $(\mathscr{P}_m)$*, then for any* $K>0$*, we have*

$(1)\quad \sigma^m(x,T_K(u_m),\nabla T_K(u_m)) \ \ is \ \ bounded \ \ in \ \ \mathring{W}^1_{\overline{M}}(\Omega),$

$(2)\quad M(|\nabla T_K(u_m)|) \longrightarrow M(|\nabla T_K(u)|)$ *is strongly in* $L^1(\Omega)$, *Proof.* 1)

$$
\begin{aligned}
\|\sigma^m(x,T_K(u_m),\nabla T_K(u_m))\|_{\overline{M},\Omega} &= \sum_{i=1}^{N}\|\sigma_i^m(x,T_K(u_m),\nabla T_K(u_m))\|_{\overline{M_i},\Omega}\\
&\leq \sum_{i=1}^{N}\int_\Omega M_i(|\nabla T_K(u_m)|)\,dx + \|\varphi\|_1 + N,
\end{aligned}
$$

from Proposition 3.1.2 we obtain :

$$
\|\sigma^m(x,T_K(u_m),\nabla T_K(u_m))\|_{\overline{M},\Omega} \leq c(K) + \|\varphi\|_1 + N.
$$

Hence, $\sigma^m(x,T_K(u_m),\nabla T_K(u_m))$ is bounded in $\mathring{W}^1_{\overline{M}}(\Omega)$.

2) Showing that $M(|\nabla T_K(u_m)|) \longrightarrow M(|\nabla T_K(u)|)$ strongly in $L^1(\Omega)$ that's why, let's introduce the following functions of a variable K defined as $h_j(K) = \begin{cases} 1 & \text{if } |K| \leq j\,, \\ 0 & \text{if } |K| \geqslant j+1\,, \\ j+1+|K| & \text{if } j<|K|<j+1, \end{cases}$ with j as a non-negative real parameter, $\Omega_K = \{x\in\Omega : |\nabla T_K(u(x))| \leq K\}$ and we note that χ_K is a characteristic function of Ω_K. It's clear that $\Omega_K \subset \Omega_{K+1}$ and $\text{meas}(\Omega\setminus\Omega_K)\to 0$ since $K\to\infty$ shows that the following assertions are true.

Assertion 1 :

$$
\lim_{j\to\infty}\lim_{m\to\infty}\int_{\{\Omega:\, j<|K|<j+1\}} \sigma^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\,dx = 0.
$$

Assertion 2 :

$$
T_K(u_m) \longrightarrow T_K(u) \text{ modular convergence in } \mathring{W}^1_M(\Omega).
$$

Proof of assertion 1. Let

$$
v = u_m + \exp(G(|u_m|))\,.\,T_1(u_m - T_j(u_m)),
$$

with $G(s) = \int_0^s \frac{l(t)}{\bar{a}}\,dt$ as a test function in $(\mathscr{P}_m)$ then we get :

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega \sigma_i^m(x,u_m,\nabla u_m)\,.\,\nabla(\exp(G(|u_m|))\,.\,T_1(u_m-T_j(u_m)))\,dx\\
&\quad+\sum_{i=1}^{N}\int_\Omega \mathfrak{b}_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,.\,T_1(u_m-T_j(u_m))\,dx\\
&\quad+\int_\Omega m\,.\,T_m(u_m-\zeta)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(|u_m|))\,.\,T_1(u_m-T_j(u_m))\,dx\\
&\quad=\int_\Omega f^m(x)\,.\,\exp(G(|u_m|))\,.\,T_1(u_m-T_j(u_m))\,dx,
\end{aligned}$$

by (3.2) and (3.4) we obtain :

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega \sigma_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,.\,\nabla(T_1(u_m-T_j(u_m)))\,dx\\
&\quad+\int_\Omega m\,.\,T_m(u_m-\zeta)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(|u_m|))\,.\,T_1(u_m-T_j(u_m))\,dx\\
&\quad\le\int_\Omega \left[f^m(x)+h(x)+\phi(x)\,.\,\frac{l(u_m)}{\bar{a}}\right].\,\exp(G(|u_m|))\,.\,T_1(u_m-T_j(u_m))\,dx,
\end{aligned}$$

since $f,\,\phi\in L^1(\Omega)$, $l\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$, $\exp(G(\pm\infty))\le\exp\left(\frac{||l(u_m)||_{L^1(\mathbb{R})}}{\bar{a}}\right)$ and by proposition 3.1.3 we obtain

$$\lim_{m\to\infty}\lim_{j\to\infty}\int_\Omega \left[f^m(x)+h(x)+\phi(x)\,.\,\frac{l(u_m)}{\bar{a}}\right].\,\exp(G(|u_m|))\,.\,T_1(u_m-T_j(u_m))\,dx=0.$$

Hence,

$$\lim_{m\to\infty}\lim_{j\to\infty}\sum_{i=1}^{N}\int_{\{\Omega:\,j<|u_m|<j+1\}}\sigma_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\,dx=0,$$

and

$$\lim_{m\to\infty}\lim_{j\to\infty}\int_\Omega m\,.\,T_m(u_m-\zeta)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(|u_m|))\,.\,T_1(u_m-T_j(u_m))\,dx=0.$$

Proof of assertion 2. Let $j\ge K>0$, we consider

$$v=u_m+\exp(G(|u_m|))\,.\,(T_K(u_m)-T_K(u))\,.\,h_j(u_m),$$

as a test function in $(\mathscr{P}_m)$ we obtain :

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega}\sigma_i^m(x,u_m,\nabla u_m)\,.\,\nabla(\exp(G(|u_m|))\,.\,(T_K(u_m)-T_K(u))\,.\,h_j(u_m))\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\Omega}\mathfrak{b}_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,.\,(T_K(u_m)-T_K(u))\,.\,h_j(u_m)\,dx\\
&\quad+\int_{\Omega}m\,.\,T_m(u_m-\zeta)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(|u_m|))\,.\,(T_K(u_m)-T_K(u))\,.\,h_j(u_m)\,dx\\
&\quad=\int_{\Omega}f^m(x)\,.\,\exp(G(|u_m|))\,.\,(T_K(u_m)-T_K(u))\,.\,h_j(u_m)\,dx,
\end{aligned}$$

by (3.2) and (3.4) we have :

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\{|u_m|\leq K\}}\sigma_i^m(x,T_K(u_m),\nabla T_K(u_m))\,.\,(\nabla T_K(u_m)-\nabla T_K(u))\,.\,\exp(G(|u_m|))\,dx\\
&\quad+\int_{\Omega}m\,.\,T_m(u_m-\zeta)^-\,.\,sg_{\frac{1}{m}}(u_m)\exp(G(|u_m|))\,.\,(T_K(u_m)-T_K(u))\,.\,h_j(u_m)\,dx\\
&\quad\leq\int_{\Omega}\left[f^m(x)+h(x)+\phi(x)\,.\,\frac{l(u_m)}{\bar{a}}\right].\,\exp(G(|u_m|))\,.\,(T_K(u_m)-T_K(u))\,.\,h_j(u_m)\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\{\Omega:K<|u_m|<j+1\}}|\,\sigma_i^m(x,T_{j+1}(u_m),\nabla T_{j+1}(u_m))\,|\,.\,|\,\nabla T_K(u_m)\,|\,.\,\exp(G(|u_m|))\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\{\Omega:K<|u_m|<j+1\}}\sigma_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\,.\,|\,T_K(u_m)-T_K(u)\,|\,.\,\exp(G(|u_m|))\,dx, \qquad (3.6)
\end{aligned}$$

and since $T_K(u_m)\rightharpoonup T_K(u)$ is weakly in $\mathring{W}_M^1(\Omega)$, we have :

$$\int_{\Omega}\left[f^m(x)+h(x)+\phi(x)\,.\,\frac{l(u_m)}{\bar{a}}\right].\,\exp(G(|u_m|))\,.\,(T_K(u_m)-T_K(u))\,.\,h_j(u_m)\,dx\longrightarrow 0,$$

and

$$\int_{\Omega}m\,.\,T_m(u_m-\zeta)^-\,.\,sg_{\frac{1}{m}}(u_m)\exp(G(|u_m|))\,.\,(T_K(u_m)-T_K(u))\,.\,h_j(u_m)\,dx\longrightarrow 0,$$

since $|\,\sigma_i^m(x,T_{j+1}(u_m),\nabla T_j(u_m))\,|$ is bounded in $L_{\overline{M}}(\Omega)$, then there exist $\tilde{\sigma}^m\in L_{\overline{M}}(\Omega)$ such that

$$|\,\sigma_i^m(x,T_{j+1}(u_m),\nabla T_{j+1}(u_m))\,|\rightharpoonup\tilde{\sigma}^m \text{ in } L_{\overline{M}}(\Omega), \qquad (3.7)$$

$$\sum_{i=1}^{N}\int_{\{\Omega:K<|u_m|<j+1\}}|\,\sigma_i^m(x,T_{j+1}(u_m),\nabla T_{j+1}(u_m))\,|\,.\,|\,\nabla T_K(u_m)\,|\,.\,\exp(G(|u_m|))\,dx$$
$$\leq \exp\left(\frac{||\,l\,||_{L^1(\mathbb{R})}}{\bar{a}}\right).\sum_{i=1}^{N}\int_{\{\Omega:K<|u_m|<j+1\}}|\,\sigma_i^m(x,T_{j+1}(u_m),\nabla T_{j+1}(u_m))\,|\,.\,|\,\nabla T_K(u_m)\,|\,dx$$
$$\longrightarrow \exp\left(\frac{||\,l\,||_{L^1(\mathbb{R})}}{\bar{a}}\right).\sum_{i=1}^{N}\int_{\{\Omega:K<|u_m|<j+1\}}\tilde{\sigma}^m\,.\,|\,\nabla T_K(u)\,|\,dx=0 \text{ as } m\to\infty, \tag{3.8}$$

according to assertion 1, we get :

$$\sum_{i=1}^{N}\int_{\{\Omega:K<|u_m|<j+1\}}\sigma_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\,.\,|\,T_K(u_m)-T_K(u)\,|\,.\,\exp(G(|u_m|))\,dx$$
$$\leq 2K\,.\,\exp\left(\frac{||\,l\,||_{L^1(\mathbb{R})}}{\bar{a}}\right).\sum_{i=1}^{N}\int_{\{\Omega:K<|u_m|<j+1\}}\sigma_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\,dx\longrightarrow 0 \text{ as } j\to\infty, \tag{3.9}$$

combine (3.6) - (3.9) we obtain :

$$\sum_{i=1}^{N}\int_{\Omega}\Big[\sigma_i^m(x,T_K(u_m),\nabla T_K(u_m))-\sigma_i^m(x,T_K(u_m),\nabla T_K(u))\Big]\,.\,(\nabla T_K(u_m)-\nabla T_K(u)\,)\,dx\longrightarrow 0$$
$$\text{as } m\longrightarrow\infty. \tag{3.10}$$

According to proposition 3.1.1, we conclude that

$$M(|\,\nabla T_K(u_m)\,|)\longrightarrow M(|\,\nabla T_K(u)\,|) \text{ in } L^1(\Omega).$$

□

Proposition 3.1.5. *(See [26]) Suppose that the conditions* (3.1) - (3.4) *and* (1.18) *are true, and* $u^j,u\in \mathring{W}_M^1(\Omega)$

$$||\,u^j\,||_{\mathring{W}_M^1(\Omega)}\leq c\,,\;\; j=1,\cdots,\infty \tag{3.11}$$

$$u^j\longrightarrow u \;\; in \;\; L_M(\Omega), \tag{3.12}$$

with $M(z)$ *is a N-function. Let's assume the following functions :*

$$\mathfrak{A}^j(x)=\sum_{i=1}^{N}\left(\sigma_i^m(x,u^j,\nabla u^j)-\sigma_i^m(x,u,\nabla u)\right)\nabla(u^j-u)+\sum_{i=1}^{N}\left(\mathfrak{b}_i^m(x,u^j,\nabla u^j)-\mathfrak{b}_i^m(x,u,\nabla u)\right)(u^j-u),$$

$j=1,\cdots$ *satisfying the condition*

$$\int_{\Omega}\mathfrak{A}^j(x)\,dx\longrightarrow 0,\qquad j\longrightarrow\infty. \tag{3.13}$$

Then, there exists a sequence of natural numbers $J \subset \mathbb{N}$ such that as $j \to \infty$, $j \in J$

$$\sigma_i^m(x,u^j,\nabla u^j) \rightharpoonup \sigma_i^m(x,u,\nabla u) \ \ in \ \ L_{\overline{M}_i}(\Omega), \qquad i=1,\cdots,N. \tag{3.14}$$

Theorem 3.1.2. *Under assumptions* (3.1)–(3.4)*, the problem* $(\mathscr{P})$ *has at least one entropy solution.*

Proof. We divide our proof in six steps :

Step 1 : A priori estimate of $\{u^m\}$.

We consider the following test function :

$$v = u_m + \eta \, \exp(G(|u_m|))\, T_1(u_m - T_j(u_m)),$$

with η small enough, we get :

$$\begin{aligned}
&\sum_{i=1}^N \int_\Omega \sigma_i^m(x,u_m,\nabla u_m)\,.\,\nabla\big[\exp(G(|u_m|))\,T_1(u_m - T_j(u_m))\big]\,dx \\
&\quad + \sum_{i=1}^N \int_\Omega \mathfrak{b}_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,T_1(u_m - T_j(u_m))\,dx \\
&\quad + \int_\Omega m\,.\,T_m(u_m-\zeta)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(|u_m|))\,T_1(u_m - T_j(u_m)) \\
&\quad \leq \int_\Omega f^m(x)\,.\,\exp(G(|u_m|))\,T_1(u_m - T_j(u_m))\,dx,
\end{aligned}$$

according to (3.2) and (3.4) we obtain :

$$\begin{aligned}
&\sum_{i=1}^N \int_\Omega \sigma_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,\nabla T_1(u_m - T_j(u_m))\,dx \\
&\quad + \int_\Omega m\,.\,T_m(u_m-\zeta)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(|u_m|))\,T_1(u_m - T_j(u_m)) \\
&\quad \leq \int_\Omega \left[f^m(x) + h(x) + \phi(x)\,.\,\frac{l(u_m)}{\bar a}\right].\,\exp(G(|u_m|))\,T_1(u_m - T_j(u_m))\,dx,
\end{aligned}$$

since $f, h, \phi \in L^1(\Omega)$, $l \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $\exp(G(\pm\infty)) \leq \exp\left(\frac{||l||_{L^1(\mathbb{R})}}{\bar a}\right)$ and the fact $T_1(u_m - T_j(u_m)) \longrightarrow 0$ is weakly in $\mathring{W}_M^1(\Omega)$ as $j \longrightarrow \infty$ (proposition 3.1.3). We have :

$$\int_\Omega \left[f^m(x) + h(x) + \phi(x)\,.\,\frac{l(u_m)}{\bar a}\right].\,\exp(G(|u_m|))\,T_1(u_m - T_j(u_m))\,dx \longrightarrow 0 \text{ as } m \to \infty,$$

then,

$$\sum_{i=1}^{N}\int_{\Omega}\sigma_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,.\nabla T_1(u_m)-T_j(u_m))\,dx$$
$$+\int_{\Omega}m\,.\,T_m(u_m-\zeta)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(|u_m|))\,.\,T_1(u_m)-T_j(u_m))\,dx$$
$$\leq 0.$$

Hence,

$$\lim_{m\to\infty}\lim_{j\to\infty}\sum_{i=1}^{N}\int_{\{\Omega:K<|u_m|<j+1\}}\sigma_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\,dx=0,$$

and

$$\lim_{m\to\infty}\int_{\Omega}m\,.\,T_m(u_m-\zeta)^-\,.\,sg_{\frac{1}{m}}(u_m)\,dx=0. \tag{3.15}$$

Step 2 : Almost everywhere convergence of the gradient.

In this step we consider again the following test function :

$$v=u_m+\eta\,\exp(G(|u_m|))\,(T_K(u_m)-T_K(u))\,h_j(u_m),$$

$$\text{with, } h_j(u_m))=1-|\,T_1(u_m-T_j(u_m))\,|=\begin{cases}1 & \text{if } \{|u_m|\geq j\}\ ,\\ 0 & \text{if } \{|u_m|\geq j+1\}\ ,\\ j+1-|u_m| & \text{if } \{j<|u_m|<j+1\},\end{cases}$$

and $|T_K(u_m)-T_K(u)|$ at the same sign when $u_m\in\{|u_m|>K\}$ where $j\geq K>0$ and η are small enough, we obtain

$$\sum_{i=1}^{N}\int_{\Omega}\sigma_i^m(x,u_m,\nabla u_m)\,.\,\nabla(\exp(G(|u_m|))\,(T_K(u_m)-T_K(u))\,h_j(u_m))\,dx$$
$$+\int_{\Omega}m\,.\,T_K(u_m-\zeta)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(|u_m|))\,(T_K(u_m)-T_K(u))\,h_j(u_m)\,dx$$
$$+\sum_{i=1}^{N}\int_{\Omega}\mathfrak{b}_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,(T_K(u_m)-T_K(u))\,h_j(u_m)\,dx$$
$$\leq\int_{\Omega}f^m(x)\,.\,\exp(G(|u_m|))\,(T_K(u_m)-T_K(u))\,h_j(u_m)\,dx,$$

by (3.2), (3.4) and the fact $j \geq K > 0$ we have this :

$$\sum_{i=1}^{N} \int_{\{|u_m| \leq K\}} \sigma_i^m(x, T_K(u_m), \nabla T_K(u_m)) \,.\, \exp(G(|u_m|)) \,.\, \nabla(T_K(u_m) - T_K(u)) \, dx$$
$$+ \sum_{i=1}^{N} \int_{\Omega} \sigma_i^m(x, u_m, \nabla u_m) \,.\, \exp(G(|u_m|)) \,.\, (T_K(u_m) - T_K(u)) \,.\, \nabla h_j(u_m) \, dx$$
$$+ \int_{\Omega} m \,.\, T_m(u_m - \zeta)^- \,.\, sg_{\frac{1}{m}}(u_m) \,.\, \exp(G(|u_m|)) \,.\, (T_K(u_m) - T_K(u)) \,.\, h_j(u_m) \, dx$$
$$\leq \int_{\Omega} \left[f^m(x) + h(x) + \phi(x) \,.\, \frac{l(|u_m|)}{\bar{a}} \right] \,.\, \exp(G(|u_m|)) \,.\, (T_K(u_m) - T_K(u)) \,.\, h_j(u_m) \, dx, \quad (3.16)$$

then, by the condition (c) in proposition 3.1.3 we have $T_K(u_m) \longrightarrow T_K(u)$ weakly in $\mathring{W}_M^1(\Omega)$, and since $f^m, h(x), \phi \in L^1(\Omega)$ we get

$$\int_{\Omega} \left[f^m(x) + h(x) + \phi(x) \,.\, \frac{l(|u_m|)}{\bar{a}} \right] \,.\, \exp(G(|u_m|)) \,.\, (T_K(u_m) - T_K(u)) \,.\, h_j(u_m) \, dx \longrightarrow 0, \quad (3.17)$$

and

$$\int_{\Omega} m \,.\, T_m(u_m - \zeta)^- \,.\, sg_{\frac{1}{m}}(u_m) \,.\, \exp(G(|u_m|)) \,.\, (T_K(u_m) - T_K(u)) \,.\, h_j(u_m) \, dx = 0,$$

and

$$\sum_{i=1}^{N} \int_{\Omega} \sigma_i^m(x, u_m, \nabla u_m) \,.\, \exp(G(|u_m|)) \,.\, (T_K(u_m) - T_K(u)) \,.\, \nabla h_j(u_m) \, dx$$
$$= \sum_{i=1}^{N} \int_{\{\Omega :\, j < |u_m| < j+1\}} \sigma_i^m(x, u_m, \nabla u_m) \,.\, \nabla u_m \,.\, \exp(G(|u_m|)) \,.\, (T_K(u_m) - T_K(u)) \, dx$$
$$\leq 2K \,.\, \exp\left(\frac{||l||_{L^1(\Omega)}}{\bar{a}} \right) \,.\, \sum_{i=1}^{N} \int_{\{\Omega :\, j < |u_m| < j+1\}} \sigma_i^m(x, u_m, \nabla u_m) \,.\, \nabla u_m \, dx \longrightarrow 0 \text{ as } j \to \infty, \quad (3.18)$$

combining (3.16) – (3.18), we get

$$\sum_{i=1}^{N} \int_{\{|u_m| \leq K\}} \sigma_i^m(x, T_K(u_m), \nabla T_K(u_m)) \,.\, \exp(G(|u_m|)) \,.\, \nabla(T_K(u_m) - T_K(u)) \, dx \leq \varepsilon(i, j, m),$$

thus,

$$\sum_{i=1}^{N}\int_{\Omega}\left[\sigma_i^m(x,T_K(u_m),\nabla T_K(u_m))-\sigma_i^m(x,T_K(u_m),\nabla T_K(u))\right].\nabla(T_K(u_m)-T_K(u)).\exp(G(|u_m|))\,dx$$
$$\leq -\sum_{i=1}^{N}\int_{\Omega}\sigma_i^m(x,T_K(u_m),\nabla T_K(u)).\nabla(T_K(u_m)-T_K(u)).\exp(G(|u_m|))\,dx$$
$$-\sum_{i=1}^{N}\int_{\{|u_m|>K\}}\sigma_i^m(x,T_K(u_m),\nabla T_K(u_m)).\nabla(T_K(u_m)-T_K(u)).\exp(G(|u_m|))\,dx+\varepsilon(i,j,m),$$

letting i, j, m tend to infinity, we have :

$$\sum_{i=1}^{N}\int_{\Omega}\left[\sigma_i^m(x,T_K(u_m),\nabla T_K(u_m))-\sigma_i^m(x,T_K(u_m),\nabla T_K(u))\right].\nabla(T_K(u_m)-T_K(u)).\exp(G(|u_m|))\,dx$$
$$\longrightarrow 0 \text{ as } m\to\infty, \tag{3.19}$$

which is implied by proposition 3.1.1,

$$M(|\nabla u_m|)\longrightarrow M(|\nabla u|) \text{ in } L^1(\Omega). \tag{3.20}$$

Hence, we obtain for a subsequence :

$$\nabla u_m \longrightarrow \nabla u \text{ a.e in } \Omega. \tag{3.21}$$

Step 3 : The equi-integrability of $\mathfrak{b}_i^m(x,u_m,\nabla u_m)$.

In this step, we will show that :

$$\mathfrak{b}_i^m(x,u_m,\nabla u_m)\longrightarrow \mathfrak{b}_i(x,u,\nabla u). \tag{3.22}$$

Therefore, it is enough to show that $\mathfrak{b}_i^m(x,u_m,\nabla u_m)$ is uniformly equi-integrable. We take the following test function :

$$v=u_m+\eta\,\exp(G(|u_m|))\int_{|u_m|}^{0}\int_{\{|s|>j\}}l(s)\,ds\,dx,$$

we obtain :

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega}\sigma_i^m(x,u_m,\nabla u_m)\,.\,\nabla(\exp(G(|u_m|))\int_{|u_m|}^{0}\int_{\{|s|>j\}}l(s)\,ds\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\Omega}\mathfrak{b}_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\int_{|u_m|}^{0}\int_{\{|s|>j\}}l(s)\,ds\,dx\\
&\quad+\int_{\Omega}m\,.\,T_m(u_m-\zeta)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(|u_m|))\int_{|u_m|}^{0}\int_{\{|s|>j\}}l(s)\,ds\,dx\\
&\quad=\int_{\Omega}f^m(x)\,.\,\exp(G(|u_m|))\int_{|u_m|}^{0}\int_{\{|s|>j\}}l(s)\,ds\,dx,
\end{aligned}$$

by (3.2) and (3.4) we get :

$$\begin{aligned}
&\bar{a}\sum_{i=1}^{N}\int_{\Omega}M_i(|\nabla u_m|)\,.\,\exp(G(|u_m|))\,.\int_{\{|u_m|>j\}}l(u_m)\,dx\\
&\quad+\int_{\Omega}m\,.\,T_m(u_m-\zeta)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(|u_m|))\int_{|u_m|}^{0}\int_{\{|s|>j\}}l(s)\,ds\,dx\\
&\quad\leq\int_{\Omega}\left[f^m+h(x)+\phi(x)\,.\,\frac{l(|u_m|)}{\bar{a}}\right].\,\exp(G(|u_m|))\int_{|u_m|}^{0}\int_{\{|s|>j\}}l(s)\,ds\,dx\\
&\quad+\int_{\Omega}\phi(x)\,.\,\exp(G(|u_m|))\,.\int_{\{|u_m|>j\}}l(u_m)\,dx,
\end{aligned}$$

which implies :

$$\bar{a}\sum_{i=1}^{N}\int_{\Omega}M_i(|\nabla u_m|)\,.\,\exp(G(|u_m|))\,.\,l(u_m)\,.\,\chi_{\{|u_m|>j\}}\,dx\leq c_1\int_{|u_m|}^{0}l(u_m)\,.\,\chi_{\{|u_m|>j\}}\,dx.$$

Therefore,

$$\sum_{i=1}^{N}\int_{\{|u_m|>j\}}l(u_m)\,.\,M_i(|\nabla u_m|)\,dx\leq c_2\int_{|u_m|}^{0}l(|u_m|)\,.\,\chi_{\{|u_m|>j\}}\,dx,$$

and

$$0\leq\int_{\Omega}m\,.\,T_m(u_m-\zeta)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(|u_m|))\int_{|u_m|}^{0}\int_{\{|s|>j\}}l(s)\,ds\,dx\leq c_3, \tag{3.23}$$

and since $l\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$ we deduce that :

$$\lim_{j\to\infty}\sup_{m\in\{1,\cdots,N\}}\sum_{i=1}^{N}\int_{\{|u_m|\}}l(|u_m|)\,.\,M_i(|\nabla u_m|)\,dx=0,$$

by (3.19) and (3.13) we conclude (3.12)

Step 4 : Passing to the limit.

Let $\varphi \in \mathring{W}_M^1(\Omega) \cap L^\infty(\Omega)$ we take the following test function :

$$v = u_m - \eta\, T_j(u_m - \varphi),$$

and $|u_m| - ||\varphi||_\infty < |u_m - \varphi| \leq j$. Then, $\{|u_m - \varphi| \leq j\} \subset \{|u_m| \leq j + ||\varphi||_\infty\}$ we obtain :

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega \sigma_i^m(x, u_m, \nabla u_m) \,.\, \nabla T_j(u_m - \varphi)\, dx + \sum_{i=1}^{N} \int_\Omega \mathfrak{b}_i^m(x, u_m, \nabla u_m) \,.\, T_j(u_m - \varphi)\, dx \\
&\quad + \int_\Omega m \,.\, T_m(u_m - \zeta)^- \,.\, sg_{\frac{1}{m}}(u_m) \,.\, T_j(u_m - \varphi)\, dx \\
&\quad \leq \int_\Omega f^m(x) \,.\, T_j(u_m - \varphi)\, dx,
\end{aligned}$$

which implies that :

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega \sigma_i^m(x, u_m, \nabla u_m) \,.\, \nabla T_j(u_m - \varphi)\, dx \\
&\quad = \sum_{i=1}^{N} \int_\Omega \big[\, \sigma_i^m(x, T_{j+||\varphi||_\infty}(u_m), \nabla T_{j+||\varphi||_\infty}(u_m)) - \sigma_i^m(x, T_{j+||\varphi||_\infty}(u_m), \nabla \varphi) \big] \\
&\qquad\quad \times \nabla T_{j+||\varphi||_\infty}(u_m - \varphi) \,.\, \chi_{\{|u_m - \varphi| < j\}}\, dx \\
&\quad + \sum_{i=1}^{N} \int_\Omega \sigma_i^m(x, T_{j+||\varphi||_\infty}(u_m), \nabla \varphi)\, \nabla T_{j+||\varphi||_\infty}(u_m - \varphi) \,.\, \chi_{\{|u_m - \varphi| < j\}}\, dx,
\end{aligned}$$

by Fatou's Lemma we get :

$$\begin{aligned}
&\liminf_{m\to\infty} \sum_{i=1}^{N} \int_\Omega \sigma_i^m(x, T_{j+||\varphi||_\infty}(u_m), \nabla \varphi)\, \nabla T_{j+||\varphi||_\infty}(u_m - \varphi) \,.\, \chi_{\{|u_m - \varphi| < j\}}\, dx \\
&\qquad = \sum_{i=1}^{N} \int_\Omega \sigma_i^m(x, T_{j+||\varphi||_\infty}(u), \nabla \varphi)\, \nabla T_{j+||\varphi||_\infty}(u - \varphi) \,.\, \chi_{\{|u - \varphi| < j\}}\, dx,
\end{aligned}$$

and the fact that

$$\sum_{i=1}^{N} \sigma_i^m(x, T_{j+||\varphi||_\infty}(u_m), \nabla T_{j+||\varphi||_\infty}(u_m)) \rightharpoonup \sum_{i=1}^{N} \sigma_i^m(x, T_{j+||\varphi||_\infty}(u), \nabla T_{j+||\varphi||_\infty}(u)), \tag{3.24}$$

weakly in $\mathring{W}_M^1(\Omega)$. And since $T_j(u_m - \varphi) \rightharpoonup T_j(u - \varphi)$ weakly in $\mathring{W}_M^1(\Omega)$, and by (3.21) we obtain :

$$\sum_{i=1}^{N} \int_\Omega \mathfrak{b}_i^m(x, u_m, \nabla u_m)\, T_j(u_m - \varphi)\, dx \longrightarrow \sum_{i=1}^{N} \int_\Omega \mathfrak{b}_i(x, u, \nabla u)\, T_j(u - \varphi)\, dx,$$

and

$$\int_\Omega f^m(x)\,T_j(u_m-\varphi)\,dx \longrightarrow \int_\Omega f(x)\,T_j(u-\varphi)\,dx,$$

which completes the proof of Theorem 3.1.2. □

Remark 11. *For the demonstration of the uniqueness solution to this problem ($\mathscr{P}$) in unbounded domain is obtained in [26] with the operator $\mathfrak{b}_i(x,u,\nabla u):\Omega\times\mathbb{R}\times\mathbb{R}^N\longrightarrow\mathbb{R}$ are strictly monotonic, at least for a broad class of lower order term, and in [28] with the operator $\mathfrak{b}_i(x,u,\nabla u):\Omega\times\mathbb{R}\times\mathbb{R}^N\longrightarrow\mathbb{R}$ for $i=1,\cdots,N$ are contraction Lipschitz continuous functions which do not satisfy any sign condition, and*

$$\sum_{i=1}^N\big[\,\sigma_i(x,\xi,\nabla\xi)-\sigma_i(x,\xi',\nabla\xi')\,\big].(\nabla\xi-\nabla\xi')\,dx>0.$$

3.1.4 appendix

Let

$$\begin{aligned}\mathfrak{A}^m:\mathring{W}^1_M(\Omega)&\longrightarrow(\mathring{W}^1_M(\Omega))'\\ v\;&\longmapsto\langle\mathfrak{A}^m(u),v\rangle=\int_\Omega\sum_{i=1}^N\left(\sigma_i^m(x,u,\nabla u).\frac{\partial v}{\partial x_i}+\mathfrak{b}_i^m(x,u,\nabla u).v\right)dx-\int_\Omega f^m(x).v\,dx,\end{aligned}$$

and let denote $L_{\overline{M}}(\Omega)=\prod_{k=1}^N L_{\overline{M}_i}(\Omega)$ with the norm :

$$||\,v\,||_{L_{\overline{M}}(\Omega)}=\sum_{i=1}^N||\,v_i\,||_{\overline{M}_i,\Omega}\qquad v=(v_1,\cdots,v_N)\in L_{\overline{M}}(\Omega).$$

Where $\overline{M_i}(t)$ are N-functions satisfying the Δ_2-conditions.

Sobolev-space $\mathring{W}^1_M(\Omega)$ is the completion of the space $C_0^\infty(\Omega)$.

$$\sigma^m(x,s,\xi)=\big(\,\sigma_1^m(x,s,\xi),\cdots,\sigma_N^m(x,s,\xi)\,\big),$$

and

$$\mathfrak{b}^m(x,s,\xi)=\big(\,\mathfrak{b}_1^m(x,s,\xi),\cdots,\mathfrak{b}_N^m(x,s,\xi)\,\big).$$

Let's show that operator $\mathfrak{A}^m$ is bounded. So, for $u \in \mathring{W}^1_M(\Omega)$, according to (1.9) and (3.5) we get :

$$\begin{aligned}
||\,\sigma^m(x,u,\nabla u)\,||_{L_{\overline{M}}(\Omega)} &= \sum_{i=1}^{N} ||\,\sigma_i^m(x,u,\nabla u)\,||_{L_{\overline{M}_i}(\Omega)} \\
&\leq \sum_{i=1}^{N} \int_\Omega \overline{M}_i(\,\sigma_i^m(x,u,\nabla u)\,)\,dx + N \\
&\leq \tilde{a}(\Omega)\,.\,||\,M(u)\,||_{1,\Omega} + ||\,\varphi||_{1,\Omega} + N. \qquad (3.25)
\end{aligned}$$

Further, for $\sigma^m(x,u,\nabla u) \in L_{\overline{M}_i}(\Omega),\ v \in \mathring{W}^1_M(\Omega)$ using Hölder's inequality we have :

$$\begin{aligned}
|\,\langle \mathfrak{A}^m(u),\,v\rangle_\Omega\,| &\leq 2\,||\,\sigma^m(x,u,\nabla u)\,||_{L_{\overline{M}}(\Omega)}\,.\,||\,v\,||_{\mathring{W}^1_M(\Omega)} \\
&\quad + 2\,||\,\mathfrak{b}^m(x,u,\nabla u)\,||_{L_M(\Omega)}\,.\,||\,v\,||_{\mathring{W}^1_M(\Omega)} + c_0\,.\,||\,v\,||_{\mathring{W}^1_M(\Omega)}. \qquad (3.26)
\end{aligned}$$

Thus, $\mathfrak{A}^m$ is bounded.

And that $\mathfrak{A}^m$ is coercive. So, for $u \in \mathring{W}^1_M(\Omega)$,

$$\begin{aligned}
\langle \mathfrak{A}^m(u),u\rangle_\Omega &= \sum_{i=1}^{N} \int_\Omega \sigma_i^m(x,u,\nabla u)\,.\,\frac{\partial u}{\partial x_i}\,dx + \sum_{i=1}^{N} \int_\Omega \mathfrak{b}_i^m(x,u,\nabla u)\,.\,u^m\,dx \\
&\quad - \int_\Omega f^m(x)\,.\,u\,dx.
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{\langle \mathfrak{A}^m(u),\,u\rangle_\Omega}{||\,u\,||_{\mathring{W}^1_M(\Omega)}} &\geq \frac{1}{||\,u\,||_{\mathring{W}^1_M(\Omega)}}\,.\,\Bigg[\,\bar{a}\sum_{i=1}^{N}\int_\Omega M_i\Bigg(\,\Bigg|\,\frac{\partial u}{\partial x_i}\,\Bigg|\,\Bigg)\,dx - c_1 - c_0 \\
&\qquad - l(u)\,.\,\sum_{i=1}^{N}\int_\Omega M_i\Bigg(\,\Bigg|\,\frac{\partial u}{\partial x_i}\,\Bigg|\,\Bigg)\,dx - \int_\Omega f^m(x)\,dx\Bigg] \\
&\geq \frac{1}{||\,u\,||_{\mathring{W}^1_M(\Omega)}}\,.\,\Bigg[\,(\,\bar{a}(\Omega) - c_2\,)\,.\,\sum_{i=1}^{N}\int_\Omega M_i\Bigg(\,\Bigg|\,\frac{\partial u}{\partial x_i}\,\Bigg|\,\Bigg)\,dx - c_0 - c_1 - c_3\Bigg].
\end{aligned}$$

According to (1.18) we have for all $k > 0$, $\exists\,\alpha_0 > 0$ such that :

$$\mathfrak{b}_i(\,|\,u_{x_i}\,|\,) > k\,\mathfrak{b}_i\Bigg(\frac{|\,u_{x_i}\,|}{||\,u_{x_i}\,||_{M_i,\Omega}}\Bigg), \qquad i = 1,\cdots,N.$$

We take $||\,u_{x_i}\,||_{M_i,\Omega} > \alpha_0 \quad i = 1,\cdots,N.$

Suppose that $||\,u_{x_i}\,||_{\mathring{W}^1_M(\Omega)} \longrightarrow 0$ as $j \to \infty$, we can assume that :

$$||\,u^j_{x_1}\,||_{M_1,\Omega} + \cdots + ||\,u^j_{x_N}||_{M_N,\Omega} \geq N\,\alpha_0.$$

According to (1.9) for $c > 1$, we have :

$$|u^j|\,\mathfrak{b}(|u^j|) < c M(u^j),$$

then, by (1.8) we obtain :

$$\begin{aligned}
\frac{\langle \mathfrak{A}(u^j), u^j\rangle_\Omega}{||u^j||_{\mathring{W}^1_M(\Omega)}} &\geq \frac{\bar{a}(\Omega)-c_2}{N\alpha_0}\cdot\sum_{i=1}^{N}\int_\Omega M_i\left(\left|\frac{\partial u}{\partial x_i}\right|\right)dx - \frac{c_4}{N\alpha_0}\\
&\geq \frac{\bar{a}(\Omega)-c_2}{N\alpha_0}\cdot\sum_{i=1}^{N}\int_\Omega |u^j_{x_i}|\,\mathfrak{b}(|u^j_{x_i}|)\,dx - \frac{c_4}{N\alpha_0}\\
&\geq \frac{(\bar{a}(\Omega)-c_2).k}{cN||u^j_{x_i}||_{M_i}}\cdot\sum_{i=1}^{N}\int_\Omega |u^j_{x_i}|\,\mathfrak{b}_i\left(\frac{|u^j_{x_i}|}{||u^j_{x_i}||_{M_i,\Omega}}\right)dx - \frac{c_4}{N\alpha_0}\\
&\geq \frac{(\bar{a}(\Omega)-c_2).k}{cN}\cdot\sum_{i=1}^{N}\int_\Omega M_i\left(\frac{|u^j_{x_i}|}{||u^j_{x_i}||_{M_i,\Omega}}\right)dx - \frac{c_4}{N\alpha_0}\\
&\geq \frac{(\bar{a}(\Omega)-c_2).k}{cN} - \frac{c_4}{N\alpha_0},
\end{aligned}$$

which shows that $\mathfrak{A}$ is coercive because k is arbitrary.

And finally that $\mathfrak{A}$ is pseudo-monotonic. Following up this assumption and since the space $\mathring{W}^1_M(\Omega)$ is separable, then $\exists (u^j) \in C_0^\infty(\Omega)$ such as :

$$u^j \rightharpoonup u \text{ in } \mathring{W}^1_M(\Omega), \tag{3.27}$$

and

$$\mathfrak{A}(u^j) \rightharpoonup y \text{ in } \big(\mathring{W}^1_M(\Omega)\big)'; \tag{3.28}$$

according to (3.27), we have for all subsequences denoted again by u^j,

$$||u^j||_{\mathring{W}^1_M(\Omega)} \leq c_2,\ j \in \mathbb{N}.$$

Then $(u^j)_{j\in\mathbb{N}}$ is bounded in $\mathring{W}^1_M(\Omega)$, and since $\mathring{W}^1_M(\Omega)$ is continuously and compactly injected into $L_M(\Omega)$, we have that

$$u^j \rightharpoonup u \text{ weakly in } L_M(\Omega),$$

$$u^j \longrightarrow u \text{ a.e. in } \Omega,\ j \in \mathbb{N},$$

and according to (3.21), we have :

$$\sigma_i^m(x,u^j,\nabla u^j) \longrightarrow \sigma_i^m(x,u,\nabla u) \text{ a.e. in } \Omega,\ j\in\mathbb{N},$$

and

$$\mathfrak{b}_i^m(x,u^j,\nabla u^j) \longrightarrow \mathfrak{b}_i^m(x,u,\nabla u) \text{ a.e. in } \Omega,\ j\in\mathbb{N},$$

and

$$m\,.\,T_m(u^j-\zeta)^-\,.\,sg_{\frac{1}{m}}(u^j) \longrightarrow m\,.\,T_m(u-\zeta)^-\,.\,sg_{\frac{1}{m}}(u) \text{ a.e. in } \Omega,\ j\in\mathbb{N},$$

from (3.27) and (3.28), there exist $\tilde{\sigma}^m$ in $L_{\overline{M}}(\Omega)$ such as :

$$\sigma_i^m(x,u^j,\nabla u^j) \rightharpoonup \tilde{\sigma}^m,\ j\in\mathbb{N}, \tag{3.29}$$

and there exist $\tilde{\mathfrak{b}}^m$ in $L_M(\Omega)$ such as :

$$\mathfrak{b}_i^m(x,u^j,\nabla u^j) \rightharpoonup \tilde{\mathfrak{b}}^m,\ j\in\mathbb{N}. \tag{3.30}$$

By (3.15) it is clear that for any $v\in \mathring{W}_M^1(\Omega)$, we get :

$$\begin{aligned}\langle y,v\rangle &= \lim_{j\to\infty}\sum_{i=1}^N\int_\Omega \sigma_i^m(x,u^j,\nabla u^j)\,.\,\nabla v\,dx + \lim_{j\to\infty}\sum_{i=1}^N\int_\Omega \mathfrak{b}_i^m(x,u^j,\nabla u^j)\,.\,v\,dx\\ &= \int_\Omega \tilde{\sigma}^m\,.\,\nabla v\,dx + \int_\Omega \tilde{\mathfrak{b}}^m\,.\,v\,dx,\end{aligned} \tag{3.31}$$

whereof :

$$\begin{aligned}\limsup_{j\to\infty}\langle\mathfrak{A}(u^j),u^j\rangle &= \limsup_{j\to\infty}\{\sum_{i=1}^N\int_\Omega \sigma_i^m(x,u^j,\nabla u^j)\nabla u^j\,dx\\ &\quad + \lim_{j\to\infty}\sum_{i=1}^N\int_\Omega \mathfrak{b}_i^m(x,u^j,\nabla u^j)\,u^j\,dx\} \le \int_\Omega \tilde{\sigma}^m\,\nabla u^j\,dx + \int_\Omega \tilde{\mathfrak{b}}^m\,u^j\,dx.\end{aligned} \tag{3.32}$$

By (3.30), we have :

$$\int_\Omega \mathfrak{b}^m(x,u^j,\nabla u^j)\,u^j\,dx \longrightarrow \int_\Omega \tilde{\mathfrak{b}}^m\,u\,dx. \tag{3.33}$$

Consequently,

$$\limsup_{j\to\infty}\sum_{i=1}^N\int_\Omega \sigma_i^m(x,u^j,\nabla u^j)\nabla u^j\,dx \le \int_\Omega \tilde{\sigma}^m\,\nabla u^j\,dx. \tag{3.34}$$

On the other hand, we have by the condition of monotony :

$$\sum_{i=1}^{N}\left(\sigma_i^m(x,u^j,\nabla u^j) - \sigma_i^m(x,u^j,\nabla u)\right).\nabla(u^j-u) \geq 0,$$

which implies

$$\sum_{i=1}^{N}\left(\sigma_i(x,T_m(u^j),\nabla u^j) - \sigma_i(x,T^j(u^j),\nabla u)\right).\nabla(u^j-u) \geq 0, \tag{3.35}$$

then,

$$\sum_{i=1}^{N}\sigma_i(x,T_m(u^j),\nabla u^j).\nabla u^j \geq \sum_{i=1}^{N}\sigma_i(x,T_m(u^j),\nabla u).\nabla(u^j-u) + \sum_{i=1}^{N}\sigma_i(x,T_m(u^j),\nabla u^j).\nabla u,$$

and by Step 2, we get :

$$\sum_{i=1}^{N}\sigma_i(x,T_m(u^j),\nabla u) \longrightarrow \sum_{i=1}^{N}\sigma_i(x,T_m(u),\nabla u) \quad \text{in } L_{\overline{M}}(\Omega),$$

according to (3.29), we obtain :

$$\liminf_{j\to\infty}\sum_{i=1}^{N}\int_\Omega \sigma_i^m(x,u^j,\nabla u^j).\nabla u^j\,dx \geq \int_\Omega \tilde{\sigma}^m.\nabla u^j\,dx. \tag{3.36}$$

Therefore, from (3.34), we have :

$$\lim_{j\to\infty}\sum_{i=1}^{N}\int_\Omega \sigma_i^m(x,u^j,\nabla u^j).\nabla u^j\,dx = \int_\Omega \tilde{\sigma}^m.\nabla u^j\,dx, \tag{3.37}$$

according to (3.31), (3.33) and (3.36) we get :

$$\langle \mathfrak{A}(u^j),u^j\rangle \longrightarrow \langle y,u\rangle \quad \text{as} \quad j\to\infty.$$

Hence, from (3.37), and (3.21) we obtain :

$$\lim_{j\to\infty}\sum_{i=1}^{N}\int_\Omega\left(\sigma_i^m(x,u^j,\nabla u^j) - \sigma_i^m(x,u^j,\nabla u)\right).\nabla(u^j-u)\,dx = 0.$$

By (3.31) we can conclude that

$$\langle y\,,\,u\rangle = \langle \mathfrak{A}(u)\,,\,u\rangle \quad \forall u\in \mathring{W}_M^1(\Omega).$$

3.2 Existence and Uniqueness of entropy solution in unbounded domain

3.2.1 Introduction

Let ω be an arbitrary domain of $\mathbb{R}^N$, $(N \geq 2)$. In this section, we investigate the existence and uniqueness solution of the following problem

$$(\mathscr{P})\begin{cases} A(u) + \sum_{i=1}^{N} b_i(x,u,\nabla u) = f & \text{in } \omega \ , \\ u \geq \psi & \text{a.e in } \omega \ , \end{cases}$$

where, $A(u) = \sum_{i=1}^{N} (a_i(x,u,\nabla u))_{x_i}$ is a Leray-Lions operator defined on $\mathring{W}^1_B(\omega)$ (is defined as the adherence space $C_0^\infty(\omega)$) into its dual, $B(t) = (B_1(t),\cdots,B_N(t))$ are N-uplet Orlicz functions that satisfy Δ_2-condition, the obstacle ψ is a measurable function belongs to $L^\infty(\omega) \cap \mathring{W}^1_B(\omega)$, and for $i = 1,\cdots,N$, $b_i(x,s,\xi) : \omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ are Carathéodory functions (measurable with respect to x in ω for every (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$ and continuous with respect to (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$ for almost every x in ω) does not satisfy any sign condition and the growth described by the vector N-function $B(t)$. As well as $f \in L^1(\omega)$.

We recall that in the last few decades a tremendous popularity has achieved the investigation of a class of nonlinear unilateral elliptic problem due to their fundamental role in describing several phenomena such as the study of fluid filtration in porous media, constrained heating, elastoplasticity, optimal control and financial mathematics and others, among these large numbers of mathematical articles, see [14, 19, 23, 45] for more details.

When Ω is an unbounded domain, namely without expecting any assumptions on the behaviour when $|x| \longrightarrow +\infty$, Domanska in [51] was investigated the well-posedness of nonlinear elliptic systems of equations generalizing the model equation

$$-\sum_{i=1}^{N} \left(|u_{x_i}(x)|^{p_i-2} u_{x_i}(x) \right)_{x_i} + |u(x)|^{p_0-2} u(x) = f(x),$$

with corresponding indices of non-linearity $p_i > 1$ $(i = \overline{0,n})$. In [22] Bendahmann et al., were solved the problem $(\mathscr{P})$ with $b(x,u,\nabla u) = div(g(u))$ and $g(u)$ a polynomial growth like u^q in

L^p-spaces. For more outcomes concerning the existence of solutions of this class in the Lebesgue Sobolev spaces, we quote [36]. And [26, 28, 29, 48, 72, 74–76] for the classical anisotropic Space.

The oddity of this section, is to continue in this direction and to show the existence and uniqueness of entropy solution for equations $(\mathscr{P})$ governed with growth and described by an N-uplet of N-functions satisfying the Δ_2-condition, within the fulfilling of anisotropic Orlicz spaces. Besides, the challenges coming about due to the absence of some topological properties like the density of bounded or smooth functions.

3.2.2 Statement of the problems

Suppose they have non-negative measurable functions ϕ, $\varphi \in L^1(\omega)$ and $\bar{a}$, $\tilde{a}$ are two constants positive such as for $\xi = (\xi_1, \cdots, \xi_N) \in \mathbb{R}^N$ and $\xi' = (\xi'_1, \cdots, \xi'_N) \in \mathbb{R}^N$ we have

$$\sum_{i=1}^{N} \left[a_i(x,s,\xi) - a_i(x,s,\xi') \right] . (\xi_i - \xi'_i) > 0, \tag{3.38}$$

$$\sum_{i=1}^{N} a_i(x,s,\xi) . \xi_i \geq \bar{a} \sum_{i=1}^{N} B_i(|\xi_i|) - \phi(x), \tag{3.39}$$

$$\sum_{i=1}^{N} |a_i(x,s,\xi)| \leq \tilde{a} \sum_{i=1}^{N} \bar{B}_i^{-1} B_i(|\xi|) + \varphi(x), \tag{3.40}$$

and

$$\sum_{i=1}^{N} |b_i(x,s,\xi)| \leq h(x) + l(s) . \sum_{i=1}^{N} B_i(|\xi|), \tag{3.41}$$

with $\bar{B}(t)$ are the complementary function of $B(t)$, $h \in L^1(\omega)$ and $l : \mathbb{R} \longrightarrow \mathbb{R}^+$ a positive continuous function such that $l \in L^1(\omega) \cap L^\infty(\omega)$.

3.2.3 Existence of entropy solution

This part is devoted to the proofs of our main results which will be split into different steps. For $m \in \mathbb{N}^*$, we define the truncation at height m, $T_m(u) : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$T_m(u) = \begin{cases} u & \text{if } |u| \leq m , \\ m & \text{if } |u| > m . \end{cases}$$

Definition 3.2.1. *A measurable function u is said to be an entropy solution for the problem* $(\mathscr{P})$*, if* $u \in \mathring{W}_B^1(\omega)$ *such that* $u \geq \psi$ *a.e in* ω *and*

$$\sum_{i=1}^{N} \int_{\omega} a_i(x,u,\nabla u) \,.\, \nabla(u-v)\, dx + \sum_{i=1}^{N} \int_{\omega} b_i(x,u,\nabla u) \,.\, (u-v)\, dx$$
$$\leq \int_{\omega} f(x) \,.\, (u-v)\, dx \qquad \forall v \in K_{\psi} \cap L^{\infty}(\omega),$$

where, $K_{\psi} = \{\, u \in \mathring{W}_B^1(\omega)/u \geq \psi \, a.e \text{ in } \omega \,\}$.

We have $f^m \longrightarrow f$ in $L^1(\omega)$, $m \to \infty$, $|\, f^m(x)\,| \leq |\, f(x)\,|$ and for $i = 1,\cdots,N$, $a_i^m(x,u_m,\nabla u_m)$: $(\,\mathring{W}_B^1(\omega)\,)^N \longrightarrow (\,\mathring{W}_{\bar{B}}^{-1}(\omega)\,)^N$ be Carathéodory functions with

$$a_i^m(x,u,\nabla u) = a_i(x,T_m(u),\nabla u),$$

and $b_i^m(x,u_m,\nabla u_m) : \omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ again be Carathéodory functions not satisfying any sign condition, with

$$b^m(x,u,\nabla u) = \frac{b(x,u,\nabla u)}{1+\frac{1}{m}\,|\,b(x,u,\nabla u)\,|},$$

and

$$|\,b^m(x,u,\nabla u)\,| = |\,b(x,T_m(u),\nabla u)\,| \leq m \text{ for all } m \in \mathbb{N}^*, \tag{3.42}$$

and for all $v \in \mathring{W}_B^1(\omega)$, we consider the following approximate problem

$$(\mathscr{P}_m) : \sum_{i=1}^{N} \int_{\omega} a_i^m(x,u_m,\nabla u_m) \,.\, \nabla(u_m-v)\, dx + \sum_{i=1}^{N} \int_{\omega} b_i^m(x,u_m,\nabla u_m) \,.\, (u_m-v)\, dx$$
$$+ \int_{\omega} m \,.\, T_m(u_m-\psi)^- \,.\, sg_{\frac{1}{m}}(u_m) \,.\, (u_m-v)\, dx = \int_{\omega} f^m(x) \,.\, (u_m-v)\, dx,$$

with $sg_m(s) = \frac{T_m(s)}{m}$.

Theorem 3.2.1. *Assume that conditions* (3.38) - (3.41) *and* (2.17) *hold true, then there exists at least one solution of the approximate problem* $(\mathscr{P}_m)$.

Proof. See appendix. □

Theorem 3.2.2. *Under assumptions* (3.38) - (3.41)*, the problem* $(\mathscr{P})$ *has at least one entropy solution.*

Proof. Let $R > 0$ and $\omega(R) = \{\, x \in \omega : |\,x\,| \leq R \,\}$. Note by $\mathrm{h}(t) = \left(\prod_{i=1}^{N} \frac{B_i^{-1}(t)}{t} \right)^{\frac{1}{N}}$ and we assume that $\int_0^1 \frac{\mathrm{h}(t)}{t}\,dt$ converge, so we consider the N-functions $B^*(z)$ defined by

$$(B^*)^{-1}(z) = \int_0^{|z|} \frac{\mathrm{h}(t)}{t}\,dt\,.$$

Lemma 3.2.1. *[72] Let $u \in \mathring{W}_B^1(\omega(R))$. If*

$$\int_1^{\infty} \frac{\mathrm{h}(t)}{t}\,dt = \infty, \tag{3.43}$$

then, $\mathring{W}_B^1(\omega(R)) \subset L_{B^*}(\omega(R))$ and $||u||_{B^*,\omega(R)} \leq \frac{N-1}{N}\,||u||_{\mathring{W}_B^1(\omega(R))}$.
If

$$\int_1^{\infty} \frac{\mathrm{h}(t)}{t}\,dt < \infty, \tag{3.44}$$

then, $\mathring{W}_B^1(\omega(R)) \subset L_\infty(\omega(R))$ and $||u||_{\infty,\omega(R)} \leq \beta\,||u||_{\mathring{W}_B^1(\omega(R))}$, with $\beta = \int_0^{\infty} \frac{h(t)}{t}\,dt$.

Step 1 : A priori estimate of $\{u_m\}$.

Let $v = u_m - \eta\, \exp(G(u_m))\,.\,T_k(u_m - v_0)^+$ where $G(s) = \int_0^s \frac{l(t)}{\bar{a}}\,dt$, $k > 0$ and $\eta \geq 0$, we have $v \in \mathring{W}_B^1(\Omega)$ and for η small enough we deduce that $v \geq \psi$. Thus v is an admissible test function in $(\mathscr{P}_m)$ and we get for all $v_0 \in K_\psi \cap L^\infty(\Omega)$ that

$$\begin{aligned}
&\sum_{i=1}^N \int_\omega a_i^m(x,u_m,\nabla u_m)\,.\,\nabla(\,\exp(G(u_m))\,.\,T_k(u_m - v_0)^+\,)\,dx\\
&\quad + \sum_{i=1}^N \int_\omega b_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(u_m))\,.\,T_k(u_m - v_0)^+\,dx\\
&\quad + \int_\omega m\,.\,T_m(u_m - \psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(u_m))\,.\,T_k(u_m - v_0)^+\,dx\\
&\quad \leq \int_\omega f^m(x)\,.\,\exp(G(u_m))\,.\,T_k(u_m - v_0)^+\,dx,
\end{aligned}$$

then,

$$\begin{aligned}
&\sum_{i=1}^N \int_\omega a_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\,.\,\frac{l(u_m)}{\bar{a}}\,.\,\exp(G(u_m))\,.\,T_k(u_m - v_0)^+\,dx\\
&\quad + \sum_{i=1}^N \int_\omega a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(u_m))\,.\,\nabla T_k(u_m - v_0)^+\,dx\\
&\quad + \int_\omega m\,.\,T_m(u_m - \psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(u_m))\,.\,T_k(u_m - v_0)^+\,dx\\
&\leq \sum_{i=1}^N \int_\omega |\,b_i^m(x,u_m,\nabla u_m)\,|\,.\,\exp(G(u_m))\,.\,T_k(u_m - v_0)^+\,dx + \int_\omega f^m(x)\,.\,\exp(G(u_m))\,.\,T_k(u_m - v_0)^+\,dx,
\end{aligned}$$

by (3.39) and (3.41), we obtain

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(u_m))\,.\,\nabla T_k(u_m-v_0)^+\,dx\\
&\quad+\int_{\omega} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(u_m))\,.\,T_k(u_m-v_0)^+\,dx\\
&\quad\leq\int_{\omega}\big[h(x)+f^m(x)+\phi(x)\,.\,\frac{l(u_m)}{\bar{a}}\big]\,.\,\exp(G(u_m))\,.\,T_k(u_m-v_0)^+\,dx,
\end{aligned}$$

so,

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\{|u_m-v_0|\leq k\}} a_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\,.\,\exp(G(u_m))\,dx\\
&\quad-c\sum_{i=1}^{N}\int_{\{|u_m-v_0|\leq k\}} a_i^m(x,u_m,\nabla u_m)\,.\,\frac{\nabla v_0}{c}\,.\,\exp(G(u_m))\,dx\\
&\quad+\int_{\omega} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(u_m))\,.\,T_k(u_m-v_0)^+\,dx\\
&\quad\leq\int_{\omega}\big[h(x)+f^m(x)+\phi(x)\,.\,\frac{l(u_m)}{\bar{a}}\big]\,.\,\exp(G(u_m))\,.\,T_k(u_m-v_0)^+\,dx,
\end{aligned}$$

where, c is a constant such that $0<c<1$, and since h, f^m, $\phi\in L^1(\omega)$ we deduce that

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\{|u_m-v_0|\leq k\}} a_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\,.\,\exp(G(u_m)\,dx\\
&+\int_{\omega} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(u_m))\,.\,T_k(u_m-v_0)^+\,dx\\
&\leq -c\sum_{i=1}^{N}\int_{\{|u_m-v_0|\leq k\}}\Big[a_i^m(x,u_m,\nabla u_m)-a_i^m(x,u_m,\frac{\nabla v_0}{c})\Big]\,.\,\nabla(u_m-\frac{\nabla v_0}{c})\,.\,\exp(G(u_m))\,dx\\
&+c\sum_{i=1}^{N}\int_{\{|u_m-v_0|\leq k\}} a_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\,.\,\exp(G(u_m))\,dx\\
&+c\sum_{i=1}^{N}\int_{\{|u_m-v_0|\leq k\}}|a_i^m(x,u_m,\frac{\nabla v_0}{c})|\,.\,|\nabla(u_m-\frac{\nabla v_0}{c})|\,.\,\exp(G(u_m))\,dx+c_1,
\end{aligned}$$

by (3.38)

$$\begin{aligned}
&(1-c)\sum_{i=1}^{N}\int_{\{|u_m-v_0|\leq k\}} a_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\,.\,\exp(G(u_m))\,dx\\
&\quad+\int_{\omega} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(u_m))\,.\,T_k(u_m-v_0)^+\,dx\\
&\leq c\sum_{i=1}^{N}\int_{\{|u_m-v_0|\leq k\}}|a_i^m(x,u_m,\frac{\nabla v_0}{c})|\,.\,|\nabla u_m|\,.\,\exp(G(u_m))\,dx\\
&\quad+c\sum_{i=1}^{N}\int_{\{|u_m-v_0|\leq k\}}|a_i^m(x,u_m,\frac{\nabla v_0}{c})|\,.\left|\frac{\nabla v_0}{c}\right|.\,\exp(G(u_m))\,dx+c_1,
\end{aligned}$$

since $\frac{\nabla v_0}{c} \in \mathring{W}^1_B(\omega)$, and by (1.4), (3.40), (1.2) and the fact that $\exp(G(\pm\infty)) \leq \exp\left(\frac{||l||_{L^1(\mathbb{R})}}{\bar{a}}\right)$ we have

$$(1-c)\sum_{i=1}^{N}\int_{\{|u_m-v_0|\leq k\}} a_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\,dx+\int_{\{|u_m-v_0|\geq 0\}} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,dx$$
$$\leq \frac{\tilde{a}\,(1-c)}{2}\sum_{i=1}^{N}\int_{\{|u_m-v_0|\leq k\}} B_i(\nabla u_m)\,dx+c_2(k)\,.\,c_1,$$

where, $c_2(k)$ is a positive constant which depends only on k.

Finally, by (3.39) we obtain

$$\sum_{i=1}^{N}\int_{\{|u_m-v_0|\leq k\}} B_i(\nabla u_m)\,dx \leq c_3\,.\,k, \tag{3.45}$$

and

$$0 \leq \int_{\{|u_m-v_0|\geq 0\}} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,dx \leq c_1. \tag{3.46}$$

Similarly, taking $v = u_m - \eta\,.\,\exp(G(u_m))\,.\,T_k(u_m-v_0)^-$ as a test function in $(\mathscr{P}_m)$, we obtain

$$\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u_m,\nabla u_m)\,.\,\nabla(\exp(G(u_m))\,.\,T_k(u_m-v_0)^-)\,dx$$
$$+\sum_{i=1}^{N}\int_{\omega} b_i^m(x,u_m,\nabla u_m)\,.\,\exp(-G(u_m))\,.\,T_k(u_m-v_0)^-\,dx$$
$$+\int_{\omega} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(-G(u_m))\,.\,T_k(u_m-v_0)^-\,dx$$
$$\leq \int_{\omega} f^m(x)\,.\,\exp(-G(u_m))\,.\,T_k(u_m-v_0)^-\,dx,$$

and using same techniques, we obtain also

$$\sum_{i=1}^{N}\int_{\{|u_m-v_0|\geq k\}} B_i(\nabla u_m)\,dx \leq c_4\,.\,k, \tag{3.47}$$

and

$$0 \leq \int_{\{|u_m-v_0|\leq 0\}} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,dx \leq c_5. \tag{3.48}$$

And by (3.45), (3.46), (3.47) and (3.48) we conclude that

$$\int_{\Omega} B(\nabla T_k(u_m))\,dx \leq c\,.\,k, \tag{3.49}$$

with c_3, c_4, c_5, c_6 are positive constants. □

Step 2 : Almost everywhere convergence of $\{u_m\}$.

Firstly we prove that meas$\{x \in \omega : |u_m| \geq k\} \to 0$.

According to Lemma 3.2.1, we have

$$\begin{aligned} ||T_k(u_m)||_{B^*} &\leq c\,||\nabla T_k(u_m)||_B \\ &\leq c\,.\,\varepsilon(k) \int_\omega B(\nabla T_k(u_m))\,dx \\ &\leq c\,.\,\varepsilon(k)\,.\,k \quad \text{for } k > 1, \end{aligned} \tag{3.50}$$

with c be a positive constant and $\varepsilon(k) \to 0$ when $k \to \infty$. By (3.50) we obtain

$$\begin{aligned} B^*\left(\frac{k}{||T_k(u_m)||_{B^*}}\right) \text{meas}\{x \in \omega : |u_m| \geq k\} &\leq \int_\omega B^*\left(\frac{T_k(u_m)}{||T_k(u_m)||_{B^*}}\right) dx \\ &\leq \int_\omega B^*\left(\frac{k}{||T_k(u_m)||_{B^*}}\right) dx. \end{aligned} \tag{3.51}$$

Thus, we deduce that

$$B^*\left(\frac{k}{||T_k(u_m)||_{B^*}}\right) \longrightarrow \infty \text{ when } k \to \infty.$$

Hence

$$\text{meas}\{x \in \omega : |u_m| \geq k\} \to 0 \text{ as } k \to \infty \text{ for all } m \in \mathbb{N}.$$

Secondly we show that for all $\{u_m\}$ measurable function on ω such that

$$T_k(u_m) \in \mathring{W}^1_B(\omega) \qquad \forall k \geq 1,$$

we have

$$\text{meas}\{x \in \omega : B(\nabla u_m) \geq \alpha\} \to 0 \text{ as } \alpha \to \infty. \tag{3.52}$$

In the beginning

$$\begin{aligned} \text{meas}\{x \in \omega : B(\nabla u_m) \geq 0\} = &\,\text{meas}\{\{x \in \omega : |u_m| \geq k, B(\nabla u_m) \geq \alpha\} \\ &\cup \{x \in \omega : |u_m| < k, B(\nabla u_m) \geq \alpha\}\}, \end{aligned}$$

if we denote

$$g(\alpha, k) = \text{meas}\{x \in \omega : |u_m| \geq k, B(\nabla u_m) \geq \alpha\},$$

we have

$$\text{meas}\{x \in \omega : |u_m| < k, B(\nabla u_m) \geq \alpha\} = g(\alpha, 0) - g(\alpha, k),$$

then

$$\int_{\{x\in\omega:|u_m|<k\}} B(\nabla u_m)\, dx = \int_0^\infty \big(g(\alpha,0) - g(\alpha,k)\big)\, d\alpha \leq c\,.k, \tag{3.53}$$

with $\alpha \to g(\alpha,k)$ is a decreasing map, then

$$\begin{aligned} g(\alpha,0) &\leq \frac{1}{\alpha}\int_0^\alpha g(\alpha,0)\, d\alpha \\ &\leq \frac{1}{\alpha}\int_0^\alpha \big(g(\alpha,0) - g(\alpha,k)\big)\, d\alpha + \frac{1}{\alpha}\int_0^\alpha g(\alpha,k)\, d\alpha \\ &\leq \frac{1}{\alpha}\int_0^\alpha \big(g(\alpha,0) - g(\alpha,k)\big)\, d\alpha + g(0,k), \end{aligned} \tag{3.54}$$

according to (3.53) and (3.54) we have

$$g(\alpha,0) \leq \frac{c\,.k}{\alpha} + g(0,k),$$

like [67] we obtain

$$\lim_{k\to\infty} g(0,k) = 0.$$

Hence

$$g(\alpha,0) \to 0 \text{ as } \alpha \to \infty.$$

We must now demonstrate the convergence almost everywhere of $\{u_m\}$

$$u_m \longrightarrow u \text{ almost everywhere in } \omega.$$

Let $g(k) = \sup_{m\in\mathbb{N}} \text{meas}\{x \in \omega : |u_m| > k\} \to 0$ as $k \to \infty$. Since ω is unbounded domain in $\mathbb{R}^N$, we define

$$\eta_R(x) = \begin{cases} 1 & \text{if } x < R\,, \\ R+1-\alpha & \text{if } R \leq x < R+1\,, \\ 0 & \text{if } x \geq R+1\,. \end{cases}$$

For R, $k > 0$, we have by (1.6)

$$\begin{aligned}\int_{\omega} B(\nabla\eta_R(|x|)\,.\,T_k(u_m))\,dx &\leq c\int_{\{x\in\omega:\,|u_m|<k\}} B(\nabla u_m)\,dx\\ &\quad + c\int_{\omega} B(T_k(u_m)\,.\,\nabla\eta_R(|x|)\,dx\\ &\leq c(k,R),\end{aligned}$$

which implies that the sequence $\{\,\eta_R(|x|)\,T_k(u_m)\}$ is bounded in $\mathring{W}^1_B(\omega(R+1))$ and by embedding Theorem, for an N-function P with $P \ll B$ we have

$$\mathring{W}^1_B(\omega(R+1)) \hookrightarrow L_P(\omega(R+1)),$$

and since $\eta_R = 1$ in $\omega(R)$, we have

$$\eta_R\,T_k(u_m) \longrightarrow v_k \text{ in } L_P(\omega(R+1)) \text{ as } m \longrightarrow \infty.$$

For $k \in \mathbb{N}^*$,

$$T_k(u_m) \longrightarrow v_k \text{ in } L_P(\omega(R+1)) \text{ as } m \longrightarrow \infty,$$

by diagonal process, we prove that there is $u : \omega \longrightarrow \mathbb{R}$ measurable such that $u_m \longrightarrow u$ a.e in ω.

Lemma 3.2.2. *[79] Let an N-function $\bar{B}(t)$ satisfy the Δ_2-condition and u_m, $m \geq 1$, and u be two functions of $L_B(\omega)$ such as*

$$||\,u_m\,||_B \leq c \qquad m = 1,2,\cdots.$$

$$u_m \longrightarrow u \textit{ almost everywhere in } \omega,\ m \to \infty.$$

Then,

$$u_m \rightharpoonup u \textit{ weakly in } L_B(\omega) \textit{ as } m \to \infty.$$

Hence,

$$meas\{\,x \in \omega : |\,u_m\,| \geq k\,\} \longrightarrow 0 \textit{ when } k \to \infty \textit{ for all } m \in \mathbb{N}.$$

Step 3 : <u>Weak convergence of the gradient.</u>

Since $\mathring{W}^1_B(\omega)$ reflexive, then, there exists a subsequence

$$T_k(u_m) \rightharpoonup v \text{ weakly in } \mathring{W}^1_B(\omega), \quad m \to \infty,$$

and since

$$\mathring{W}_B^1(\omega) \hookrightarrow L_B(\omega),$$

we have

$$\nabla T_k(u_m) \rightharpoonup \nabla u_m \text{ in } L_B(\omega) \text{ as } m \to \infty,$$

since

$$u_m \longrightarrow u \text{ almost everywhere in } \omega \text{ as } m \to \infty, \tag{3.55}$$

implies the local convergence in measure and, therefore, the local Cauchy property of u_m in measure

$$meas\{\,\omega(R) : |\,u_m - u_n\,| \geq k\,\} \to 0 \text{ as } m, n \to \infty \text{ for any } k > 0. \tag{3.56}$$

Proving that

$$\nabla u_m \longrightarrow \nabla u \text{ locally in measure as } m \to \infty. \tag{3.57}$$

For that, we use the ideas of Evans [57], Demangel-Hebey [51] and Koznikova L. M. [74, 75]. Let $\delta > 0$ be given. By Egoroff's Theorem, there exists $E_{\delta,k,\alpha} \subset\subset \Omega$ such that

$$\begin{aligned} E_{\delta,k,\alpha}(R) = \{\,\omega(R) : |\,u_m - u_n\,| < k, B(\nabla u_m) < \alpha, B(\nabla u_n) < \alpha, |\,u_m\,| \leq \alpha, \\ |\,u_n\,| \leq \alpha, |\,\nabla(u_m - u_n)\,| \geq \delta\,\}. \end{aligned}$$

$$\begin{aligned} \{\,\omega(R) : |\,\nabla(u_m - u_n)\,| \geq \delta\,\} \subset & \{\,\omega : B(\nabla u_m) > \alpha\,\} \cup \{\,\omega : B(\nabla u_n) > \alpha\,\} \\ & \cup \{\,\omega(R) : |\,u_m - u_n\,| \geq k\,\} \cup \{\,\omega : |\,u_m\,| > \alpha\,\} \\ & \cup \{\,\omega : |\,u_n\,| > \alpha\,\} \cup E_{\delta,k,\alpha}(R). \end{aligned}$$

Then, by Lemma 3.2.2 and (3.52) we obtain that

$$\begin{aligned} meas\,\{\,\omega(R) : |\,\nabla(u_m - u_n)\,| \geq \delta\,\} \leq & 4\varepsilon + meas\, E_{\delta,k,\alpha}(R) \\ & + meas\,\{\,\omega(R) : |\,u_m - u_n\,| \geq k\,\} \ \ \forall n, m \in \mathbb{N}^*. \end{aligned} \tag{3.58}$$

According to (3.38) and the fact that a continuous function on a compact set achieves the lowest value, then, there exists a function $\theta(x) > 0$ almost everywhere in ω, such as, for $B(\xi) \leq \alpha$, $B(\xi') \leq$

α, $|s| \leq \alpha$ and for $i = 1, \cdots, N$, $|\xi_i - \xi_i'| \geq k$, we have

$$\sum_{i=1}^{N} \left[a_i^m(x,s,\xi) - a_i^m(x,s,\xi') \right].(\xi_i - \xi_i') \geq \theta(x), \tag{3.59}$$

holds. Writing $(\mathscr{P}_m)$ twice for $\{u_m\}$ and $\{u_n\}$ and subtracting the second relation from the first and according to (3.42), (3.46), (3.48) and (3.55) we obtain

$$\sum_{i=1}^{N} \int_{\omega} \left[a_i^m(x,u_m,\nabla u_m) - a_i^m(x,u_n,\nabla u_n) \right].\nabla(u_m - u_n - v)\, dx = 0.$$

Considering the following test function

$$v = u_m - u_n - \eta_R(|x|)\,\eta_\alpha(|u_n|)\,\eta_\alpha(|u_m|)\,\exp(G(|u_m - u_n|))\,T_\delta(u_m - u_n).$$

Further on, applying (3.59), we get

$$\begin{aligned}
\int_{E_{\delta,k,\alpha}(R)} \theta(x)\, dx \leq & \sum_{i=1}^{N} \int_{E_{\delta,k,\alpha}(R)} \left[a_i^m(x,u_m,\nabla u_m) - a_i^m(x,u_m,\nabla u_n) \right] \\
& \times \nabla(\, \eta_R(|x|)\,\eta_\alpha(|u_n|)\,\eta_\alpha(|u_m|)\,\exp(G(|u_m - u_n|))\,T_\delta(u_m - u_n)\,)\, dx \\
\leq & \sum_{i=1}^{N} \int_{\{\omega:|u_m - u_n| < k\}} \eta_R(|x|)\,\eta_\alpha(|u_n|)\,\eta_\alpha(|u_m|)\,\exp(G(|u_m - u_n|)) \\
& \times \left[a_i^m(x,u_m,\nabla u_m) - a_i^m(x,u_m,\nabla u_n) \right].\nabla(u_m - u_n)\, dx \\
= & A_1^m(x) + A_2^m(x),
\end{aligned}$$

with,

$$\begin{aligned}
A_1^m(x) = & \sum_{i=1}^{N} \int_{\{|u_m - u_n| < k\}} \eta_R(|x|)\,\eta_\alpha(|u_n|)\,\eta_\alpha(|u_m|)\,\exp(G(|u_m - u_n|)) \\
& \times \left[a_i^m(x,u_m,\nabla u_m) - a_i^m(x,u_n,\nabla u_n) \right].\nabla(u_m - u_n)\, dx,
\end{aligned}$$

and

$$\begin{aligned}
A_2^m(x) = & \sum_{i=1}^{N} \int_{\{|u_m - u_n| < k\}} \eta_R(|x|)\,\eta_\alpha(|u_n|)\,\eta_\alpha(|u_m|)\,\exp(G(|u_m - u_n|)) \\
& \times \left[a_i^m(x,u_n,\nabla u_n) - a_i^m(x,u_m,\nabla u_n) \right].\nabla(u_m - u_n)\, dx.
\end{aligned}$$

Since $B(u)$ satisfy the Δ_2-condition, then by (1.9) we have

$$\int_{\omega} B(u)\,dx \leq c_0\,||\,u\,||_{B,\omega}. \tag{3.60}$$

According to Lemma 3.2.2, we get

$$||\,u_m\,||_{\mathring{W}^1_B(\omega)} \leq c_1 \qquad m \in \mathbb{N}^*, \tag{3.61}$$

and

$$||\,B(\nabla u_m)\,||_1 \leq c_2 \qquad m \in \mathbb{N}^*. \tag{3.62}$$

And using (1.9) and (3.40) we have

$$\begin{aligned} ||\,a(x,u,\nabla u)\,||_{L_{\bar{B}}(\omega)} &= \sum_{i=1}^{N} ||\,a_i(x,u,\nabla u)\,||_{L_{\bar{B}_i}(\omega)} \\ &\leq \sum_{i=1}^{N} \int_{\omega} \bar{B}_i(\,a_i(x,u,\nabla u)\,)\,dx + N \\ &\leq c_3\,||\,B(u)\,||_{1,\omega} + ||\,\varphi\,||_{1,\omega} + N \\ &\leq c_4. \end{aligned} \tag{3.63}$$

Hence,

$$\begin{aligned} A_1^m(x) = &\sum_{i=1}^{N} \int_{\{\,\omega:|\,u_m-u_n\,|<k,|x|<R,|\,u_m\,|<\alpha,|\,u_n\,|<\alpha\}} \exp(G(|\,u_m-u_n\,|)) \\ &\times \left[\,a_i^m(x,u_m,\nabla u_m) - a_i^m(x,u_n,\nabla u_n)\,\right].\nabla(u_m-u_n)\,dx \\ &+ \sum_{i=1}^{N} \int_{\{\,\omega:\,|\,u_m-u_n\,|<k, R\leq|x|\leq R+1,\,\alpha\leq|\,u_m\,|\leq\alpha+1,\,\alpha\leq|\,u_n\,|\leq\alpha+1\}} (R+1-|\,x\,|) \\ &\times (\alpha+1-|\,u_n\,|)\,.\,(\alpha+1-|\,u_m\,|)\,.\,\exp(G(|\,u_m-u_n\,|)) \\ &\times \left[\,a_i^m(x,u_m,\nabla u_m) - a_i^m(x,u_n,\nabla u_n)\,\right].\nabla(u_m-u_n)\,dx, \end{aligned}$$

since, $\exp(G(\pm\infty)) \leq \exp\left(\frac{||\,l\,||_{L^1(\mathbb{R})}}{\bar{a}}\right)$ and according to (3.61), (3.62), (3.63) and (1.10) we obtain that

$$A_1^m(x) \leq c_1(R,\,\alpha)\,.\,k,$$

the same for $A_2^m(x)$ we get

$$A_2^m(x) \leq c_2(R,\,\alpha)\,.\,k.$$

Then,

$$\int_{E_{\delta,k,\alpha}(R)} \theta(x)\, dx \leq c_3(R,\alpha)\,.k. \tag{3.64}$$

For any arbitrary $\delta > 0$ for fixed m and α, choosing k from (3.64) we establish the following inequality

$$\int_{E_{\delta,k,\alpha}(R)} \theta(x)\, dx < \delta.$$

Applying the Lemma 1.1.3, for any $\varepsilon > 0$, we find

$$meas\, E_{\delta,k,\alpha}(R) < \varepsilon. \tag{3.65}$$

In addition, according to (3.56), we have

$$meas\{\,\omega(R) : |\,u_m - u_n\,| \geq k\} < \varepsilon,\;\; m,n > 0. \tag{3.66}$$

Combining (3.58), (3.65) and (3.66) we deduce the inequality

$$meas\{\,\omega(R) : |\,\nabla(u_m - u_n)\,| \geq \delta\,\} < 6\varepsilon,\;\; n,m > 0.$$

Hence, the sequence $\{\,\nabla u_m\,\}$ is fundamental in measure on the set $\omega(R)$ for any $R > 0$. This implies (3.57) and the selective convergence,

$$\nabla u_m \longrightarrow \nabla u \text{ almost everywhere in } \omega,\;\; m \to \infty. \tag{3.67}$$

Then, we obtain for any fixed $k > 0$

$$\nabla T_k(u_m) \longrightarrow \nabla T_k(u) \text{ almost everywhere in } \omega \text{ as } m \to \infty.$$

Applying Lemma 3.2.2, we have the following weak convergence

$$\nabla T_k(u_m) \rightharpoonup \nabla T_k(u) \text{ in } L_B(\omega) \text{ as } m \to \infty.$$

Proposition 3.2.1. *Suppose that conditions* (3.38) - (3.41) *are satisfied and let* $(u_m)_{m\in\mathbb{N}}$ *be a sequence in* $\mathring{W}^1_B(\omega(R))$ *such that*
(a) $u_m \rightharpoonup u$ *in* $\mathring{W}^1_B(\omega(R))$.

(b) $a^m(x,u_m,\nabla u_m)$ *is bounded in* $L_{\bar{B}}(\omega(R))$.

(c) $\sum_{i=1}^{N}\int_{\omega(R)}\left[a_i^m(x,u_m,\nabla u_m)-a_i^m(x,u_m,\nabla u\chi_\varepsilon)\,.\nabla(u_m-u\chi_\varepsilon)\;dx\to 0\right.$ *as* $\varepsilon\to+\infty$ *(* χ_ε *the characteristic function of* $\omega_\varepsilon(R)=\{x\in\omega;\,|\nabla u|\le\varepsilon\}$ *). Then,*

$$B(|\nabla u_m|)\longrightarrow B(|\nabla u|)\ \textit{in}\ L^1(\omega(R)).$$

Proof. Let $\varepsilon>0$ fixed, and $\eta>\varepsilon$, then from (3.38) we have

$$\begin{aligned}
0&\le\sum_{i=1}^{N}\int_{\omega_\eta(R)}\left[a_i^m(x,u_m,\nabla u_m)-a_i^m(x,u_m,\nabla u)\right].\nabla(u_m-u)\;dx\\
&\le\sum_{i=1}^{N}\int_{\omega_\varepsilon(R)}\left[a_i^m(x,u_m,\nabla u_m)-a_i^m(x,u_m,\nabla u)\right].\nabla(u_m-u)\;dx\\
&=\sum_{i=1}^{N}\int_{\omega_\varepsilon(R)}\left[a_i^m(x,u_m,\nabla u_m)-a_i^m(x,u_m,\nabla u\chi_\varepsilon)\right].\nabla(u_m-u\chi_\varepsilon)\;dx\\
&\le\sum_{i=1}^{N}\int_{\omega(R)}\left[a_i^m(x,u_m,\nabla u_m)-a_i^m(x,u_m,\nabla u\chi_\varepsilon)\right].\nabla(u_m-u\chi_\varepsilon)\;dx,
\end{aligned}$$

using the condition (c) we get

$$\lim_{m\to\infty}\sum_{i=1}^{N}\int_{\omega(R)}\left[a_i^m(x,u_m,\nabla u_m)-a_i^m(x,u_m,\nabla u\chi_\varepsilon)\right].\nabla(u_m-u\chi_\varepsilon)\;dx=0,$$

proceeding as in [67], we obtain $\nabla u_m\longrightarrow\nabla u$, by letting $\varepsilon\longrightarrow\infty$ we get

$$\nabla u_m\chi_\varepsilon\longrightarrow\nabla u.$$

So, since

$$\begin{aligned}
\sum_{i=1}^{N}\int_{\omega(R)}a_i^m(x,u_m,\nabla u_m)\,.\nabla u_m\;dx&=\sum_{i=1}^{N}\int_{\omega(R)}\left[a_i^m(x,u_m,\nabla u_m)-a_i^m(x,u_m,\nabla u\chi_\varepsilon)\right].\nabla(u_m-u\chi_\varepsilon)\;dx\\
&+\sum_{i=1}^{N}\int_{\omega(R)}a_i^m(x,u_m,\nabla\chi_\varepsilon)\,.\nabla(u_m-u\chi_\varepsilon)\;dx+\sum_{i=1}^{N}\int_{\omega(R)}a_i^m(x,u_m,\nabla u_m)\,.\nabla u\chi_\varepsilon\;dx,
\end{aligned}$$

using (b), we have

$$\sum_{i=1}^{N}a_i^m(x,u_m,\nabla u_m)\rightharpoonup\sum_{i=1}^{N}a_i^m(x,u,\nabla u)\text{ weakly in }(L_{\bar{B}}(\omega(R)))^N.$$

Therefore

$$\sum_{i=1}^{N}\int_{\omega(R)} a_i^m(x,u_m,\nabla u_m)\,.\,\nabla u\chi_\varepsilon\,dx \longrightarrow \sum_{i=1}^{N}\int_{\omega(R)} a_i^m(x,u,\nabla u)\,.\,\nabla u\,dx \text{ as } m\to\infty,\ \varepsilon\to\infty.$$

So,

$$\sum_{i=1}^{N}\int_{\omega(R)} \left[a_i^m(x,u_m,\nabla u_m)-a_i^m(x,u_m,\nabla u\chi_\varepsilon)\right].\,\nabla(u_m-\nabla u\chi_\varepsilon)\,dx \longrightarrow 0 \text{ as } m\to\infty,\ \varepsilon\to\infty,$$

and

$$\sum_{i=1}^{N}\int_{\omega(R)} a_i^m(x,u_m,\nabla u\chi_\varepsilon)\,.\,\nabla(u_m-u\chi_\varepsilon)\,dx \longrightarrow 0 \text{ as } m\to\infty,\ \varepsilon\to\infty.$$

Thus,

$$\lim_{m\to\infty}\sum_{i=1}^{N}\int_{\omega(R)} a_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\,dx = \sum_{i=1}^{N}\int_{\omega(R)} a_i^m(x,u,\nabla u)\,.\,\nabla u\,dx,$$

from (3.39), and the vitali's Theorem, we get

$$\bar{a}\sum_{i=1}^{N}\int_{\omega(R)} B_i(|\nabla u_m|)\,dx-\int_{\omega(R)}\phi(x)\,dx \geq \bar{a}\sum_{i=1}^{N}\int_{\omega(R)} B_i(|\nabla u|)\,dx-\int_{\omega(R)}\phi(x)\,dx.$$

Consequently, by Lemma 2.6 in [12] and (3.67), we get

$$B(|\nabla u_m|)\longrightarrow B(|\nabla u|) \text{ in } \mathring{W}_B^1(\omega(R)),$$

thanks to lemma 1 (see [72]) and (3.67), we have

$$B(|\nabla u_m|)\longrightarrow B(|\nabla u|) \text{ in } L^1(\omega(R)).$$

□

Step 4 : **Strong convergence of the gradient.**

In this step we consider again the following test function

$$v = u_m+\eta\,\exp(G(|u_m|))\,(T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m),$$

with, $h_j(u_m)) = 1 - |\, T_1(u_m - T_j(u_m))\,| = \begin{cases} 1 & \text{if } \{|u_m| \geq j\}\ , \\ 0 & \text{if } \{|u_m| \geq j+1\}\ , \\ j+1-|u_m| & \text{if } \{j \leq |u_m| \leq j+1\}, \end{cases}$

and, $|T_k(u_m) - T_k(u)|$ at the same sign when $u_m \in \{\, |u_m| > k\}$ where, $j \geq k > 0$ and η small enough, we obtain

$$
\begin{aligned}
&\sum_{i=1}^{N} \int_{\omega} a_i^m(x,u_m,\nabla u_m)\,.\,\nabla(\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m))\,dx \\
&\quad + \int_{\omega} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m)\,dx \\
&\quad + \sum_{i=1}^{N} \int_{\omega} b_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m)\,dx \\
&\quad \leq \int_{\omega} f^m(x)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m)\,dx,
\end{aligned}
$$

which implies,

$$
\begin{aligned}
&\sum_{i=1}^{N} \int_{\omega} a_i^m(x,u_m,\nabla u_m)\,.\,\nabla(\exp(G(|u_m|)))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m)\,dx \\
&\quad + \sum_{i=1}^{N} \int_{\omega} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,\nabla((\,T_k(u_m)-T_k(u)))\,\eta_j(|u_m|)\,h_j(u_m)\,dx \\
&\quad + \sum_{i=1}^{N} \int_{\omega} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\nabla\eta_j(|u_m|)\,h_j(u_m)\,dx \\
&\quad + \sum_{i=1}^{N} \int_{\omega} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|)))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,\nabla h_j(u_m)\,dx \\
&\quad + \sum_{i=1}^{N} \int_{\omega} b_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m)\,dx \\
&\quad + \int_{\omega} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m)\,dx \\
&\quad \leq \int_{\omega} f^m(x)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m)\,dx,
\end{aligned}
$$

then,

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\,.\,\frac{l(|u_m|)}{\bar{a}}\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u)\,)\,\eta_j(|u_m|)\,h_j(u_m)\,dx\\
&+\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,\nabla((\,T_k(u_m)-T_k(u)))\,\eta_j(|u_m|)\,h_j(u_m)\,dx\\
&+\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\nabla\eta_j(|u_m|)\,h_j(u_m)\,dx\\
&+\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|)))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,\nabla h_j(u_m)\,dx\\
&+\int_{\omega} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m)\,dx\\
&\leq\sum_{i=1}^{N}\int_{\omega} |\,b_i^m(x,u_m,\nabla u_m)\,|\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m)\,dx\\
&\;+\int_{\omega} f^m(x)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m)\,dx,
\end{aligned}$$

by (3.39) and (3.41) we get

$$\begin{aligned}
&\bar{a}\sum_{i=1}^{N}\int_{\omega} B_i(|\nabla u_m|)\,.\,\frac{l(|u_m|)}{\bar{a}}\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u)\,)\,\eta_j(|u_m|)\,h_j(u_m)\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,\nabla((\,T_k(u_m)-T_k(u)))\,\eta_j(|u_m|)\,h_j(u_m)\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\nabla\eta_j(|u_m|)\,h_j(u_m)\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|)))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,\nabla h_j(u_m)\,dx\\
&\quad+\int_{\omega} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m)\,dx\\
&\quad\leq\sum_{i=1}^{N}\int_{\omega} B_i(|\nabla u_m|)\,l(|u_m|)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u)\,)\,\eta_j(|u_m|)\,h_j(u_m)\,dx\\
&\quad\;+\int_{\omega}\big(h(x)+f^m(x)+\phi(x)\,.\,\frac{l(|u_m|)}{\bar{a}}\big)\,.\,\exp(G(|u_m|))\,(\,T_k(u_m)-T_k(u)\,)\,\eta_j(|u_m|)\,h_j(u_m)\,dx,
\end{aligned}$$

we obtain

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{\{|u_m|\leq j\}} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,(\nabla T_k(u_m)-\nabla T_k(u))\,\eta_j(|u_m|)\,dx\\
&\quad-\sum_{i=1}^{N}\int_{\{j\leq|u_m|\leq j+1\}} a_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\,.\,(j+1+|u_m|)\,.\,\nabla T_k(u)\,\eta_j(|u_m|)\exp(G(|u_m|))\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,(T_k(u_m)-T_k(u))\,\nabla\eta_j(|u_m|)\,h_j(u_m)\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\{j\leq|u_m|\leq j+1\}} a_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\,.\,\exp(G(|u_m|))\,(T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,dx\\
&\quad+\int_{\omega} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(|u_m|))\,(T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m)\,dx\\
&\quad\leq\int_{\omega}\big(h(x)+f^m(x)+\phi(x)\,.\,\frac{l(|u_m|)}{\bar a}\big)\,.\,\exp(G(|u_m|))\,(T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m)\,dx,
\end{aligned}
$$

by (3.39) we get

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{\{|u_m|\leq j\}} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,(\nabla T_k(u_m)-\nabla T_k(u))\,\eta_j(|u_m|)\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,(T_k(u_m)-T_k(u))\,\nabla\eta_j(|u_m|)\,h_j(u_m)\,dx\\
&\quad+\int_{\omega} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(|u_m|))\,(T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m)\,dx\\
&\quad\leq\int_{\omega}\big(h(x)+f^m(x)+\phi(x)\,.\,\frac{l(|u_m|)}{\bar a}\big)\,.\,\exp(G(|u_m|))\,(T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,h_j(u_m)\,dx\\
&\quad\quad-\sum_{i=1}^{N}\int_{\{j\leq|u_m|\leq j+1\}}\phi(x)\,.\,(j+1+|u_m|)\,.\,\nabla T_k(u)\exp(G(|u_m|))\,\eta_j(|u_m|)\,dx\\
&\quad\quad+\sum_{i=1}^{N}\int_{\{j\leq|u_m|\leq j+1\}}\phi(x)\,.\,\exp(G(|u_m|))\,(T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,dx\\
&\quad\quad+\bar a\sum_{i=1}^{N}\int_{\{j\leq|u_m|\leq j+1\}}B_i(|\nabla u_m|)\,(j+1+|u_m|)\,.\,\nabla T_k(u)\exp(G(|u_m|))\,\eta_j(|u_m|)\,dx\\
&\quad\quad-\bar a\sum_{i=1}^{N}\int_{\{j\leq|u_m|\leq j+1\}}B_i(|\nabla u_m|)\exp(G(|u_m|))\,(T_k(u_m)-T_k(u))\,\eta_j(|u_m|)\,dx.
\end{aligned}
$$

According to (3.46), (3.48) and the fact that $T_k(u_m)\rightharpoonup T_k(u)$ weakly in $\mathring{W}_B^1(\omega)$, and $h_j\geq 0$, $\eta_j(|u_m|)\geq 0$, and $u_m\,(T_k(u_m)-T_k(u))\geq 0$ and $\exp(G(\pm\infty))\leq\exp\left(\frac{||l||_{L^1(\mathbb{R})}}{\bar a}\right)$ we deduce that,

$$
\sum_{i=1}^{N}\int_{\{|u_m|\leq j\}} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(G(|u_m|))\,(\nabla T_k(u_m)-\nabla T_k(u))\,\eta_j(|u_m|)\,dx\leq C(k,j,m). \quad (3.68)
$$

Then,

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\omega}\big[a_i(x,T_k(u_m),\nabla T_k(u_m)) - a_i(x,T_k(u_m),\nabla T_k(u))\big] \\
&\qquad \times \exp(G(|u_m|))\,(\nabla T_k(u_m)-\nabla T_k(u))\,\eta_j(|u_m|)\,dx \\
&\quad \leq -\sum_{i=1}^{N}\int_{\omega} a_i(x,T_k(u_m),\nabla T_k(u))\,.\,\exp(G(|u_m|))\,|\,\nabla T_k(u_m)-\nabla T_k(u)\,|\,\eta_j(|u_m|)\,dx \\
&\quad -\sum_{i=1}^{N}\int_{\{|u_m|\leq k\}} a_i(x,T_k(u_m),\nabla T_k(u))\,.\,\exp(G(|u_m|))\,\nabla T_k(u)\,\eta_j(|u_m|)\,dx + C(k,j,m). \qquad (3.69)
\end{aligned}$$

By Lebesgue dominated convergence theorem, we have $T_k(u_m) \longrightarrow T_k(u)$ strongly in $\mathring{W}^1_{B,loc}(\omega)$ and $\nabla T_k(u_m) \rightharpoonup \nabla T_k(u)$ weakly in $\mathring{W}^1_B(\omega)$, then the terms on the right hand side of (3.69) goes to zeros as k, j, m tend to infinity which gives

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\omega}\big[a_i(x,T_k(u_m),\nabla T_k(u_m)) - a_i(x,T_k(u_m),\nabla T_k(u))\big] \qquad (3.70)\\
&\qquad \times (\nabla T_k(u_m)-\nabla T_k(u))\,\exp(G(|u_m|))\,\eta_j(|u_m|)\,dx \longrightarrow 0,
\end{aligned}$$

by Proposition 3.2.1 and diagonal process, we deduce for $k \longrightarrow \infty$ that

$$B(|\nabla u_m|) \longrightarrow B(|\nabla u|) \text{ in } L^1(\omega). \qquad (3.71)$$

Hence, we obtain for a subsequence

$$\nabla u_m \longrightarrow \nabla u \text{ a.e in } \omega. \qquad (3.72)$$

Step 5 : <u>The equi-integrability of $b_i^m(x,u_m,\nabla u_m)$.</u>

In this step we will show that

$$b_i^m(x,u_m,\nabla u_m) \longrightarrow b_i(x,u,\nabla u). \qquad (3.73)$$

Therefore, it is enough to show that $b_i^m(x,u_m,\nabla u_m)$ is uniformly equi-integrable. We take the following test function

$$v = u_m - \eta\,\exp(2G(|u_m|))\,\eta_j(|u_m|)\,T_1(u_m - T_j(u_m)),$$

we have

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u_m,\nabla u_m)\,.\,\nabla\big(\exp(2\,G(|u_m|))\,\eta_j(|u_m|)\,T_1(u_m-T_j(u_m))\big)\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\omega} b_i^m(x,u_m,\nabla u_m)\,.\,\exp(2\,G(|u_m|))\,\eta_j(|u_m|)\,T_1(u_m-T_j(u_m))\,dx\\
&\quad+\int_{\omega} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(2\,G(|u_m|))\,\eta_j(|u_m|)\,T_1(u_m-T_j(u_m))\,dx\\
&\quad\le\int_{\omega} f^m(x)\,.\,\exp(2\,G(|u_m|))\,\eta_j(|u_m|)\,T_1(u_m-T_j(u_m))\,dx,
\end{aligned}$$

then,

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\,.\,\frac{l(|u_m|)}{\bar a}\,\exp(2\,G(|u_m|))\,\eta_j(|u_m|)\,T_1(u_m-T_j(u_m))\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(2\,G(|u_m|))\,\nabla(\eta_j(|u_m|))\,T_1(u_m-T_j(u_m))\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(2\,G(|u_m|))\,\eta_j(|u_m|)\,\nabla T_1(u_m-T_j(u_m))\,dx\\
&\quad+\int_{\omega} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(2\,G(|u_m|))\,\eta_j(|u_m|)\,T_1(u_m-T_j(u_m))\,dx\\
&\quad\le\sum_{i=1}^{N}\int_{\omega} |\,b_i^m(x,u_m,\nabla u_m)\,|\,.\,\exp(2\,G(|u_m|))\,\eta_j(|u_m|)\,T_1(u_m-T_j(u_m))\,dx\\
&\qquad+\int_{\omega} f^m(x)\,.\,\exp(2\,G(|u_m|))\,\eta_j(|u_m|)\,T_1(u_m-T_j(u_m))\,dx,
\end{aligned}$$

by (3.39) and (3.41) we get

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(2\,G(|u_m|))\,\nabla(\eta_j(|u_m|))\,T_1(u_m-T_j(u_m))\,dx\\
&\quad+\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(2\,G(|u_m|))\,\eta_j(|u_m|)\,\nabla T_1(u_m-T_j(u_m))\,dx\\
&\quad+\int_{\omega} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(2\,G(|u_m|))\,\eta_j(|u_m|)\,T_1(u_m-T_j(u_m))\,dx\\
&\quad\le\int_{\omega}\Big(h(x)+f^m(x)+\phi(x)\,.\,\frac{l(|u_m|)}{\bar a}\Big)\,.\,\exp(2\,G(|u_m|))\,\eta_j(|u_m|)\,T_1(u_m-T_j(u_m))\,dx,
\end{aligned}$$

we deduce that

$$\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u_m,\nabla u_m)\,.\,\exp(2G(|u_m|))\,\nabla(\eta_j(|u_m|))\,T_1(u_m-T_j(u_m))\,dx$$
$$+\sum_{i=1}^{N}\int_{\{j\le|u_m|\le j+1\}} a_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\,\exp(2G(|u_m|))\,\eta_j(|u_m|)\,dx$$
$$+\int_{\omega} m\,.\,T_m(u_m-\psi)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(2G(|u_m|))\,\eta_j(|u_m|)\,T_1(u_m-T_j(u_m))\,dx$$
$$\le\int_{\omega}\big(h(x)+f^m(x)+\phi(x)\,.\,\frac{l(|u_m|)}{\bar a}\big)\,.\,\exp(2G(|u_m|))\,\eta_j(|u_m|)\,T_1(u_m-T_j(u_m))\,dx.$$

Since $a_i^m(x,u_m,\nabla u_m)$ is bounded in $\mathring{W}_B^1(\omega)$, and $\eta_j(|u_m|)\ge 0$ then by (3.46), (3.48) we obtain

$$\sum_{i=1}^{N}\int_{\{j+1<|u_m|\}} B_i(|\nabla u_m|)\,dx$$
$$\le\exp\left(2\frac{||l||_{L^1(\mathbb{R})}}{\bar a}\right).\int_{\{j<|u_m|\}}\big[|C(x)|+\phi(x)+h(x)+f^m(x)+\phi(x)\,.\,\frac{l(|u_m|)}{\bar a}\big]\,dx.$$

So, $\forall\varepsilon>0\,\exists\, j(\varepsilon)>0$ such that

$$\sum_{i=1}^{N}\int_{\{j+1<|u_m|\}} B_i(|\nabla u_m|)\,dx\le\frac{\varepsilon}{2}\qquad\forall j>j(\varepsilon). \tag{3.74}$$

Let $\mathring{V}(\omega(R))$ be an arbitrary bounded subset for ω then, for any measurable set $E\subset\mathring{V}(\omega(R))$ we have

$$\sum_{i=1}^{N}\int_{E} B_i(|\nabla u_m|)\,dx\le\sum_{i=1}^{N}\int_{E} B_i(|\nabla T_k(u_m)|)\,dx+\sum_{i=1}^{N}\int_{\{j+1<|u_m|\}} B_i(|\nabla u_m|)\,dx. \tag{3.75}$$

We conclude that $\forall E\subset\mathring{V}(\omega(R))$ with $\text{meas}(E)<\beta(\varepsilon)$, and $T_k(u_m)\longrightarrow T_k(u)$ in $\mathring{W}_B^1(\omega)$,

$$\sum_{i=1}^{N}\int_{E} B_i(|\nabla T_k(u_m)|)\,dx\le\frac{\varepsilon}{2}. \tag{3.76}$$

Finally, combining the last formulas we obtain

$$\sum_{i=1}^{N}\int_{E} B_i(|\nabla u_m|)\,dx\le\varepsilon\qquad\forall E\subset\mathring{V}(\omega(R))\text{ such that meas}(E)<\beta(\varepsilon),$$

giving the assumed results.

Step 6 : <u>Passing to the limit.</u>

Let $\varphi \in \mathring{W}^1_B(\omega) \cap L^\infty(\omega)$ we take the following test function

$$v = u_m - \psi_k\, T_k(u_m - \varphi),\ \psi_K \in \mathscr{D}(\omega),$$

such that

$$\psi_k(x) = \begin{cases} 1 & \text{for } \omega(R)\ , \\ \\ 0 & \text{for } \omega(R+1)\backslash\omega(R)\ , \end{cases}$$

and $|u_m| - ||\varphi||_\infty < |u_m - \varphi| \le j$. Then, $\{|u_m - \varphi| \le j\} \subset \{|u_m| \le j + ||\varphi||_\infty\}$ we obtain

$$\begin{aligned}
&\sum_{i=1}^N \int_{\omega(R+1)} a_i(x, T_m(u_m), \nabla u_m)\, \psi_k\, \nabla T_k(u_m - \varphi)\, dx \\
&\quad + \sum_{i=1}^N \int_{\omega(R+1)} a_i(x, T_m(u_m), \nabla u_m)\, T_k(u_m - \varphi)\, \nabla \psi_k\, dx \\
&\quad + \sum_{i=1}^N \int_{\omega(R+1)} b_i^m(x, u_m, \nabla u_m)\, \psi_k\, T_k(u_m - \varphi)\, dx \\
&\quad + \int_{\omega(R+1)} m\,.\, T_m(u_m - \psi)^-\,.\, sg_{\frac{1}{m}}(u_m)\,.\, \psi_k\, T_k(u_m - \varphi)\ dx \\
&\quad \le \int_{\omega(R+1)} f^m(x)\, \psi_k\, T_k(u_m - \varphi)\, dx,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\sum_{i=1}^N \int_{\omega(R+1)} a_i(x, T_m(u_m), \nabla u_m)\, \psi_k\, \nabla T_k(u_m - \varphi)\, dx \\
&\quad = \sum_{i=1}^N \int_{\omega(R+1)} a_i(x, T_{j+||\varphi||_\infty}(u_m), \nabla T_{j+||\varphi||_\infty} u_m)\, \psi_k\, \nabla T_k(u_m - \varphi)\, dx \\
&\quad = \sum_{i=1}^N \int_{\omega(R+1)} \left[a_i(x, T_{j+||\varphi||_\infty}(u_m), \nabla T_{j+||\varphi||_\infty}(u_m)) - a_i(x, T_{j+||\varphi||_\infty}(u_m), \nabla \varphi) \right] \\
&\qquad \times \nabla T_{j+||\varphi||_\infty}(u_m - \varphi)\,.\, \chi_{\{|u_m - \varphi| < j\}}\, dx \\
&\quad + \sum_{i=1}^N \int_{\omega(R+1)} a_i(x, T_{j+||\varphi||_\infty}(u_m), \nabla \varphi)\, \nabla T_{j+||\varphi||_\infty}(u_m - \varphi)\,.\, \chi_{\{|u_m - \varphi| < j\}}\, dx.
\end{aligned}$$

By Fatou's Lemma we get

$$\begin{aligned}
&\liminf_{m \to \infty} \sum_{i=1}^N \int_{\omega(R+1)} a_i(x, T_{j+||\varphi||_\infty}(u_m), \nabla \varphi)\, \nabla T_{j+||\varphi||_\infty}(u_m - \varphi)\,.\, \chi_{\{|u_m - \varphi| < j\}}\, dx \\
&\quad = \sum_{i=1}^N \int_{\omega(R+1)} a_i(x, T_{j+||\varphi||_\infty}(u), \nabla \varphi)\, \nabla T_{j+||\varphi||_\infty}(u - \varphi)\,.\, \chi_{\{|u - \varphi| < j\}}\, dx,
\end{aligned}$$

and the fact that

$$a_i(x, T_{j+||\varphi||_\infty}(u_m), \nabla T_{j+||\varphi||_\infty}(u_m)) \rightharpoonup a_i(x, T_{j+||\varphi||_\infty}(u), \nabla T_{j+||\varphi||_\infty}(u)), \tag{3.77}$$

weakly in $\mathring{W}^1_B(\omega)$. And since $\psi_k T_k(u_m - \varphi) \rightharpoonup \psi_k T_k(u - \varphi)$ weakly in $\mathring{W}^1_B(\omega)$, and by (3.72) we obtain

$$\sum_{i=1}^{N} \int_{\omega(R+1)} b_i^m(x, u_m, \nabla u_m)\, \psi_k\, T_k(u_m - \varphi)\, dx \longrightarrow \sum_{i=1}^{N} \int_{\omega(R+1)} b_i(x, u, \nabla u)\, \psi_k\, T_k(u - \varphi)\, dx,$$

and

$$\int_{\omega(R+1)} f^m(x)\, \psi_k\, T_k(u_m - \varphi)\, dx \longrightarrow \int_{\omega(R+1)} f(x)\, \psi_k\, T_k(u - \varphi)\, dx,$$

we get

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\omega(R+1)} a_i(x, u, \nabla u)\, \psi_k\, \nabla T_k(u - \varphi)\, dx + \sum_{i=1}^{N} \int_{\omega(R+1)} a_i(x, u, \nabla u)\, T_k(u - \varphi)\, \nabla \psi_k\, dx \\
&\quad + \sum_{i=1}^{N} \int_{\omega(R+1)} b_i(x, u, \nabla u)\, \psi_k\, T_k(u - \varphi)\, dx \\
&\quad \leq \int_{\omega(R+1)} f(x)\, \psi_k\, T_k(u - \varphi)\, dx.
\end{aligned}$$

Now passing to the limit to infinity in k, we obtain the entropy solution of the problem.

3.2.4 Uniqueness of entropy solution

Theorem 3.2.3. *Suppose that conditions* (3.38) - (3.40) *are true, and* $b_i(x, u, \nabla u) : \omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ *is strictly monotone operators at least for broad class of lower order terms. Then, the problem* $(\mathscr{P})$ *has a unique solution.*

Proof. Let u and $\bar{u}$ belongs to $K_\psi \cap L^\infty(\omega)$ be two solutions of problem $(\mathscr{P})$ with $u \neq \bar{u}$. According with Definition 3.2.1, we obtain

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\omega} a_i(x, u, \nabla u)\,.\, \nabla(u - v)\, dx + \sum_{i=1}^{N} b_i(x, u, \nabla u)\,.\,(u - v)\, dx \\
&\quad \leq \int_{\omega} f(x)\,.\,(u - v)\, dx,
\end{aligned} \tag{3.78}$$

and

$$\sum_{i=1}^{N}\int_{\omega} a_i(x,\bar{u},\nabla\bar{u})\,.\,\nabla(\bar{u}-v)\,dx+\sum_{i=1}^{N} b_i(x,\bar{u},\nabla\bar{u})\,.\,(\bar{u}-v)\,dx$$
$$\leq \int_{\omega} f(x)\,.\,(\bar{u}-v)\,dx. \tag{3.79}$$

Denote $v=u-\mu(x)(u-\bar{u})(x)$ and $v=\bar{u}-\mu(x)(u-\bar{u})(x)$ with

$$\mu(x)=\begin{cases} 0 & \text{if } \ x\geq k\ , \\ k-\frac{|x|^2}{k} & \text{if } \ |x|<k\ , \\ 0 & \text{if } \ x\leq -k\ , \end{cases}$$

as test functions in (3.78) and (3.79) respectively. Using (3.38), (3.46), (3.48) and the condition of strictly monotone for the operator $b_i(x,u,\nabla u)$, we substract the equations to obtain

$$\sum_{i=1}^{N}\int_{\omega}\big[a_i(x,u,\nabla u)-a_i(x,\bar{u},\nabla\bar{u})\big]\,.\,(u-\bar{u})\nabla\mu(x)\,dx\leq 0.$$

According to (1.2), we obtain

$$\sum_{i=1}^{N}\int_{\omega}\bar{B}_i\big(a_i(x,u,\nabla u)-a_i(x,\bar{u},\nabla\bar{u})\big)\,dx+\sum_{i=1}^{N}\int_{\omega}B_i\big((u-\bar{u})\,.\,\nabla\mu(x)\big)\,dx$$
$$\leq\sum_{i=1}^{N}\int_{\omega}\bar{B}_i\big(a_i(x,u,\nabla u)-a_i(x,\bar{u},\nabla\bar{u})\big)\,dx+2\sum_{i=1}^{N}\int_{\omega}B_i(u-\bar{u})\,dx$$
$$\leq 0. \tag{3.80}$$

Since the N-functions $\bar{B}_i$ verified the same conditions and properties of B_i, then by (1.6), we have

$$\sum_{i=1}^{N}\int_{\omega}\bar{B}_i\big(a_i(x,u,\nabla u)-a_i(x,\bar{u},\nabla\bar{u})\big)\,dx$$
$$\leq c\sum_{i=1}^{N}\int_{\omega}|\bar{B}_i(a_i(x,u,\nabla u))|\,dx-c\sum_{i=1}^{N}\int_{\omega}|\bar{B}_i(a_i(x,\bar{u},\nabla\bar{u}))|\,dx,$$

according to (3.40), we obtain

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\omega}\bar{B}_i\big(a_i(x,u,\nabla u)-a_i(x,\bar{u},\nabla\bar{u})\big)\,dx\\
&\quad\leq \tilde{a}c\sum_{i=1}^{N}\int_{\omega}B_i(\nabla(u-\bar{u}))\,dx\\
&\quad\leq \tilde{a}c\,||B(u-\bar{u})||_{1,\omega}. \qquad (3.81)
\end{aligned}$$

Combine with (3.80) and (3.81) we get

$$0\leq(\tilde{a}c+2)\,.\,||B(u-\bar{u})||_{1,\omega}\leq 0.$$

Finally, $||B(u-\bar{u})||_{1,\omega}=0$, therefore $u=\bar{u}$ a.e in ω. □

3.2.5 appendix

Let S^m be the operator defined by

$$\begin{aligned}
S^m(u)=&\sum_{i=1}^{N}\int_{\omega}a_i^m(x,u,\nabla u)\,dx+\sum_{i=1}^{N}\int_{\omega}b_i^m(x,u,\nabla u)\,dx+\int_{\omega}m\,.\,T_m(u-\psi)^-\,.\,sg_{\frac{1}{m}}(u)\,dx\\
&-\int_{\omega}f^m(x)\,dx,
\end{aligned}$$

and for any $v\in\mathring{W}_B^1(\omega)$, $u\in\mathring{W}_{B,loc}^1(\overline{\omega(R)})$ we have

$$\begin{aligned}
\langle S^m(u),v\rangle=&\sum_{i=1}^{N}\int_{\omega(R)}a_i^m(x,u,\nabla u)\,.\,\nabla v\,dx+\sum_{i=1}^{N}\int_{\omega(R)}b_i^m(x,u,\nabla u)\,.\,v\,dx\\
&+\int_{\omega(R)}m\,.\,T_m(u-\psi)^-\,.\,sg_{\frac{1}{m}}(u)\,.\,v\,dx-\int_{\omega(R)}f^m(x)\,.\,v\,dx. \qquad (3.82)
\end{aligned}$$

In order to show the result of the Theorem 3.2.1, it is sufficient to show that operator S is bounded, coercive and pseudo-monotonic.

Let's start by demonstrating that S is bounded. And according to (3.82), (1.10), (3.46) and (3.48) we obtain

$$\begin{aligned}
|\langle S^m(u),v\rangle|\leq& 2\,||a^m(x,u,\nabla u)||_{\bar{B},\omega(R)}\,.\,||v||_{\mathring{W}_B^1(\omega)}+2\,||b^m(x,u,\nabla u)||_{B,\omega(R)}\,.\,||v||_{\mathring{W}_B^1(\omega)}\\
&-c_0\,.\,||v||_{\mathring{W}_B^1(\omega)},
\end{aligned}$$

or

$$||a^m(x,u,\nabla u)||_{\bar{B},\omega(R)} \le \sum_{i=1}^{N}\int_{\omega(R)} \bar{B}_i(\,a_i^m(x,u,\nabla u)\,)\,dx + N,$$

and by diagonal process we obtain

$$||a^m(x,u,\nabla u)||_{\bar{B},\omega(R)} \le \bar{a}\,||u||_{\mathring{W}_B^1(\omega)} + ||\varphi||_1 + N,$$

and by (3.42) we deduce that

$$b^m(x,u,\nabla u) \text{ bounded in } L_B(\omega)$$

if u with bounded support and if supp $u = \overline{\omega(R)}$, then (3.82) is bounded.

Next, we will move to proving that S is coercive. By (3.46), (3.48) for any $u \in \mathring{W}_B^1(\omega)$,

$$\langle S^m(u),u\rangle \ge \bar{a}\sum_{i=1}^{N}\int_\omega B_i\left(\left|\frac{\partial u}{\partial x_i}\right|\right)dx - \int_\omega \varphi(x)\,dx + \sum_{i=1}^{N}\int_\omega b_i^m(x,u,\nabla u)\,.\,u\,dx - \int_\omega f^m(x)\,.\,u\,dx,$$

then

$$\frac{\langle S^m(u),u\rangle}{||u||_{\mathring{W}_B^1(\omega)}} \ge \frac{1}{||u||_{\mathring{W}_B^1(\omega)}}\left[\bar{a}\sum_{i=1}^{N}\int_\omega B_i\left(\left|\frac{\partial u}{\partial x_i}\right|\right)dx - c_1 - c_0\right] + \frac{1}{||u||_{\mathring{W}_B^1(\omega)}}\sum_{i=1}^{N}\int_\omega b_i^m(x,u,\nabla u)\,.\,u\,dx,$$

using (3.42), we obtain

$$\frac{1}{||u||_{\mathring{W}_B^1(\omega)}}\sum_{i=1}^{N}\int_\omega b_i^m(x,u,\nabla u)\,.\,u\,dx \ge -2\,c(m).$$

Thus,

$$\frac{\langle S^m(u),u\rangle}{||u||_{\mathring{W}_B^1(\omega)}} \ge \frac{1}{||u||_{\mathring{W}_B^1(\omega)}}\left[\bar{a}\sum_{i=1}^{N}\int_\omega B_i\left(\left|\frac{\partial u}{\partial x_i}\right|\right)dx - c_1 - c_0\right] - 2\,c(m)$$

according to (1.18) page 5, we have for all $k > 0$, $\exists \alpha_0 > 0$ such that

$$b_i(\,|u_{x_i}|\,) > k\,b_i\left(\frac{|u_{x_i}|}{||u_{x_i}||_{B_i,\omega}}\right), \qquad i = 1,\cdots,N$$

we take $||u_{x_i}||_{B_i,\omega} > \alpha_0$, $i = 1, \cdots, N$. And since ω is unbounded domain, then we can assume that $||u^j||_{\mathring{W}^1_B(\omega)} \longrightarrow \infty$ as $j \longrightarrow \infty$. We suppose

$$||u^j_{x_1}||_{B_1,\omega} + \cdots + ||u^j_{x_N}||_{B_N,\omega} \geq N\alpha_0$$

according to (1.5) we get

$$|u^j|\, b(|u^j|) < c B(u^j)$$

then,

$$\begin{aligned}
\frac{\langle S^m(u^j), u^j\rangle}{||u^j||_{\mathring{W}^1_B(\omega)}} &\geq \frac{\bar{a}}{N\alpha_0}\sum_{i=1}^{N}\int_\omega B_i\left(\left|\frac{\partial u^j}{\partial x_i}\right|\right) dx - \frac{c_2}{N\alpha_0} - 2c(m)\\
&\geq \frac{\bar{a}}{cN||u^j_{x_i}||_{B_i}}\sum_{i=1}^{N}\int_\omega |u^j_{x_i}|\,.\,b_i(|u^j_{x_i}|)\, dx - \frac{c_2}{N\alpha_0} - 2c(m)\\
&\geq \frac{\bar{a}k}{cN}\sum_{i=1}^{N}\int_\omega B_i\left(\frac{|u^j_{x_i}|}{||u^j_{x_i}||_{B_i,\omega}}\right) dx - \frac{c_2}{N\alpha_0} - 2c(m),
\end{aligned}$$

with $c_2 = c_0 + c_1$. Now, by the Luxembourg norm, we have

$$||u^j||_B = \inf\{k > 0 / \int_\omega B\left(\frac{u^j(x)}{k}\right) dx \leq 1\}$$

then

$$\sum_{i=1}^{N}\int_\omega B_i\left(\frac{|u^j_{x_i}|}{||u^j_{x_i}||_{B_i,\omega}}\right) dx \geq \sum_{i=1}^{N}||u^j_{x_i}||_{B_i}.$$

Hence,

$$\frac{\langle S^m(u^j), u^j\rangle}{||u^j||_{\mathring{W}^1_B(\omega)}} \geq \frac{\bar{a}k}{cN}||u^j||_{\mathring{W}^1_B(\omega)} - \frac{c_2}{N\alpha_0} - 2c(m) \longrightarrow \infty \text{ as } ||u^j||_{\mathring{W}^1_B(\omega)} \longrightarrow \infty$$

which gives the coercivity of the operator S^m.

Finally, we will end it by the demonstration of pseudo-monotonic of S^m. Following up this assumption, since the space $\mathring{W}^1_B(\omega)$ is separable, then $\exists(u^j) \in C_0^\infty(\omega)$ such that

$$u^j \rightharpoonup u \text{ in } \mathring{W}^1_B(\omega), \tag{3.83}$$

and

$$S^m(u^j) \rightharpoonup y \text{ in } (\mathring{W}^1_B(\omega))', \tag{3.84}$$

according to (3.83), we have for all subsequence noted again by u^j,

$$||u^j||_{\mathring{W}^1_B(\omega)} \leq c_2, \; j \in \mathbb{N}$$

then, $(u^j)_{j\in\mathbb{N}}$ is bounded in $\mathring{W}^1_B(\omega)$, and since $\mathring{W}^1_B(\omega)$ is continuously and compactly injected into $L_B(\omega)$,

$$u^j \rightharpoonup u \text{ in } L_B(\omega),$$

thus,

$$u^j \longrightarrow u \text{ a.e in } \omega, \; j \in \mathbb{N},$$

and according to (3.72) , we have

$$a_i^m(x,u^j,\nabla u^j) \longrightarrow a_i^m(x,u,\nabla u) \text{ a.e in } \omega, \; j \in \mathbb{N}$$

and

$$b_i^m(x,u^j,\nabla u^j) \longrightarrow b_i^m(x,u,\nabla u) \text{ a.e in } \omega, \; j \in \mathbb{N},$$

form (3.83) and (3.84), $\exists \tilde{a}^m \in L_{\bar{B}}(\omega)$ such that

$$a_i^m(x,u^j,\nabla u^j) \rightharpoonup \tilde{a}^m, \; j \in \mathbb{N} \tag{3.85}$$

and $\exists \tilde{b}^m \in L_B(\omega)$ such that

$$b_i^m(x,u^j,\nabla u^j) \rightharpoonup \tilde{b}^m, \; j \in \mathbb{N} \tag{3.86}$$

by (3.46) and (3.48) it's clear that for any $v \in \mathring{W}^1_B(\omega)$, we get

$$\begin{aligned}\langle y,v\rangle &= \lim_{j\to\infty}\sum_{i=1}^N \int_\omega a_i^m(x,u^j,\nabla u^j)\,.\,\nabla v\,dx + \lim_{j\to\infty}\sum_{i=1}^N \int_\omega b_i^m(x,u^j,\nabla u^j)\,.\,v\,dx \\ &= \int_\omega \tilde{a}^m\,.\,\nabla v\,dx + \int_\omega \tilde{b}^m\,.\,v\,dx, \end{aligned} \tag{3.87}$$

hence

$$\begin{aligned}\limsup_{j\to\infty}\langle S^m(u^j),u^j\rangle \;=\; &\limsup_{j\to\infty}\{\sum_{i=1}^N \int_\omega a_i^m(x,u^j,\nabla u^j)\,\nabla u^j\,dx \\ &+ \lim_{j\to\infty}\sum_{i=1}^N \int_\omega b_i^m(x,u^j,\nabla u^j)\,u^j\,dx\} \leq \int_\omega \tilde{a}^m\,\nabla u^j\,dx + \int_\omega \tilde{b}^m\,u^j\,dx, \end{aligned} \tag{3.88}$$

by (3.86), we have

$$\int_{\omega} b^m(x,u^j,\nabla u^j)\,u^j\,dx \longrightarrow \int_{\omega} \tilde{b}^m\,u\,dx, \tag{3.89}$$

consequently,

$$\lim_{j\to\infty}\sup\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u^j,\nabla u^j)\nabla u^j\,dx \leq \int_{\omega} \tilde{a}^m\,\nabla u^j\,dx. \tag{3.90}$$

On the other hand, we have by the condition of monotony,

$$\sum_{i=1}^{N}\big(\,a_i^m(x,u^j,\nabla u^j\,) - a_i^m(x,u^j,\nabla u\,)\,\big)\,.\,\nabla(u^j - u) \geq 0,$$

which implies

$$\sum_{i=1}^{N}\big(\,a_i(x,T_m(u^j),\nabla u^j) - a_i(x,T^j(u^j),\nabla u)\,\big)\,.\,\nabla(u^j - u) \geq 0, \tag{3.91}$$

then,

$$\sum_{i=1}^{N} a_i(x,T_m(u^j),\nabla u^j)\,.\,\nabla u^j \geq \sum_{i=1}^{N} a_i(x,T_m(u^j),\nabla u)\,.\,\nabla(u^j - u) + \sum_{i=1}^{N} a_i(x,T_m(u^j),\nabla u^j)\,.\,\nabla u,$$

by Step 3, we get

$$\sum_{i=1}^{N} a_i(x,T_m(u^j),\nabla u) \longrightarrow \sum_{i=1}^{N} a_i(x,T_m(u),\nabla u) \quad \text{in } L_{\bar{B}}(\omega),$$

according to (3.85), we obtain

$$\lim_{j\to\infty}\inf\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u^j,\nabla u^j)\,.\,\nabla u^j\,dx \geq \int_{\omega} \tilde{a}^m\,.\,\nabla u^j\,dx, \tag{3.92}$$

therefore, from (3.90), we have

$$\lim_{j\to\infty}\sum_{i=1}^{N}\int_{\omega} a_i^m(x,u^j,\nabla u^j)\,.\,\nabla u^j\,dx = \int_{\omega} \tilde{a}^m\,.\,\nabla u^j\,dx, \tag{3.93}$$

according to (3.87), (3.89) and (3.92) we get

$$\langle S^m(u^j),u^j\rangle \longrightarrow \langle y,u\rangle \;\; \text{as} \;\; j\to\infty.$$

Hence, from (3.93), and (3.72) we obtain

$$\lim_{j\to\infty}\sum_{i=1}^{N}\int_{\omega}\big(\,a_i^m(x,u^j,\nabla u^j\,) - a_i^m(x,u^j,\nabla u\,)\,\big)\,.\,\nabla(u^j - u)\,dx = 0.$$

By (3.87) we can conclude that

$$\langle y , u \rangle = \langle S(u) , u \rangle \quad \forall u \in \mathring{W}_B^1(\omega).$$

4

Existence Results for double phase obstacle problems with variable exponents

4.1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$. In this chapter, we introduce a new class of the approximating problems corresponding to a quasilinear obstacle equations, which involves a general variable exponents elliptic operator in divergence form, called double phase obstacle operator with variable exponents, defined as follows :

$$(\mathscr{P}) \begin{cases} -\operatorname{div}A(x,\nabla u) + |u|^{\alpha(x)-2}u = \lambda\, b(x)|u|^{\beta(x)-2}u & \text{in } \Omega, \\ u(x) \leq \Phi(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

Where $\lambda > 0$ is a parameter, $A : \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ admits a potential $\mathscr{A}$, with respect to its second variable ζ, satisfying the following assumptions :

$(\mathscr{A}_1)$ The potential $\mathscr{A} = \mathscr{A}(x, \varepsilon)$ is a continuous function in $\Omega \times \Omega$, with continuous derivative with respect to ζ, $A = \partial_\varepsilon \mathscr{A}(x, \varepsilon)$, and verifies :

(i) $\mathscr{A}(x, \cdot)$ is strictly convex in Ω for all $x \in \Omega$.

(ii) $\mathscr{A}(x, 0) = 0$ and $\mathscr{A}(x, \zeta) = \mathscr{A}(x, -\zeta)$, for all $(x, \zeta) \in \Omega \times \Omega$.

(iii) There exist positive constants c_1, c_2 and variable exponents $p(\cdot), q(\cdot) : \Omega \to \mathbb{R}$ such that for all $(x, \zeta) \in \Omega \times \Omega$

$$A(x, \zeta) \,.\, \zeta \geq \begin{cases} c_1 \, |\zeta|^{p(x)} & ; \text{if } \ |\zeta| \gg 1, \\ \\ c_1 \, |\zeta|^{q(x)} & ; \text{if } \ |\zeta| \ll 1, \end{cases}$$

and

$$|A(x, \zeta)| \leq \begin{cases} c_2 \, |\zeta|^{p(x)-1} & ; \text{if } \ |\zeta| \gg 1, \\ \\ c_2 \, |\zeta|^{q(x)-1} & ; \text{if } \ |\zeta| \ll 1, \end{cases}$$

(iv) $1 \ll p(\cdot) \ll q(\cdot) \ll \min\{N, p^*(\cdot)\}$, and $p(\cdot), q(\cdot)$ are Lipschitz continuous in $\mathbb{R}^N$, satisfy

$$\frac{q(\cdot)}{p(\cdot)} < 1 + \frac{1}{N},$$

where

$$p^*(x) = \begin{cases} \frac{Np(x)}{N - p(x)} & ; \text{if } \ p(x) < N, \\ \\ \infty & ; \text{if } \ p(x) \geq N, \end{cases}$$

(v) $A(x, \zeta) \,.\, \zeta \leq s(\cdot) \mathscr{A}(x, \zeta)$ for any $(x, \zeta) \in \Omega \times \Omega$, where $s(\cdot) : \Omega \to \mathbb{R}$ is Lipschitz continuous and satisfies $q(\cdot) \leq s(\cdot) \ll p^*(\cdot)$.

$(\mathscr{A}_2)$ $\mathscr{A}$ is uniformly convex, that is, for any $0 < \varepsilon < 1$, there exists $\delta(\varepsilon) \in (0, 1)$ such that $|u - v| \leq \varepsilon \max\{|u|, |v|\}$ or

$$\mathscr{A}\left(x, \frac{u+v}{2}\right) \leq \frac{1}{2} (1 - \delta(\varepsilon)) \, (\mathscr{A}(x, u) + \mathscr{A}(x, v)),$$

for any $x, u, v \in \Omega$

(vi) $\Phi : \Omega \longrightarrow \mathbb{R}^+$ is a given function satisfy : $\Phi \in L^{q(x)}(\Omega)$.

(viii) $w \in L^{r(\cdot)}(\Omega)$, $w > 0$ a.e in Ω, $1 \ll r(x) \ll \infty$, and

$$r'(x) \leq \frac{p^*(x)}{\gamma(x)}, \ \forall x \in \Omega,$$

where $r'(x)$ is the conjugate function of $r(x)$, namely $\frac{1}{r(x)} + \frac{1}{r'(x)} = 1$ and $\gamma(\cdot)$ is Lipschitz continuous satisfy $\alpha \leq \gamma(\cdot) \ll p^*(\cdot)$.

(ix) $1 \ll \beta(x) \leq \beta^+ < \alpha^-$, $b \in L^{r^*(\cdot)}(\Omega)$, $b > 0$ a.e in Ω, $1 \ll r^*(x) \ll \infty$, and

$$\alpha(x) \leq \frac{r^*(x)}{r^*(x)-1}\beta(x) \leq p^*(x) \ \forall x \in \Omega.$$

Recently, the existence of solutions to nonlinear problems with non-homogeneous structures have received much attention, particularly the existence of solutions to double phase problems with variable exponents. These operators are the natural extension of the classical double phase problems when p and q are constants. For example, the existence solutions of such problems when $p = q =$ constant and using the surjectivity theorem, multivalued mapping Kluge's fixed point principle and tools from non-smooth analysis, can be found in [100–102] and the references given there.
Let us mention some relevant papers in this direction. Zhikov in [104] describe models of strongly anisotropic materials by treating the functional

$$u \longmapsto \int_\Omega (|\nabla u|^p + \mu(x)|\nabla u|^q)\, dx. \tag{4.1}$$

Li, Yao, and Zhou in [78] proved the existence and uniqueness of entropy solutions and the uniqueness of renormalized solutions to the general nonlinear elliptic equations in Musielak-Orlicz spaces. Moreover, they also obtain the equivalence of entropy solutions and renormalized solutions. For a deeper comprehension, we refer the reader to [18, 49, 80, 81, 87, 88] and the references therein for more background.

Moving on to another novel aspect; the double phase problem with variable exponents that few author consider. Ragusa and Tachikawa in [91, 93–95] and reference therein, are the first ones who have achieved the regularity theory for minimizers of (4.1) with variable exponents. Moreover, Zhang and Rădulescu in [103] proved the existence of multiple solution for the quasilinear equation

$$\operatorname{div} A(x,\nabla u) + V(x)|u|^{\alpha(x)-2}u = f(x,u) \text{ in } \mathbb{R}^N,$$

which involves a general variable exponent elliptic operator in divergence form. This type of problem corresponds to double phase anisotropic phenomena, in the sense that the differential operator has behaviors like $|\zeta|^{q(x)-2}\zeta$ for small $|\zeta|$ and like $|\zeta|^{p(x)-2}\zeta$ for large $|\zeta|$ where $1 < \alpha(\cdot) \leq p(\cdot) < q(\cdot) < N$. We refer to other methods to solve this type of problems which can be found in the work of Shi, Rădulescu, Repovš, and Zhang [97] and the references therein.

The novelty of this chapter is the fact that we combine several different phenomena in one problem. More precisely, our problem $(\mathscr{P})$ contains : Quasilinear equation ; which involves a general variable exponents elliptic operator in divergence form, an obstacle restriction, and double phase operators ; the reason why it is called double phase, is that (4.1) is defined by the fact that the energy density changes its ellipticity and growth properties depending on the point in the domain. To be specific, its behavior depends on the values of the weight function $\mu(\cdot)$. Actually, on the set $\{x \in \Omega/\mu(x) = 0\}$ it will be composed by the gradient of order $p(\cdot)$ and on the set $\{x \in \Omega/\mu(x) \neq 0\}$ it is the gradient of order $q(\cdot)$.

To the best of our knowledge, no previous research has investigated the double phase obstacle operator with variable exponents given in the general form $(\mathscr{P})$. Besides, we address the challenges that come about due to the non-homogeneities of the growths, and the presence of several non-linear terms.

4.2 Properties of functionals and mains results

In this section, we begin by presenting some results that can be concluded from the previous assumptions in the section 4.1.

- The conditions $(\mathscr{A}_1), (i)$ and (ii) imply that

$$\mathscr{A}(x,\zeta) \leq A(x,\zeta)\,.\,\zeta \text{ for all } (x,\zeta) \in \Omega \times \Omega. \tag{4.2}$$

- By $(\mathscr{A}_1), (i)$ and (iii), we get

$$\mathscr{A}(x,\zeta) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta}\mathscr{A}(x,\theta\zeta)\,\mathrm{d}\theta = \int_0^1 \frac{1}{\theta}A(x,\theta\zeta)\,.\,\theta\zeta\,\mathrm{d}\theta \geq \begin{cases} c_1|\zeta|^{p(x)} & ;\text{if } |\zeta| > 1, \\ c_1|\zeta|^{q(x)} & ;\text{if } |\zeta| \leq 1, \end{cases} \tag{4.3}$$

• According to (iii), (4.2) and (4.3) we obtain that

$$\begin{cases} c_1\,|\zeta|^{p(x)} & ;\text{if } \ |\zeta|>1, \\ \\ c_1\,|\zeta|^{q(x)} & ;\text{if } \ |\zeta|\leq 1, \end{cases} \leq \mathscr{A}(x,\zeta) \leq A(x,\zeta)\,.\,\zeta \leq \begin{cases} c_2\,|\zeta|^{p(x)} & ;\text{if } \ |\zeta|>1, \\ \\ c_2\,|\zeta|^{q(x)} & ;\text{if } \ |\zeta|\leq 1, \end{cases} \tag{4.4}$$

for all $(x,\zeta) \in \Omega \times \Omega$.

By (g) of Proposition 1.3.5, we deduce that $\mathscr{A}(x,\nabla u)$ is integrable on Ω for all $u \in X$. Thus, $\displaystyle\int_\Omega \mathscr{A}(x,\nabla u)\,dx$ is well defined. For $u \in X$, it follows by (4.4) that

$$\begin{aligned} &\int_\Omega A(x,\nabla u)\,.\,\nabla u\,dx + \int_\Omega |u|^{\alpha(x)}\,dx \\ &\quad \geq c_1 \left(\int_{\Omega\cap\mathscr{D}_{\nabla u}} |\nabla u|^{p(x)}\,dx + \int_{\Omega\cap\mathscr{D}^c_{\nabla u}} |\nabla u|^{q(x)}\,dx + \int_\Omega |u|^{\alpha(x)}\,dx \right), \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} &\int_\Omega A(x,\nabla u)\,.\,\nabla u\,dx + \int_\Omega |u|^{\alpha(x)}\,dx \\ &\quad \leq c_2 \left(\int_{\Omega\cap\mathscr{D}_{\nabla u}} |\nabla u|^{p(x)}\,dx + \int_{\Omega\cap\mathscr{D}^c_{\nabla u}} |\nabla u|^{q(x)}\,dx + \int_\Omega |u|^{\alpha(x)}\,dx \right), \end{aligned}$$

where c_1 and c_2 are positive constants.

Similarly, using (4.4), we get for all $u \in X$

$$\begin{aligned} &\int_\Omega \mathscr{A}(x,\nabla u)\,dx + \int_\Omega \frac{1}{\alpha(x)}\,|u|^{\alpha(x)}\,dx \\ &\quad \geq c_1 \left(\int_{\Omega\cap\mathscr{D}_{\nabla u}} |\nabla u|^{p(x)}\,dx + \int_{\Omega\cap\mathscr{D}^c_{\nabla u}} |\nabla u|^{q(x)}\,dx + \int_\Omega \frac{1}{\alpha(x)}|u|^{\alpha(x)}\,dx \right), \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} &\int_\Omega \mathscr{A}(x,\nabla u)\,dx + \int_\Omega \frac{1}{\alpha(x)}\,|u|^{\alpha(x)}\,dx \\ &\quad \leq c_2 \left(\int_{\Omega\cap\mathscr{D}_{\nabla u}} |\nabla u|^{p(x)}\,dx + \int_{\Omega\cap\mathscr{D}^c_{\nabla u}} |\nabla u|^{q(x)}\,dx + \int_\Omega \frac{1}{\alpha(x)}|u|^{\alpha(x)}\,dx \right), \end{aligned}$$

where,

$$\mathscr{D}_{\nabla u} = \{x \in \Omega \,/\, |\nabla u(x)| > 1\} \text{ and } \mathscr{D}^c_{\nabla u} = \{x \in \Omega \,/\, |\nabla u(x)| \leq 1\}.$$

Let K be a subset of $X(\Omega)$ defined by

$$K = \{u \in X(\Omega)/u(x) \leq \Phi(x) \, \text{for a.a} \, x \in \Omega\}. \tag{4.7}$$

Remark 12. *(a) The set K is nonempty, closed and convex subset of $X(\Omega)$.*

(b) From assumption (vi) we see that $0 \in K$.

Definition 4.2.1. *(a) We say that $u \in K$ is a weak solution of problem $(\mathscr{P})$ if*

$$\int_\Omega A(x, \nabla u) \,.\, \nabla(v-u)\, dx + \int_\Omega |u|^{\alpha(x)-2} u \,.\, (v-u)\, dx \\ = \int_\Omega \lambda\, b(x)\, |u|^{\beta(x)-2} u\, (v-u)\, dx,$$

for all $v \in K$, where K is given by (4.2).

(b) We say that $u \in X$ is a weak solution of problem $(\mathscr{P})$ if for all $\varphi \in X$, we have

$$\int_\Omega A(x, \nabla u) \,.\, \nabla \varphi(x)\, dx + \frac{1}{\rho_n} \int_\Omega (u(x) - \Phi(x))^+ \,.\, \varphi(x)\, dx \\ + \int_\Omega |u|^{\alpha(x)-2} u \,.\, \varphi(x)\, dx = \int_\Omega \lambda\, b(x)\, |u|^{\beta(x)-2} u \,.\, \varphi(x)\, dx,$$

where $\{\rho_n\}$ is a sequence with $\rho_n > 0$ for each $n \in \mathbb{N}$ such that $\rho_n \longrightarrow 0$ when $n \to \infty$.

It is easy to prove the following lemma.

Lemma 4.2.1. *If hypotheses (vi) holds, then the function $I : X \longrightarrow X^*$ given by*

$$\langle Iu, \varphi \rangle_X = \int_\Omega (u(x) - \Phi(x))^+ \,.\, \varphi(x)\, dx \qquad \text{for all } u, \varphi \in X,$$

is bounded, demi-continuous and monotone, where $\langle \cdot, \cdot \rangle_X$ denotes the duality pairing between X and its dual space X^*.

Proof. From (vi), we deduce that the function Φ is nonnegative. Next, according to the Proposition 1.3.5 we get that the function I is bounded, monotone and for the demi-continuous of the function I, we consider $\{u_n\}_{n \in \mathbb{N}^*}$ a bounded sequence in Ω such as $u_n \to u$ for all $u \in \Omega$, we get that $Iu_n \rightharpoonup Iu$ in $X(\Omega)$. □

Theorem 4.2.1. *Assume that $1 \ll \beta(\cdot) \leq \alpha(\cdot) \leq p(\cdot) \ll q(\cdot) \ll min\{N, p^*\}$, $1 \ll \alpha(\cdot) \ll p^*(\cdot)\, \frac{q'(\cdot)}{p'(\cdot)}$, λ is small enough, and hypotheses $(\mathscr{A}_1) - (\mathscr{A}_2), (vi), (vii)$ and (ix) hold. Then, the problem $(\mathscr{P})$ possesses a weak solution.*

Proof. It follows that solutions of $(\mathscr{P})$ correspond to the critical points of the Euler-Lagrange energy functional $J : X \longrightarrow \mathbb{R}$

$$J(u) = \int_\Omega \mathscr{A}(x,\nabla u)\,dx + \int_\Omega \frac{1}{\alpha(x)}\,|u|^{\alpha(x)}\,dx - \frac{1}{\rho_n}\int_\Omega (u(x)-\Phi(x))^+\,dx \\ - \int_\Omega \lambda\,\frac{b(x)}{\beta(x)}\,|u|^{\beta(x)}\,dx.$$

Lemma 4.2.2. *Assume that* $1 \ll \beta(\cdot) \le \alpha(\cdot) \le p(\cdot) \ll q(\cdot) \ll min\{N, p^*\}$, $1 \ll \alpha(\cdot) \ll p^*(\cdot)\,\frac{q'(\cdot)}{p'(\cdot)}$, λ *is small enough, and hypotheses* $(\mathscr{A}_1)-(\mathscr{A}_2), (vi), (vii)$ *and* (ix) *hold. The functional J satisfies mountain pass geometry in the sense that :*

(1) $J(0) = 0$.

(2) *There exists* γ, $\delta > 0$ *such that* $J(u) \ge \delta$ *if* $||u||_X > \gamma$.

(3) *There exists* u, $||u||_X > \gamma$ *such that* $J(u) \le 0$.

Proof. (1) According to (ii) and the nonnegativity of Φ, we deduce that $J(0) = 0$.
(2) According to Lemma 4.2.1, (vi), (vii), (ix) and the monotonicity of the function $s \longmapsto s^+$, we get

$$\begin{aligned} J(u) &= \int_\Omega \mathscr{A}(x,\nabla u)\,dx + \int_\Omega \frac{1}{\alpha(x)}|u|^{\alpha(x)}\,dx - \frac{1}{\rho_n}\int_\Omega (u(x)-\Phi(x))^+\,dx \\ &\quad - \int_\Omega \lambda\,\frac{b(x)}{\beta(x)}|u|^{\beta(x)}\,dx \\ &\ge c_1\left(\int_{\Omega\cap\Lambda_{\nabla u}} |\nabla u|^{p(x)}\,dx + \int_{\Omega\cap\Lambda^c_{\nabla u}} |\nabla u|^{q(x)}\,dx + \int_\Omega \frac{1}{\alpha(x)}|u|^{\alpha(x)}\,dx\right) \\ &\quad - \frac{1}{\rho_n}\int_\Omega (u(x)-\Phi(x))^+\,dx - \int_\Omega \lambda\,\frac{b(x)}{\beta(x)}|u|^{\beta(x)}\,dx \\ &\ge c_1\left(\frac{1}{p^+}+\frac{1}{q^+}+\frac{1}{\alpha^-}\right)||u||_X^{\alpha^-} - \lambda\,c_2\,||u||_X^{\beta^+} \longrightarrow +\infty, \text{ when } ||u||_X \to +\infty. \end{aligned}$$

Then, $\exists\eta > 0$ be small enough. Which for that, we have

$$J(u) \ge \delta > 0 \qquad \text{for all } ||u||_X = \eta.$$

(3) Let K be a real fixed, choosing Υ_k a k-dimensional linear subspace of X such that $\Upsilon_k \subset C_0^\infty(B_R)$, which the norms on Υ_k are equivalent. Then, for any $\delta_0 > 0$ given, there exists $\sigma_k \in (0, 1)$ such that $u \in \Upsilon_k$ with $||u|| \le \sigma_k$ implies $|u|_{L^\infty} \le \delta_0$. Consider the following set :

$$F^{(k)}_{\sigma_k} = \{u \in \Upsilon_k : ||u|| = \sigma_k\}.$$

For $u \in F^{(k)}_{\sigma_k}$ and $0 < t < 1$, we get

$$\begin{aligned} J(tu) &= \int_\Omega \mathscr{A}(x, \nabla tu)\, dx + \int_\Omega \frac{1}{\alpha(x)}\, |tu|^{\alpha(x)}\, dx - \frac{1}{\rho_n} \int_\Omega (tu(x) - \Phi(x))^+\, dx \\ &\quad - \int_\Omega \lambda\, \frac{b(x)}{\beta(x)}\, |tu|^{\beta(x)}\, dx \\ &\leq c_1 \frac{t^{\alpha^-}}{\alpha^-}\, \sigma_k^{\alpha^-} - \frac{1}{\rho_n}\, t\, \sigma_k - t^{\beta^+}\, \theta_k, \end{aligned}$$

because, by the compactness of the set $F^{(k)}_{\sigma_k}$, $\exists\, \theta_k > 0$ such that

$$\int_\Omega \lambda\, \frac{b(x)}{\beta(x)} |tu|^{\beta(x)}\, dx \geq \theta_k \ \text{ for all } u \in F^{(k)}_{\sigma_k}.$$

Hence, since $1 \ll \beta^+ < \alpha^-$, we find $t_k \in (0, 1)$ and $\varepsilon_k > 0$ such that

$$J(t_k u) \leq -\varepsilon_k < 0 \qquad \text{for all } u \in F^{(k)}_{\sigma_k}.$$

Then,

$$J(u) \leq -\varepsilon_k < 0 \qquad \text{for all } u \in F^{(k)}_{t_k\, \sigma_k}.$$

Which completes the proof of the Lemma. □

Lemma 4.2.3. *Assume that* $1 \ll \beta(\cdot) \leq \alpha(\cdot) \leq p(\cdot) \ll q(\cdot) \ll min\{N, p^*\}$, $1 \ll \alpha(\cdot) \ll p^*(\cdot)\, \frac{q'(\cdot)}{p'(\cdot)}$, λ *is small enough, and hypotheses* $(\mathscr{A}_1) - (\mathscr{A}_2), (vi), (vii)$ *and* (ix) *hold. The functional J satisfies Palais-Smale condition.*

Proof. Let $\{u_n\}$ be a Palais-Smale condition sequence, such that the associated sequence of real numbers $\{J(u_n)\}$ is bounded, and $J'(u_n) \longrightarrow 0$ in X^*. For that we will demonstrate the (u_n) is bounded in X, and we will argue it by contradiction.

We suppose that $||u_n||_X \to \infty$ when $n \to \infty$. Then, we have

$$
\begin{aligned}
J(u_n) - \frac{1}{\mu}\langle J'(u_n), u_n\rangle = & \int_\Omega \mathscr{A}(x,\nabla u_n)\,dx + \int_\Omega \frac{1}{\alpha(x)}\,|u_n|^{\alpha(x)}\,dx \\
& - \frac{1}{\rho_n}\int_\Omega (u_n(x)-\Phi(x))^+\,dx - \int_\Omega \lambda\,\frac{b(x)}{\beta(x)}\,|u_n|^{\beta(x)}\,dx \\
& - \frac{1}{\mu}\int_\Omega A(x,\nabla u_n)\,.\,\nabla u_n\,dx - \frac{1}{\mu}\int_\Omega |u_n|^{\alpha(x)}\,dx \\
& + \frac{1}{\mu}\,\frac{1}{\rho_n}\int_\Omega (u_n(x)-\Phi(x))^+\,dx + \frac{1}{\mu}\int_\Omega \lambda\,b(x)\,|u_n|^{\beta(x)}\,dx \\
\geq & \left(1-\frac{s^-}{\mu}\right)\int_\Omega \mathscr{A}(x,\nabla u_n)\,dx + \left(\frac{1}{\alpha^-}-\frac{1}{\mu}\right)\int_\Omega |u_n|^{\alpha(x)}\,dx \\
& + \left(\frac{1}{\mu}-1\right)\frac{1}{\rho_n}\int_\Omega (u_n(x)-\Phi(x))^+\,dx \\
& + \left(\frac{1}{\mu}-\frac{1}{\beta^+}\right)\int_\Omega \lambda\,b(x)\,|u_n|^{\beta(x)}\,dx \\
\geq & \, c_1\int_\Omega \mathscr{A}(x,\nabla u_n) + |u_n|^{\alpha(x)}\,dx \\
& + \left(\frac{1}{\mu}-1\right)\frac{1}{\rho_n}\int_\Omega (u_n(x)-\Phi(x))^+\,dx \\
& + \left(\frac{1}{\mu}-\frac{1}{\beta^+}\right)\int_\Omega \lambda\,b(x)|u_n|^{\beta(x)}\,dx \\
\geq & \, c_1\,||u_n||_X^{\alpha^-} - c_2\,||u_n||_X - c_3\,||u_n||_X^{\beta^+} - c_4,
\end{aligned} \tag{4.8}
$$

where, c_1, c_2, c_3 and c_4 are positive constants.

Dividing both sides of (4.8) by $||u_n||_X$ and passing to the limit $n \to +\infty$ with the fact that $\alpha^- > \beta^+ > 1$, we get $0 \geq \infty$. Which is a contradiction.

Hence, the functional J satisfies the Palais-Smale condition. □

□

Conclusion

According to the Lemmas 4.2.2 and 4.2.3, we conclude that the problem $(\mathscr{P})$ possesses a weak solution.

5

Existence Results for double phase problem in Sobolev-Orlicz spaces with variable exponents in Complete Manifold

5.1 Introduction

Let (M, g) be a smooth, complete compact Riemannian n-manifold. In this paper, we focused on the existence of non-trivial solutions of the following double phase problem

$$(\mathscr{P})\begin{cases} -\mathrm{div}(|\nabla u(x)|^{p(x)-2}\nabla u+\mu(x)|\nabla u(x)|^{q(x)-2}\nabla u) \\ \quad =\lambda|u(x)|^{q(x)-2}u(x)-|u(x)|^{p(x)-2}u(x)+f(x,u(x)) & \text{in } M\,, \\ u=0 & \text{on } \partial M\,, \end{cases}$$

where $-\Delta_{p(x)}u(x) = -div(|\nabla u(x)|^{p(x)-2}.\nabla u(x))$, $-\Delta_{q(x)}u(x) = -div(|\nabla u(x)|^{q(x)-2}.\nabla u(x))$ are the $p(x)$-laplacian and $q(x)$-laplacian in (M,g) respectively, $\lambda > 0$ is a parameter specified later, the function $\mu : \overline{M} \to \mathbb{R}_*^+$ is supposed to be Lipschitz continuous, and the variables exponents $p, q \in C(\overline{M})$ satisfy the assumption (5.1) in section 5.2.

The perturbation $f(x,u)$ is a Caratheodory function which satisfies the Ambrosetti-Rabinowitz type condition :

(f_1) : There exists $\beta > p^+$ and some $A > 0$ such as for each $|\alpha| > A$ we have

$$0 < \int_M F(x,\alpha)\, dv_g(x) \leq \int_M f(x,\alpha)\,.\,\frac{\alpha}{\beta}\, dv_g(x) \text{ a.e } x \in M,$$

where $F(x,\alpha) = \displaystyle\int_0^\alpha f(x,t)\,dt$ being the primitive of $f(x,\alpha)$ and $dv_g = \sqrt{\det(g_{ij})}\,dx$ is the Riemannian volume element on (M,g), where the g_{ij} are the components of the Riemannian metric g in the chart and dx is the Lebesgue volume element of $\mathbb{R}^N$.

(f_2) : $f(x,0) = 0$.

And

(f_3) : $\displaystyle\lim_{|\alpha|\to 0} \frac{f(x,\alpha)}{|\alpha|^{q(x)-1}} = 0$ uniformly a.e $x \in M$.

Up to this day, several contributions have been devoted to study double phase problems. This kind of operator was introduced, first, by Zhikov in his relevant paper [104] in order to describe models with strongly anisotropic materials by studying the functional

$$u \longmapsto \int_\Omega (|\nabla u|^p + \mu(x)|\nabla u|^q)\, dx,$$

where $1 < p < q < N$ and with a nonnegative weight function $\mu \in L^\infty(\Omega)$, see also the works of Zhikov [105, 106] and the monograph of Zhikov-Kozlov-Oleinik [107]. Indeed, we can easily see that the previous function reduces to p-laplacian if $\mu(x) = 0$ or to the weighted laplacian $(p(x), q(x))$ if $\inf_{x\in\overline{M}} \mu(x) > 0$, respectively.

Studying this type of problems is both significant and relevant. In the one hand, we have the physical motivation ; since the double phase operator has been used to model the steady-state solutions of reaction-diffusion problems, that arise in biophysic, plasma-physic and in the study of chemical reactions. In the other hand, these operators provide a useful paradigm for describing the behaviour of strongly anisotropic materials, whose hardening properties are linked to the exponent governing the growth of the gradient change radically with the point, where the coefficient $\mu(.)$ determines the geometry of a composite made of two different materials.

The aim of this chapter, is to prove the existence of non-negative non-trivial solutions of the problem $(\mathscr{P})$ where the perturbation $f(x,u)$ is a Caratheodory function, that satisfies the Ambrosetti-Rabinowitz type condition. To the best of our knowledge, the existence result for double-phase problems $(\mathscr{P})$ in the framework of Sobolev-Orlicz spaces with variable exponents in complete manifold has not been considered in the literature. The present paper is the first study devoted to this type of problem in the setting of Sobolev-Orlicz spaces with variable exponents in a complete manifold.

We would like to draw attention to the fact that the $p(x)-$laplacian operator has more complicated non-linearity than the p-laplacian operator. For example, they are non-homogeneous. Thus, we cannot use the Lagrange Multiplier Theorem in many problems involving this operators, which prove that our problem is more difficult than the operators p-Laplacian type.

5.2 Nehari Manifold Analysis for $(\mathscr{P})$

In what follows, we note by $D(M)$ the space of C_c^∞ functions with compact support in M.

Definition 5.2.1. *$u \in W_0^{1,q(x)}(M)$ is said to be a weak solution of the problem $(\mathscr{P})$ if for every $\phi \in D(M)$ we have*

$$\begin{aligned}&\int_M \left(|\nabla u(x)|^{p(x)-2} + \mu(x)|\nabla u(x)|^{q(x)-2}\right).g(\nabla u(x), \nabla\phi(x))\, dv_g(x)\\&= \lambda \int_M |u(x)|^{q(x)-2}.u(x)\,\phi(x)\, dv_g(x) - \int_M |u(x)|^{p(x)-2}.u(x).\phi(x)\, dv_g(x)\\&\quad + \int_M f(x,u(x)).\phi(x)\, dv_g(x).\end{aligned}$$

The variable exponents $p,q \in C(\overline{M})$, are assumed to satisfy the following assumption :

$$1 < q^- \leq q^+ < p^- \leq p^+ < N. \tag{5.1}$$

Then, we have

$$\frac{p^+}{q^+ - q^-} < \frac{p^+ - q^+}{p^+ - q^-} - \frac{(q^+ - q^-).(p^+ - q^+)}{(p^+ - q^-).(p^- - q^-)}. \tag{5.2}$$

We suppose $\frac{p^-}{q^+} \leq 1 + \frac{1}{N}$, and the function $\mu : \overline{M} \to \mathbb{R}_*^+$ is Lipschitz continuous.

Let us consider the energy functional $J_\lambda : W_0^{1,q(x)}(M) \longrightarrow \mathbb{R}$ associated to problem $(\mathscr{P})$ which is defined by

$$
\begin{aligned}
J_{\lambda}(u) = & \int_M \frac{1}{p(x)} \,|\, \nabla u(x) \,|^{p(x)} \, dv_g(x) + \int_M \frac{\mu(x)}{q(x)} \,|\, \nabla u(x) \,|^{q(x)} \, dv_g(x) \\
& - \int_M \frac{\lambda}{q(x)} \,|\, u(x) \,|^{q(x)} \, dv_g(x) + \int_M \frac{1}{p(x)} \,|\, u(x) \,|^{p(x)} \, dv_g(x) \\
& - \int_M F(x, u(x)) \, dv_g(x).
\end{aligned}
$$

And, for any $u \in W_0^{1,q(x)}(M)$ with $||\, u \,||_{W_0^{1,q(x)}(M)} > 1$, we have by $(f_1), (f_2)$, (1.23), Proposition 1.4.4 and Poincaré inequality that

$$
\begin{aligned}
J_{\lambda}(u) = & \int_M \frac{1}{p(x)} \,|\, \nabla u(x) \,|^{p(x)} \, dv_g(x) + \int_M \frac{\mu(x)}{q(x)} \,|\, \nabla u(x) \,|^{q(x)} \, dv_g(x) \\
& - \int_M \frac{\lambda}{q(x)} \,|\, u(x) \,|^{q(x)} \, dv_g(x) + \int_M \frac{1}{p(x)} \,|\, u(x) \,|^{p(x)} \, dv_g(x) \\
& - \int_M F(x, u(x)) \, dv_g(x) \\
\geq & \frac{1}{p^+} \int_M |\, \nabla u(x) \,|^{p(x)} \, dv_g(x) + \frac{\mu_0}{D^{p^+}(c+1)^{p^+} q^+} \int_M |\, u(x) \,|^{p(x)} \, dv_g(x) \\
& - \frac{\lambda}{q^-} \int_M |\, u(x)|^{q(x)} \, dv_g(x) + \frac{1}{p^+} \int_M |\, u(x) \,|^{p(x)} \, dv_g(x) \\
& - \frac{1}{\beta} \int_M f(x, u(x)) \, . \, u(x) \, dv_g(x) \\
\geq & \frac{1}{cp^+} \rho_{p(\cdot)}(u) + \frac{\mu_0}{D^{p^+}(c+1)^{p^+} q^+} \rho_{p(\cdot)}(u) - \frac{\lambda}{q^-} \rho_{q(\cdot)}(u) + \frac{1}{p^+} \rho_{p(\cdot)}(u) - \frac{1}{p^+} \rho_{q(\cdot)}(u)
\end{aligned}
$$

(since $\beta > p^+$ from (f_1), and c is the Poincaré constant).

According to the proposition 1.4.6, we have that

$$
\begin{aligned}
J_{\lambda}(u) \geq & \left(\frac{1}{cp^+} + \frac{1}{p^+} + \frac{\mu_0}{D^{p^+}(c+1)^{p^+} q^+} \right) \rho_{p(\cdot)}(u) - \frac{\lambda}{q^-} \rho_{q(\cdot)}(u) - \frac{1}{p^+} \rho_{q(\cdot)}(u) \\
\geq & \left(\frac{1}{cp^+} + \frac{1}{p^+} + \frac{\mu_0}{D^{p^+}(c+1)^{p^+} q^+} \right) ||u||^{p^-}_{W_0^{1,q(x)}(M)} - \left(\frac{\lambda}{q^-} + \frac{1}{p^+} \right) ||u||^{q^+}_{W_0^{1,q(x)}(M)}.
\end{aligned}
$$

From (5.1), we have that J_{λ} is not bounded below on the whole space $W_0^{1,q(x)}(M)$, but it is bounded above on an appropriate subset of $W_0^{1,q(x)}(M)$ which is the Nehari manifold associated to J_{λ} defined by

$$
\mathcal{N}_{\lambda} = \{ u \in W_0^{1,q(x)}(M) \backslash \{0\} : \langle J'_{\lambda}(u), u \rangle = 0 \}.
$$

Indeed, if we take for example X a Banach space, and J the Euler (energy) functional associated with a variational problem on X. If J is bounded above and has a minimizer, then, this minimizer is a critical point of J. Therefore, it is a weak solution of the variational problem. However, in many problems, J is not bounded on the whole space X, but is bounded on an appropriate subset of X, which is the case of our problem.

So, it is clear that the critical points of the functional J_λ must lie on $\mathscr{N}_\lambda$ and local minimizers on $\mathscr{N}_\lambda$ are usually critical points of J_λ. Thus, $u \in \mathscr{N}_\lambda$ if and only if

$$\begin{aligned} \langle J_\lambda^{'}(u), u\rangle &= \int_M |\nabla u(x)|^{p(x)}\, dv_g(x) + \int_M \mu(x)\, |\nabla u(x)|^{q(x)}\, dv_g(x) \\ &- \lambda \int_M |u(x)|^{q(x)}\, dv_g(x) + \int_M |u(x)|^{p(x)}\, dv_g(x) \\ &- \int_M f(x,u(x))\,.\,u(x)\, dv_g(x) = 0. \end{aligned} \tag{5.3}$$

Hence, $\mathscr{N}_\lambda$ contains every nontrivial weak solution of problem ($\mathscr{P}$) (see definition 5.2.1). Moreover, we have the following result

Lemma 5.2.1. *Under assumptions $(f_1)-(f_3)$. The energy functional J_λ is coercive and bounded below on $W_0^{1,q(x)}(M)$.*

Proof. Let $u \in \mathscr{N}_\lambda$ and $||\,u\,|| > 1$, where $||\,.\,||$ is the induced norm of $W_0^{1,q(x)}(\Omega)\backslash\{0\}$. Then, by (5.3), (5.1), (1.23), (f_1), (f_3), propositions 1.4.4 and 1.4.6, we have

$$\begin{aligned} J_\lambda(u) &\geq \frac{1}{p^+}\int_M |\nabla u(x)|^{p(x)}\, dv_g(x) + \frac{1}{q^+}\int_M \mu(x)|\nabla u(x)|^{q(x)}\, dv_g(x) \\ &- \frac{\lambda}{q^-}\int_M |u(x)|^{q(x)}\, dv_g(x) + \frac{1}{p^+}\int_M |u(x)|^{p(x)}\, dv_g(x) - \int_M F(x,u(x))\, dv_g(x) \\ &= \frac{1}{p^+}\int_M \nabla u(x)|^{p(x)}\, dv_g(x) + \frac{1}{q^+}\int_M \mu(x)|\nabla u(x)|^{q(x)}\, dv_g(x) \\ &- \frac{\lambda}{q^-}\int_M |u(x)|^{q(x)}\, dv_g(x) + \frac{1}{p^+}\Big[-\int_M |\nabla u(x)|^{p(x)}\, dv_g(x) - \int_M \mu(x)|\nabla u(x)|^{q(x)}\, dv_g(x) \\ &+ \lambda\int_M |u(x)|^{q(x)}\, dv_g(x) + \int_M f(x,u(x))\,.\,u(x)\, dv_g(x)\Big] - \int_M F(x,u(x))\, dv_g(x) \\ &\geq \mu_0\left(\frac{1}{q^+} - \frac{1}{p^+}\right)\int_M |\nabla u(x)|^{q(x)}\, dv_g(x) + \lambda\left(\frac{1}{p^+} - \frac{1}{q^-}\right)\int_M |u(x)|^{q(x)}\, dv_g(x) \\ &+ \frac{1}{\beta}\int_M f(x,u(x))\,.\,u(x)\, dv_g(x) + \int_M F(x,u(x))\, dv_g(x) \end{aligned}$$

(since $\beta > p^+$, then $\frac{1}{p^+} > \frac{1}{\beta}$ and by (f_1) we get the following inequality)

$$\begin{aligned} &\geq \frac{\mu_0}{D^{p^+}(c+1)^{p^+}}\left(\frac{p^+ - q^+}{p^+q^+}\right)\rho_{p(\cdot)}(u) + \lambda\left(\frac{q^- - p^+}{p^+q^-}\right)\rho_{q(\cdot)}(u) \text{ (from (1.23))} \\ &\geq \frac{\mu_0}{D^{p^+}(c+1)^{p^+}}\left(\frac{p^+ - q^+}{p^+q^+}\right)||u||^{p^-} + \lambda\left(\frac{q^- - p^+}{p^+q^-}\right)||u||^{q^+} \text{ (from Proposition 1.4.6).} \end{aligned}$$

As $p^- > q^+$, then $J_\lambda(u) \longrightarrow +\infty$ as $||\,u\,|| \to \infty$. It follows that J_λ is coercive and bounded below on

$\mathcal{N}_\lambda$. □

Next, we consider the functional $\psi : \mathcal{N}_\lambda \longrightarrow \mathbb{R}$ defined by

$$\psi_\lambda(u) = \langle J_\lambda^{'}(u), u\rangle \text{ for all } u \in \mathcal{N}_\lambda.$$

Hence, it is natural to split $\mathcal{N}_\lambda$ into three part : the first set corresponding to local minima, the second set corresponding to local maxima, and the third one corresponding to points of inflection which defined respectively as follows

$$\mathcal{N}_\lambda^{+} = \{u \in \mathcal{N}_\lambda : \langle \psi_\lambda^{'}(u), u\rangle > 0\}.$$

$$\mathcal{N}_\lambda^{-} = \{u \in \mathcal{N}_\lambda : \langle \psi_\lambda^{'}(u), u\rangle < 0\}.$$

$$\mathcal{N}_\lambda^{0} = \{u \in \mathcal{N}_\lambda : \langle \psi_\lambda^{'}(u), u\rangle = 0\}.$$

Lemma 5.2.2. *Under assumptions $(f_1)-(f_3)$. There exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$ we have $\mathcal{N}_\lambda^0 = \emptyset$.*

Proof. Suppose otherwise, that is $\mathcal{N}_\lambda^0 \neq \emptyset$ for all $\lambda \in \mathbb{R}\backslash\{0\}$. Let $u \in \mathcal{N}_\lambda^0$ such that $||u|| > 1$. Then by (5.3), (5.1), (f_1) and the definition of $\mathcal{N}_\lambda^0$, we have

$$\begin{aligned}
0 = \langle \psi_\lambda^{'}(u), u\rangle \geq\ & p^- \int_M |\nabla u(x)|^{p(x)}\, dv_g(x) + q^- \int_M \mu(x)\,|\nabla u(x)|^{q(x)}\, dv_g(x) \\
& - q^+ \Bigg[\int_M |\nabla u(x)|^{p(x)}\, dv_g(x) + \int_M \mu(x)\,|\nabla u(x)|^{q(x)}\, dv_g(x) \\
& + \int_M |u(x)|^{p(x)}\, dv_g(x) - \int_M f(x,u(x))\,.\,u(x)\, dv_g(x) \Bigg] \\
& + p^- \int_M |u(x)|^{p(x)}\, dv_g(x) - \int_M F(x,u(x))\, dv_g(x) \\
& \geq (p^- - q^+) \int_M |\nabla u(x)|^{p(x)}\, dv_g(x) + (q^- - q^+) \int_M \mu(x)|\nabla u(x)|^{q(x)}\, dv_g(x) \\
& + (p^- - q^+) \int_M |u(x)|^{p(x)}\, dv_g(x) + q^+ \int_M f(x,u(x))\,.\,u(x)\, dv_g(x) \\
& - \int_M F(x,u(x))\, dv_g(x)
\end{aligned}$$

(since $\beta > p^+ > q^+$, and by (1.1) we have $q^+ > \frac{1}{q^+} > \frac{1}{\beta}$. Then,by (f_1) we get the following inequality)

$$\geq (q^- - q^+) \int_M \mu(x)|\nabla u(x)|^{q(x)}\, dv_g(x) + (p^- - q^+) \int_M |u(x)|^{p(x)}\, dv_g(x).$$

Then,

$$0 \geq (p^- - q^+) \int_M |u(x)|^{p(x)}\, dv_g(x) + (q^- - q^+) \int_M \mu(x) |\nabla u(x)|^{q(x)}\, dv_g(x).$$

Therefore, by Propositions 1.4.6, 1.4.8 and Theorem 1.4.2, we have

$$0 \geq (p^- - q^+)||u||^{p^-} + c_1 (q^- - q^+)||u||^{q^+},$$

where c_1 being the constant of the embedding Theorem 1.4.2.

Hence,

$$||u|| \leq \left(\frac{c_1(q^+ - q^-)}{p^- - q^+} \right)^{\frac{1}{p^- - q^+}}. \tag{5.4}$$

Analogously :

$$\begin{aligned} 0 = \langle \psi'_\lambda(u), u \rangle \leq\ & p^+ \int_M |\nabla u(x)|^{p(x)}\, dv_g(x) + q^+ \int_M \mu(x) |\nabla u(x)|^{q(x)}\, dv_g(x) \\ & - \lambda\, q^- \int_M |u(x)|^{q(x)}\, dv_g(x) + p^+ \Bigg[- \int_M |\nabla u(x)|^{p(x)}\, dv_g(x) \\ & - \int_M \mu(x) |\nabla u(x)|^{q(x)}\, dv_g(x) + \lambda \int_M |u(x)|^{q(x)}\, dv_g(x) \\ & + \int_M f(x,u(x))\,.\,u(x)\, dv_g(x) \Bigg] - \int_M F(x,u(x))\, dv_g(x) \\ \leq\ & (q^+ - p^+) \int_M \mu(x) |\nabla u(x)|^{q(x)}\, dv_g(x) \\ & + \lambda (p^+ - q^-) \int_M |u(x)|^{q(x)}\, dv_g(x) + p^+ \int_M f(x,u(x))\,.\,u(x)\, dv_g(x) \\ \leq\ & (q^+ - p^+) \int_M \mu(x) |\nabla u(x)|^{q(x)}\, dv_g(x) \\ & + \lambda (p^+ - q^-) \int_M |u(x)|^{q(x)}\, dv_g(x) + p^+ \int_M |u(x)|^{q(x)}\, dv_g(x). \end{aligned}$$

Then,

$$\begin{aligned} \mu_0 (p^+ - q^+) \int_M |\nabla u(x)|^{q(x)}\, dv_g(x) &\leq (p^+ - q^+) \int_M \mu(x) |\nabla u(x)|^{q(x)}\, dv_g(x) \\ &\leq \left[\lambda (p^+ - q^-) + p^+ \right] \int_M |u(x)|^{q(x)}\, dv_g(x). \end{aligned}$$

By (1.23) and proposition 1.4.4 we deduce that

$$\frac{\mu_0}{D^{p^+}(c+1)^{p^+}} (p^+ - q^+)\, ||u||^{p^-} \leq \left[\lambda\, (p^+ - q^-) + p^+ \right] ||u||^{q^+}.$$

Thus,

$$||u|| \leq \left(\frac{D^{p^+}(c+1)^{p^+}\left[\lambda\,(p^+-q^-)+p^+\right]}{\mu_0\,(p^+-q^+)} \right)^{\frac{1}{p^--q^+}}. \tag{5.5}$$

For λ sufficiently small $\left(\lambda < \frac{2\mu_0\,(p^+-q^+)}{D^{p^+}(c+1)^{p^+}(p^+-q^-)} - \frac{\mu_0\,c_1\,(q^+-q^-)(p^+-q^+)}{D^{p^+}(c+1)^{p^+}(p^+-q^-)(p^--q^-)} - \frac{p^+}{p^+-q^-}\right)$, if we combining (5.2), (5.4) and (5.5) we find $||u|| < 1$ for μ_0 sufficiently large, which contradicts our assumption. Consequently, we can conclude that there exists $\lambda^* > 0$ such that $\mathcal{N}_\lambda^0 = \emptyset$ for any $\lambda \in (0, \lambda^*)$. □

Remark 13. *As a consequence of Lemma 5.2.2, for* $0 < \lambda < \lambda^*$*, we can write* $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$*, and we define*

$$\theta_\lambda^+ = \inf_{u \in \mathcal{N}_\lambda^+} J_\lambda(u), \qquad \theta_\lambda^- = \inf_{u \in \mathcal{N}_\lambda^-} J_\lambda(u).$$

Lemma 5.2.3. *Suppose that* $(f_1)-(f_3)$ *are true. If* $0 < \lambda < \lambda^*$ *with* $\lambda^* > 0$*, then for all* $u \in \mathcal{N}_\lambda^+$ *we have* $J_\lambda(u) < 0$.

Proof. Suppose $u \in \mathcal{N}_\lambda^+$, from the definition of J_λ, we have

$$\begin{aligned} J_\lambda(u) \leq & \frac{1}{p^-}\int_M |\nabla u(x)|^{p(x)}\,dv_g(x) + \frac{1}{q^-}\int_M \mu(x)\,|\nabla u(x)|^{q(x)}\,dv_g(x) \\ & - \frac{\lambda}{q^+}\int_M |u(x)|^{q(x)}\,dv_g(x) + \frac{1}{p^-}\int_M |u(x)|^{p(x)}\,dv_g(x) \\ & - \int_M F(x,u(x))\,dv_g(x), \end{aligned} \tag{5.6}$$

from (5.3) and (5.6) we have

$$\begin{aligned} J_\lambda(u) \leq & \frac{1}{p^-}\int_M |\nabla u(x)|^{p(x)}\,dv_g(x) + \frac{1}{q^-}\int_M \mu(x)\,|\nabla u(x)|^{p(x)}\,dv_g(x) \\ & - \frac{1}{q^+}\Big[\int_M |\nabla u(x)|^{p(x)}\,dv_g(x) + \int_M \mu(x)\,|\nabla u(x)|^{q(x)}\,dv_g(x) \\ & + \int_M |u(x)|^{p(x)}\,dv_g(x) - \int_M f(x,u(x))\,.\,u(x)\,dv_g(x)\Big] \\ & + \frac{1}{p^-}\int_M |u(x)|^{p(x)}\,dv_g(x) - \int_M F(x,u(x))\,dv_g(x) \\ \leq & \left(\frac{1}{p^-} - \frac{1}{q^+}\right)\int_M |\nabla u(x)|^{p(x)}\,dv_g(x) + \left(\frac{1}{q^-} - \frac{1}{q^+}\right)\int_M \mu(x)\,|\nabla u(x)|^{q(x)}\,dv_g(x) \\ & + \left(\frac{1}{p^-} - \frac{1}{q^+}\right)\int_M |u(x)|^{p(x)}\,dv_g(x) + \frac{1}{q^+}\int_M f(x,u(x))\,.\,u(x)\,dv_g(x) \\ & - \int_M F(x,u(x))\,dv_g(x), \end{aligned}$$

using Poincaré inequality and Theorem 1.4.2, we find that :

$$J_\lambda(u) \leq -\left(\frac{p^- - q^+}{p^- q^+}\right)\left(\frac{1}{c}+1\right)||u||^{p^-} + \left[c_1\left(\frac{q^+ - q^-}{q^- q^+}\right) + \frac{1}{q^+}\right]||u||^{q^+}, \tag{5.7}$$

where c_1 being the constant of the embedding Theorem 1.4.2.

As $u \in \mathscr{N}_\lambda^+$ we have

$$\begin{aligned} &p^+ \int_M |\nabla u(x)|^{p(x)}\, dv_g(x) + q^+ \int_M \mu(x)|\nabla u(x)|^{q(x)}\, dv_g(x) \\ &- \lambda q^- \int_M |u(x)|^{q(x)}\, dv_g(x) + p^+ \int_M |u(x)|^{p(x)}\, dv_g(x) - \int_M F(x,u(x))\, dv_g(x) > 0, \end{aligned} \tag{5.8}$$

we multiply (5.3) by $(-p^+)$, we obtain

$$\begin{aligned} &-p^+ \int_M |\nabla u(x)|^{p(x)}\, dv_g(x) - p^+ \int_M \mu(x)|\nabla u(x)|^{q(x)}\, dv_g(x) \\ &+ \lambda p^+ \int_M |u(x)|^{q(x)}\, dv_g(x) - p^+ \int_M |u(x)|^{p(x)}\, dv_g(x) \\ &+ p^+ \int_M f(x,u(x))\,.\,u(x)\, dv_g(x) = 0, \end{aligned} \tag{5.9}$$

we add (5.8) to (5.9), then according to the fact that $\beta > p^+$ and by (5.1) we have $p^+ > \frac{1}{p^+} > \frac{1}{\beta}$. Therefore, we obtain that

$$(q^+ - p^+)\int_M \mu(x)|\nabla u(x)|^{q(x)}\, dv_g(x) + \lambda(p^+ - q^-)\int_M |u(x)|^{q(x)}\, dv_g(x) > 0.$$

Then, by (1.23) and Proposition 1.4.4 we get

$$\begin{aligned} \lambda(p^+ - q^-)\int_M |u(x)|^{q(x)}\, dv_g(x) &> (p^+ - q^+)\int_M \mu(x)\,|\nabla u(x)|^{q(x)}\, dv_g(x) \\ &> \frac{\mu_0(p^+ - q^+)}{D^{p^+}(c+1)^{p^+}}\int_M |u(x)|^{p(x)}\, dv_g(x). \end{aligned}$$

Hence,

$$||u||^{p^-} < \frac{\lambda D^{p^+}(c+1)^{p^+}(p^+ - q^-)}{\mu_0(p^+ - q^+)}||u||^{q^+}.$$

According to (5.7), we get

$$J_\lambda(u) \leq \left[-\left(\frac{p^- - q^+}{p^- q^+}\right)\left(\frac{1}{c}+1\right).\frac{\lambda D^{p^+}(c+1)^{p^+}(p^+ - q^-)}{\mu_0(p^+ - q^+)} + c_1\left(\frac{q^+ - q^-}{q^- q^+}\right) + \frac{1}{q^+}\right]||u||^{q^+}.$$

Finally, for λ sufficiently large, we deduce that $\theta_\lambda^+ = \inf_{u \in \mathscr{N}_\lambda^+} J_\lambda(u) < 0$. □

Lemma 5.2.4. *Under assumptions* $(f_1)-(f_3)$. *If* $0<\lambda<\lambda^{**}$, *then for all* $u\in\mathcal{N}_\lambda^-$ *we have* $J_\lambda(u)>0$.

Proof. Let $u\in\mathcal{N}_\lambda^-$. By (5.1), (f_1), (5.3) and the definition of J_λ, we find that

$$\begin{aligned}
J_\lambda(u) &\geq \frac{1}{p^+}\int_M |\nabla u(x)|^{p(x)}\,dv_g(x)+\frac{1}{q^+}\int_M \mu(x)\,|\nabla u(x)|^{q(x)}\,dv_g(x)\\
&-\frac{\lambda}{q^-}\int_M |u(x)|^{q(x)}\,dv_g(x)+\frac{1}{p^+}\Big[-\int_M |\nabla u(x)|^{p(x)}\,dv_g(x)\\
&-\int_M \mu(x)\,|\nabla u(x)|^{q(x)}\,dv_g(x)+\lambda\int_M |u(x)|^{q(x)}\,dv_g(x)\\
&+\int_M f(x,u(x))\,.\,u(x)\,dv_g(x)\Big]-\int_M F(x,u(x))\,dv_g(x)\\
&\geq \Big(\frac{1}{q^+}-\frac{1}{p^+}\Big)\int_M \mu(x)\,|\nabla u(x)|^{q(x)}\,dv_g(x)\\
&+\lambda\,\Big(\frac{1}{p^+}-\frac{1}{q^-}\Big)\int_M |u(x)|^{q(x)}\,dv_g(x)+\frac{1}{p^+}\int_M f(x,u(x))\,.\,u(x)\,dv_g(x)\\
&-\int_M F(x,u(x))\,dv_g(x)\\
&\geq \Big(\frac{1}{q^+}-\frac{1}{p^+}\Big)\int_M \mu(x)\,|\nabla u(x)|^{q(x)}\,dv_g(x)\\
&+\lambda\,\Big(\frac{1}{p^+}-\frac{1}{q^-}\Big)\int_M |u(x)|^{q(x)}\,dv_g(x)+\frac{1}{\beta}\int_M f(x,u(x))\,.\,u(x)\,dv_g(x)\\
&-\int_M F(x,u(x))\,dv_g(x)\\
&\text{(since } \beta>p^+\text{, then } \tfrac{1}{p^+}>\tfrac{1}{\beta} \text{ and by } (f_1) \text{ we get the following inequality)}\\
&\geq \mu_0\,\Big(\frac{1}{q^+}-\frac{1}{p^+}\Big)\int_M |\nabla u(x)|^{q(x)}\,dv_g(x)\\
&+\lambda\,\Big(\frac{1}{p^+}-\frac{1}{q^-}\Big)\int_M |u(x)|^{q(x)}\,dv_g(x),
\end{aligned}$$

according to (1.23) we deduce that

$$J_\lambda(u)\geq \frac{\mu_0}{D^{p^+}(c+1)^{p^+}}\Big(\frac{1}{q^+}-\frac{1}{p^+}\Big)\,||u||^{p^-}+\lambda\,\Big(\frac{1}{p^+}-\frac{1}{q^-}\Big)\,||u||^{q^+}.$$

Since, $p^->q^+$ we have

$$J_\lambda(u)\geq\left(\frac{\mu_0}{D^{p^+}(c+1)^{p^+}}\cdot\frac{p^+-q^+}{q^+\,p^+}+\lambda\,\frac{q^--p^+}{p^+\,q^-}\right)||u||^{p^-}.$$

Thus, if we choose $\lambda<\frac{\mu_0\,q^-\,(p^+-q^+)}{D^{p^+}(c+1)^{p^+}q^+\,(p^+-q^-)}=\lambda^{**}$, we deduce that $J_\lambda(u)>0$.
It follows that $\theta_\lambda^-=\inf_{u\in\mathcal{N}_\lambda^-}J_\lambda(u)>0$. □

Hence, $\mathcal{N}_\lambda=\mathcal{N}_\lambda^+\cup\mathcal{N}_\lambda^-$ and $\mathcal{N}_\lambda^+\cap\mathcal{N}_\lambda^-=\emptyset$, by above Lemma, we must have $u\in\mathcal{N}_\lambda^-$.

5.3 Existence of non-negative solutions

In this section, we prove the existence of two non-negative solutions of problem $(\mathscr{P})$. For this, we first show the existence of minimizers in $\mathscr{N}_\lambda^+$ and $\mathscr{N}_\lambda^-$ for all $\lambda \in (0, \bar{\lambda})$, where $\bar{\lambda} = \min\{\lambda^*, \lambda^{**}\}$.

Theorem 5.3.1. *Suppose that $(f_1)-(f_3)$ are true, then for all $\lambda \in (0, \lambda^*)$, there exists a minimizer u_0^+ of $J_\lambda(u)$ on $\mathscr{N}_\lambda^+$ such that $J_\lambda(u_0^+) = \theta_\lambda^+$.*

Proof. From Lemma 5.2.1, J_λ is bounded below on $\mathscr{N}_\lambda$, in particular is bounded below on $\mathscr{N}_\lambda^+$. Then there exists a minimizing sequence $\{u_n^+\} \subset \mathscr{N}_\lambda^+$ such that

$$\lim_{n\to+\infty} J_\lambda(u_n^+) = \inf_{u\in\mathscr{N}^+} J_\lambda(u) = \theta_\lambda^+ < 0.$$

Since, J_λ is coercive, $\{u_n^+\}$ is bounded in $W_0^{1,q(x)}(M)$. Hence we assume that, without loss generality, $u_n^+ \rightharpoonup u_0^+$ in $W_0^{1,q(x)}(M)$ and by the compact embedding (Theorem 1.4.1) we have

$$u_n^+ \longrightarrow u_0^+ \text{ in } L^{p(x)}(M). \tag{5.10}$$

Now, we shall prove $u_n^+ \longrightarrow u_0^+$ in $W_0^{1,q(x)}(M)$. Otherwise, let $u_n^+ \not\to u_0^+$ in $W_0^{1,q(x)}(M)$. Then, we have

$$\rho_{q(\cdot)}(u_0^+) < \lim_{n\to+\infty}\inf \rho_{q(\cdot)}(u_n^+), \tag{5.11}$$

using (5.10) we obtain

$$\int_M |u_0^+|^{p(x)}\, dv_g(x) = \lim_{n\to+\infty}\inf \int_M |u_n^+|^{p(x)}\, dv_g(x),$$

since $\langle J_\lambda'(u_n^+), u_n^+\rangle = 0$, and using the same technique as in Lemma 5.2.4, we get by (1.23) that

$$J_\lambda(u_n^+) \geq \frac{\mu_0}{D^{p^+}(c+1)^{p^+}}\Big(\frac{1}{q^+} - \frac{1}{p^+}\Big)\rho_{p(.)}(u_n^+) + \lambda\Big(\frac{1}{p^+} - \frac{1}{q^-}\Big)\rho_{q(.)}(u_n^+).$$

That is

$$\begin{aligned}\lim_{n\to+\infty} J_\lambda(u_n^+) \geq &\frac{\mu_0}{D^{p^+}(c+1)^{p^+}}\Big(\frac{1}{q^+} - \frac{1}{p^+}\Big)\lim_{n\to+\infty}\rho_{p(.)}(u_n^+)\\ &+\lambda\Big(\frac{1}{p^+} - \frac{1}{q^-}\Big)\lim_{n\to+\infty}\rho_{q(.)}(u_n^+).\end{aligned}$$

By (5.10) and (5.11), we have

$$\theta_\lambda^+ > \frac{\mu_0}{D^{p^+}(c+1)^{p^+}}\Big(\frac{1}{q^+}-\frac{1}{p^+}\Big)\,||u_0^+||^{p^-}+\lambda\,\Big(\frac{1}{p^+}-\frac{1}{q^-}\Big)\,||u_0^+||^{q^+},$$

since $p^- > q^+$, for $||u_0^+|| > 1$, we deduce

$$\theta_\lambda^+ = \inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u) > 0,$$

which is a contradiction with Lemma 5.2.3. Hence

$$u_n^+ \longrightarrow u_0^+ \text{ in } W_0^{1,q(x)}(M),$$

and

$$\lim_{n\to+\infty} J_\lambda(u_n^+) = J_\lambda(u_0^+) = \theta_\lambda^+.$$

Consequently, u_0^+ is a minimizer of J_λ on $\mathcal{N}_\lambda^+$. □

Theorem 5.3.2. *Suppose that conditions $(f_1)-(f_3)$ are true, and for all $\lambda\in(0,\lambda^{**})$, there exists a minimizer u_0^- of J_λ on $\mathcal{N}_\lambda^-$ such that $J_\lambda(u_0^-)=\theta_\lambda^-$.*

Proof. Since J_λ is bounded below on $\mathcal{N}_\lambda$ and so on $\mathcal{N}_\lambda^-$. Then, there exists a minimizing sequence $\{u_n^-\}\subseteq\mathcal{N}_\lambda^-$ such that

$$\lim_{n\to+\infty} J_\lambda(u_n^-) = \inf_{u\in\mathcal{N}_\lambda^-} J_\lambda(u) = \theta_\lambda^- > 0.$$

As J_λ is coercive, $\{u_n^-\}$ is bounded in $W_0^{1,q(x)}(M)$. Thus without loss of generality, we may assume that, $u_n^- \rightharpoonup u_0^-$ in $W_0^{1,q(x)}(M)$ and by Theorem 1.4.1 we have

$$u_n^- \longrightarrow u_0^- \text{ in } L^{p(x)}(M). \tag{5.12}$$

On the other hand, if $u_0^-\in\mathcal{N}_\lambda^-$, then there exists a constant $t>0$ such that $t\,u_0^-\in\mathcal{N}_\lambda^-$ and $J_\lambda(u_0^-)\geq J_\lambda(t\,u_0^-)$. According to (f_1) and the definition of ψ_λ', we have

$$
\begin{aligned}
\langle \psi_\lambda'(t\,u_0^-), t\,u_0^- \rangle = & \int_M p(x)\,|\nabla t u_0^-(x)|^{p(x)}\,dv_g(x) + q(x)\int_M \mu(x)\,|\nabla t u_0^-(x)|^{q(x)}\,dv_g(x) \\
& - \lambda\, q(x)\int_M |t u_0^-(x)|^{q(x)}\,dv_g(x) + p(x)\int_M |t u_0^-(x)|^{p(x)}\,dv_g(x) \\
& - \int_M F(x, t u_0^-(x))\,dv_g(x) \\
& \leq p^+ t^{p^+}\int_M |\nabla u_0^-|^{p(x)}\,dv_g(x) + q^+ t^{q^+}\int_M \mu(x)|\nabla u_0^-|^{q(x)}\,dv_g(x) \\
& - \lambda\, q^- t^{q^-}\int_M |u_0^-(x)|^{q(x)}\,dv_g(x) + p^+ t^{p^+}\int_M |u_0^-(x)|^{p(x)}\,dv_g(x).
\end{aligned}
$$

Since $q^- \leq q^+ < p^+$, and by (1.23), propositions 1.4.4 and 1.4.6, it follows that $\langle \psi_\lambda'(t\,u_0^-, t\,u_0^-)\rangle < 0$. Hence by the definition of $\mathcal{N}_\lambda^-$, $t\,u_0^- \in \mathcal{N}_\lambda^-$.

Next, we show that $u_n^- \longrightarrow u_0^-$ in $W_0^{1,q(x)}(M)$. Otherwise, suppose $u_n^- \not\to u_0^-$ in $W_0^{1,q(x)}(M)$. Then by Fatou's Lemma we have

$$
\int_M \mu(x)\,|\nabla u_0^-(x)|^{q(x)}\,dv_g(x) \leq \lim_{n\to+\infty}\int_M \mu(x)\,|\nabla u_n^-(x)|^{q(x)}\,dv_g(x).
$$

By (5.12) we get

$$
\int_M |u_0^-(x)|^{p(x)}\,dv_g(x) \leq \lim_{n\to+\infty}\int_M |u_n^-(x)|^{p(x)}\,dv_g(x),
$$

and

$$
\int_M |\nabla u_0^-(x)|^{p(x)}\,dv_g(x) \leq \lim_{n\to+\infty}\int_M |\nabla u_n^-(x)|^{p(x)}\,dv_g(x).
$$

Then, according the above inequalities and (f_1), we obtain

$$
\begin{aligned}
J_\lambda(t\,u_0^-) \leq & \frac{t^{p^+}}{p^-}\int_M |\nabla u_0^-(x)|^{p(x)}\,dv_g(x) + \frac{t^{q^+}}{q^+}\int_M \mu(x)\,|\nabla u_0^-(x)|^{q(x)}\,dv_g(x) \\
& - \frac{\lambda\, t^{q^-}}{q^+}\int_M |u_0^-(x)|^{q(x)}\,dv_g(x) + \frac{t^{p^+}}{p^+}\int_M |u_0^-(x)|^{p(x)}\,dv_g(x) \\
& - \int_M F(x, t\,u_0^-(x))\,dv_g(x) \\
\leq & \lim_{n\to+\infty}\Bigg[\frac{t^{p^+}}{q^-}\int_M |\nabla u_n^-(x)|^{p(x)}\,dv_g(x) \\
& + \frac{t^{q^+}}{q^+}\int_M \mu(x)\,|\nabla u_n^-(x)|^{q(x)}\,dv_g(x) - \frac{\lambda\, t^{q^-}}{q^+}\int_M |u_n^-(x)|^{q(x)}\,dv_g(x) \\
& + \frac{t^{p^+}}{p^+}\int_M |u_n^-(x)|^{p(x)}\,dv_g(x) - \int_M F(x, t\,u_0^-(x))\,dv_g(x)\Bigg] \\
\leq & \lim_{n\to+\infty} J_\lambda(t\,u_n^-) < \lim_{n\to+\infty} J_\lambda(u_n^-) = \inf_{u\in\mathcal{N}_\lambda^-} J(u) = \theta_\lambda^-.
\end{aligned}
$$

Hence, $J_\lambda(t u_0^-) < \inf_{u\in\mathcal{N}_\lambda^-} J_\lambda(u) = \theta_\lambda^-$, which is a contradiction. Consequently

$$u_n^- \longrightarrow u_0^- \text{ in } W_0^{1,q(x)}(M) \text{ and } \lim_{n\to+\infty} J_\lambda(u_n^-) = J_\lambda(u_0^-) = \theta_\lambda^- .$$

Then, we conclude that u_0^- is a minimizer of J_λ on $\mathcal{N}_\lambda^-$. □

Theorem 5.3.3. *Under assumptions $(f_1)-(f_3)$ we assume that the smooth complete compact Riemannian n-manifold (M, g) has property $B_{vol}(\lambda, v)$. Then, there exists $\bar\lambda$ such that for all $\lambda\in(0, \bar\lambda)$, the problem $(\mathscr{P})$ has at least two non-negative weak solutions.*

Proof. Form Theorems 5.3.1 and 5.3.2, we deduce that for any $\lambda \in (0, \bar\lambda)$, there exist $u_0^+ \in \mathcal{N}_\lambda^+$ and $u_0^- \in \mathcal{N}_\lambda^-$ such as

$$J_\lambda(u_0^+) = \inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u) \text{ and } J_\lambda(u_0^-) = \inf_{u\in\mathcal{N}_\lambda^-} J_\lambda(u).$$

Then, the problem $(\mathscr{P})$ has two solutions $u_0^+ \in \mathcal{N}_\lambda^+$ and $u_0^- \in \mathcal{N}_\lambda^-$ in $W_0^{1,q(x)}(M)$. By Lemma 5.2.2, it follows that $\mathcal{N}_\lambda^- \cap \mathcal{N}_\lambda^+ = \emptyset$. Then, $u_0^- \neq u_0^+$. Thus these two solutions are distinct.
Next, we prove that u_0^- and u_0^+ are non-negative in M. For this, we introduce the truncation function $h_+ : M\times\mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$h_+(x,s) = \begin{cases} 0 & \text{if } s<0\,, \\ \\ h(x,s) & \text{if } s\geq 0\,. \end{cases}$$

We set $H_+(x,s) = \int_0^s f(x,t)\,dt$ and consider the C^1-functional
$J_\lambda^+ : W_0^{1,q(x)}(M) \longrightarrow \mathbb{R}$ given by

$$\begin{aligned} J_\lambda^+(u) = &\int_M \frac{1}{p(x)}\,|\nabla u(x)|^{p(x)}\,dv_g(x) + \int_M \frac{\mu(x)}{q(x)}\,|\nabla u(x)|^{q(x)}\,dv_g(x) \\ &- \int_M H_+(x,u(x))\,dv_g(x). \end{aligned}$$

Then, by (1.23) and proposition 1.4.6 we have for all $u_- = \min\{0, u(x)\}$ that

$$\begin{aligned} 0 = \langle (J_\lambda^+)'(u_-), u_-\rangle &\geq p^-\,\rho_{p(.)}(|\nabla u_-|) + \frac{\mu_0}{D^{p^+}(c+1)^{p^+}}\,q^-\,\rho_{p(.)}(u_-) \\ &\geq \rho_{p(.)}(u_-) \geq ||u_-||^{p^-}. \end{aligned}$$

Hence, $||u_-|| = 0$, and thus $u = u_+$. Then, by taking $u = u_0^-$ and $u = u_0^+$ respectively, we deduce that u_0^- and u_0^+ are non-negative solutions of problem $(\mathscr{P})$. □

Conclusion : According to the above results, we can then say that $u^{\pm}$ are critical points of J_λ and hence are non-negative weak solutions of problem $(\mathscr{P})$.

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