Some cases of compatibility of the tangency relations of sets

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Abstract

In the present paper some cases of the compatibility of the tangency relations $T_{l_i}(a, b, k, p)$, $(i = 1, 2)$ of sets of the classes $\widetilde{M}_{p,k}$ having the Darboux property at the point p of the metric space (E, l_0) are considered. Certain sufficient conditions for the compatibility of these relations are shown here.

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§1. Introduction

Let E be an arbitrary non-empty set and let l be a non-negative real function defined on the Cartesian product $E_0 \times E_0$ of the family E_0 of all non-empty subsets of the set E.

Let l_0 be the function defined by the formula:

$$l_0(x, y) = l(\{x\}, \{y\}) \quad \text{for} \quad x, y \in E. \tag{1.1}$$

Making the certain assumptions concerning the function l, the function l_0 defined by (1.1) will be the metric of the set E. Then the pair (E, l) can be treated as a certain generalization of a metric space and we shall call it the generalized metric space (see [11]).

Similarly as in a metric space, using the formula (1.1), we may define in the space (E, l) the following notions: the sphere $S_l(p, r)$ and the open ball $K_l(p, r)$ with the centre at the point p and the radius r.

Let $S_l(p, r)_u$ denote the so-called u-neighbourhood of the sphere $S_l(p, r)$ in the generalized metric space (E, l) (see [5]).

Let a, b be arbitrary non-negative real functions defined in a certain right-hand side neighbourhood of 0 such that

$$a(r) \xrightarrow[r \to 0^+]{} 0 \quad \text{and} \quad b(r) \xrightarrow[r \to 0^+]{} 0. \tag{1.2}$$

If 0 is the cluster point of the set of all numbers $r > 0$ such that the sets $A \cap S_l(p, r)_{a(r)}$ and $B \cap S_l(p, r)_{b(r)}$ are non-empty, then we say that the pair (A, B) of sets $A, B \in E_0$ is (a, b)-clustered at the point p of the space (E, l).

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By the definition ([11]), we consider

$$(1.3)\quad T_l(a,b,k,p) = \{(A,B) \mid\ A,B \in E_0,\ \text{the pair } (A,B) \text{ is } (a,b)\text{-clustered at the point } p \text{ of the space } (E,l) \text{ and } \frac{1}{r^k} l(A \cap S_l(p,r)_{a(r)}, B \cap S_l(p,r)_{b(r)}) \xrightarrow[r\to 0^+]{} 0\}.$$

If $(A,B) \in T_l(a,b,k,p)$, then we say that the set A is (a,b)-tangent of order $k > 0$ to the set B at the point p of the space (E,l).

The set $T_l(a,b,k,p)$ defined by the formula (1.3) we call the relation of (a,b)-tangency of order k at the point p (shortly: the tangency relation) of sets in the generalized metric space (E,l).

If $(A,B) \in T_{l_1}(a_1,b_1,k,p) \Leftrightarrow (A,B) \in T_{l_2}(a_2,b_2,k,p)$ for $A,B \in E_0$, then the tangency relations $T_{l_1}(a_1,b_1,k,p)$ and $T_{l_2}(a_2,b_2,k,p)$ are called compatible in the set E.

We say that the set $A \in E_0$ has the Darboux property at the point p of the space (E,l_0), which we write: $A \in D_p(E,l_0)$ (see [6]), if there exists a number $\tau > 0$ such that $A \cap S_{l_0}(p,r) \neq \varnothing$ for $r \in (0,\tau)$.

In the present paper we consider some cases of the compatibility of the tangency relations of sets of the classes $\widetilde{M}_{p,k} \cap D_p(E,l_0)$, where l_0 is the metric generated by the functions $l \in \mathcal{F}_{f,\rho}$. The definition of the class of functions $\mathcal{F}_{f,\rho}$ we shall give in Section 2 of this paper.

§2. The compatibility of the tangency relations of sets

Let ρ be a metric of the set E and let A be any set of the family E_0. Let us put

$$(2.4)\qquad \rho(x,A) = \inf\{\rho(x,y) \mid\ y \in A\} \quad \text{for} \quad x \in E.$$

By A' we shall denote the set of all cluster points of the set $A \in E_0$. Let k be a fixed positive real number and let by the definition (see [6]) :

$$(2.5)\quad \widetilde{M}_{p,k} = \{A \in E_0 :\ p \in A' \text{ and there exists a number } \mu > 0 \text{ such that for an arbitrary } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that for every pair of points } (x,y) \in [A,p;\mu,k] \text{ if } \rho(p,x) < \delta \text{ and } \frac{\rho(x,A)}{\rho^k(p,x)} < \delta, \text{ then } \frac{\rho(x,y)}{\rho^k(p,x)} < \varepsilon\},$$

where

$$(2.6)\quad [A,p;\mu,k] = \{(x,y) \mid x \in E,\ y \in A \ \text{ and } \ \mu\rho(x,A) < \rho^k(p,x) = \rho^k(p,y)\}.$$

Let f be subadditive increasing and continuous real function defined in a certain right-hand side neighbourhood of 0 such that $f(0) = 0$. By $\mathcal{F}_{f,\rho}$ we shall denote the class of all functions l fulfilling the conditions:

1^0 $\ l : E_0 \times E_0 \longmapsto [0,\infty)$,

2^0 $\ f(\rho(A,B)) \le l(A,B) \le f(d_\rho(A \cup B))$ for $A,B \in E_0$,

where $\rho(A,B)$ is the distance of sets A,B and $d_\rho(A \cup B)$ is the diameter of the union of sets A,B in the metric space (E,ρ).

Because

$$f(\rho(x,y)) = f(\rho(\{x\},\{y\})) \leq l(\{x\},\{y\}) \leq f(d_\rho(\{x\}\cup\{y\})) = f(\rho(x,y)),$$

then from this and from (1.1) it follows that

$$l_0(x,y) = f(\rho(x,y)) \quad \text{for } \ l \in \mathcal{F}_{f,\rho} \ \text{ and } \ x,y \in E. \tag{2.7}$$

It is easy to prove that the function l_0 defined by the formula (2.7) is the metric of the set E.

In the paper [6] the following theorem was proved:

Theorem 2.1 . *If* $l_1, l_2 \in \mathcal{F}_{f,\rho}$ *and*

$$\frac{a(r)}{r^{k+1}} \xrightarrow[r\to 0^+]{} \alpha \quad \text{and} \quad \frac{b(r)}{r^{k+1}} \xrightarrow[r\to 0^+]{} \beta, \tag{2.8}$$

where $\alpha, \beta \in [0,\infty)$, *then the tangency relations* $T_{l_1}(a,b,k,p)$, $T_{l_2}(a,b,k,p)$ *are compatible in the classes of sets* $\widetilde{M}_{p,k} \cap D_p(E,l_0)$.

It appears that the assumptions of Theorem 2.1 related to the function a, b can be weakened, using the following Lemma from [10]:

Lemma 2.1 . *If the function* a *fulfils the condition*

$$\frac{a(r)}{r^k} \xrightarrow[r\to 0^+]{} 0, \tag{2.9}$$

then for an arbitrary set $A \in \widetilde{M}_{p,k}$ *having the Darboux property at the point* p *of the metric space* (E,ρ)

$$\frac{1}{r^k} d_\rho(A \cap S_\rho(p,r)_{a(r)}) \xrightarrow[r\to 0^+]{} 0. \tag{2.10}$$

From the equality (2.7) and from the assumption concerning the function f it follows that

$$f(d_\rho A) = d_{l_0} A = \sup\{l_0(x,y) \mid x,y \in A\} \quad \text{for } \ A \in E_0. \tag{2.11}$$

Because every function l belonging to the class $\mathcal{F}_{f,\rho}$ generates on the set E the metric l_0, then from this and from Lemma 2.1 we get

$$\frac{1}{r^k} d_{l_0}(A \cap S_{l_0}(p,r)_{a(r)}) \xrightarrow[r\to 0^+]{} 0, \tag{2.12}$$

if $A \in \widetilde{M}_{p,k} \cap D_p(E,l_0)$ and the function a fulfils the condition (2.9).

Using the equality (2.11) and the condition (2.12) resulted from Lemma 2.1, we prove now the following theorem:

Theorem 2.2 . *If* $l_i \in \mathcal{F}_{f,\rho}$ *for* $i = 1,2$,

$$\frac{a(r)}{r^k} \xrightarrow[r\to 0^+]{} 0 \quad \text{and} \quad \frac{b(r)}{r^k} \xrightarrow[r\to 0^+]{} 0, \tag{2.13}$$

then the tangency relations $T_{l_1}(a,b,k,p)$ *and* $T_{l_2}(a,b,k,p)$ *are compatible in the classes of sets* $\widetilde{M}_{p,k} \cap D_p(E,l_0)$.

Proof. We assume that the functions a, b fulfil the condition (2.13). From the fact that the functions $l_1, l_2 \in \mathcal{F}_{f,\rho}$ generate on the set E the metric l_0 we get the equalities

$$S_{l_1}(p,r) = S_{l_2}(p,r) = S_{l_0}(p,r), \tag{2.14}$$

where $S_{l_0}(p,r)$ is the sphere with the centre at the point p and the radius r in the metric space (E, l_0).

Let us suppose that $(A,B) \in T_{l_1}(a,b,k,p)$ for $A, B \in \widetilde{M}_{p,k} \cap D_p(E, l_0)$ and $l_1 \in \mathcal{F}_{f,\rho}$. From this it follows that the pair of sets (A,B) is (a,b)-clustered at the point p of the space (E, l_1) and

$$\frac{1}{r^k} l_1(A \cap S_{l_0}(p,r)_{a(r)}, B \cap S_{l_0}(p,r)_{b(r)}) \xrightarrow[r \to 0^+]{} 0. \tag{2.15}$$

From the inequality

$$d_\rho(A \cup B) \le d_\rho A + d_\rho B + \rho(A,B) \quad \text{for} \quad A, B \in E_0, \tag{2.16}$$

from the properties of the function f and from the fact that $l_1, l_2 \in \mathcal{F}_{f,\rho}$ we obtain

$$\left| \frac{1}{r^k} l_2(A \cap S_{l_0}(p,r)_{a(r)}, B \cap S_{l_0}(p,r)_{b(r)}) - \frac{1}{r^k} l_1(A \cap S_{l_0}(p,r)_{a(r)}, B \cap S_{l_0}(p,r)_{b(r)}) \right|$$

$$\le \frac{1}{r^k} f(d_\rho((A \cap S_{l_0}(p,r)_{a(r)}) \cup (B \cap S_{l_0}(p,r)_{b(r)}))) - \frac{1}{r^k} f(\rho(A \cap S_{l_0}(p,r)_{a(r)}, B \cap S_{l_0}(p,r)_{b(r)}))$$

$$\le \frac{1}{r^k} f(d_\rho(A \cap S_{l_0}(p,r)_{a(r)}) + d_\rho(B \cap S_{l_0}(p,r)_{b(r)}) + \rho(A \cap S_{l_0}(p,r)_{a(r)}, B \cap S_{l_0}(p,r)_{b(r)}))$$

$$- \frac{1}{r^k} f(\rho(A \cap S_{l_0}(p,r)_{a(r)}, B \cap S_{l_0}(p,r)_{b(r)}))$$

$$\le \frac{1}{r^k} f(d_\rho(A \cap S_{l_0}(p,r)_{a(r)})) + \frac{1}{r^k} f(d_\rho(B \cap S_{l_0}(p,r)_{b(r)})). \tag{2.17}$$

From the assumption (2.13), from the equality (2.11) and from the condition (2.12) we have

$$\frac{1}{r^k} f(d_\rho(A \cap S_{l_0}(p,r)_{a(r)})) \xrightarrow[r \to 0^+]{} 0, \tag{2.18}$$

and

$$\frac{1}{r^k} f(d_\rho(B \cap S_{l_0}(p,r)_{b(r)})) \xrightarrow[r \to 0^+]{} 0. \tag{2.19}$$

From (2.15), (2.18), (2.19) and from the inequality (2.17) we get

$$\frac{1}{r^k} l_2(A \cap S_{l_0}(p,r)_{a(r)}, B \cap S_{l_0}(p,r)_{b(r)}) \xrightarrow[r \to 0^+]{} 0. \tag{2.20}$$

Because the functions $l_1, l_2 \in \mathcal{F}_{f,\rho}$ generate on the set E the same metric l_0 (see (2.7)), then from the fact that the pair of sets (A,B) is (a,b)-clustered at the point p of the space (E, l_1) it follows that (A,B) is (a,b)-clustered at the point p of the space (E, l_2). Hence and from (2.20) it results that $(A,B) \in T_{l_2}(a,b,k,p)$ for $A, B \in \widetilde{M}_{p,k} \cap D_p(E, l_0)$ and $l_2 \in \mathcal{F}_{f,\rho}$.

If the pair (A, B) of sets $A, B \in \widetilde{M}_{p,k} \cap D_p(E, l_0)$ belongs to $T_{l_2}(a, b, k, p)$, then analogously we prove that $(A, B) \in T_{l_1}(a, b, k, p)$ for $l_1 \in \mathcal{F}_{f,\rho}$.

From the above considerations it follows that the tangency relation $T_{l_1}(a, b, k, p)$ and $T_{l_2}(a, b, k, p)$ are compatible in the classes of sets $\widetilde{M}_{p,k} \cap D_p(E, l_0)$ for $l_1, l_2 \in \mathcal{F}_{f,\rho}$, if the functions a, b fulfil the condition (2.13). This ends the proof.

Let us put by the definition:

$$\begin{aligned} \rho_1(A, B) &= \rho(A, B), \\ \rho_2(A, B) &= \sup\{\rho(x, B) : \ x \in A\}, \\ \rho_3(A, B) &= \inf\{d_\rho(\{x\} \cup B) : \ x \in A\}, \\ \rho_4(A, B) &= \sup\{\rho(x, y) : \ x \in A, \ y \in B\}, \\ \rho_5(A, B) &= d_\rho(A \cup B) \end{aligned} \tag{2.21}$$

for the sets A, B of the family E_0.

In the paper [4] was proved the following lemmas:

Lemma 2.2 . *For arbitrary sets* $A, B \in E_0$

$$\begin{aligned} \rho_2(A, B) &\le \rho_1(A, B) + d_\rho A, \\ \rho_4(A, B) &\le \rho_3(A, B) + d_\rho A, \\ \rho_5(A, B) &\le \rho_3(A, B) + d_\rho A. \end{aligned} \tag{2.22}$$

Lemma 2.3 . *For arbitrary sets* $A, B \in E_0$

$$\begin{aligned} \rho_3(A, B) &\le \rho_1(A, B) + d_\rho B, \\ \rho_4(A, B) &\le \rho_2(A, B) + d_\rho B, \\ \rho_5(A, B) &\le 2\rho_2(A, B) + d_\rho B. \end{aligned} \tag{2.23}$$

Let f be a subadditive increasing and continuous real function defined in a certain right-hand side neighbourhood of 0 such that $f(0) = 0$, and let l be the function fulfilling one of the inequalities :

$$\begin{aligned} f(\rho_1(A, B)) \le l(A, B) \le f(\rho_2(A, B)), \\ f(\rho_3(A, B)) \le l(A, B) \le f(\rho_4(A, B)), \\ f(\rho_3(A, B)) \le l(A, B) \le f(\rho_5(A, B)) \end{aligned} \tag{2.24}$$

for any sets $A, B \in E_0$.

It is easy to notice that every function l fulfilling the inequalities (2.24) belongs to the class $\mathcal{F}_{f,\rho}$ and generates on the set E the metric l_0.

Theorem 2.3 . *If*

$$\frac{a(r)}{r^k} \xrightarrow[r \to 0^+]{} 0 \qquad \text{and} \qquad b(r) \xrightarrow[r \to 0^+]{} 0, \tag{2.25}$$

and the functions l_1, l_2 *fulfil simultaneously one and only one of the inequalities* (2.24) *for sets of the classes* $\widetilde{M}_{p,k} \cap D_p(E, l_0)$, *then the tangency relations* $T_{l_1}(a, b, k, p)$ *and* $T_{l_2}(a, b, k, p)$ *are compatible in these classes of sets.*

Proof. Let us suppose that the functions l_1, l_2 fulfil the first of inequalities (2.24) for the sets A, B belonging to the classes $\widetilde{M}_{p,k} \cap D_p(E, l_0)$. Hence, from the first of inequalities (2.22) and from the properties of the function f we get

$$\left|\frac{1}{r^k} l_1(A \cap S_{l_0}(p,r)_{a(r)}, B \cap S_{l_0}(p,r)_{b(r)}) - \frac{1}{r^k} l_2(A \cap S_{l_0}(p,r)_{a(r)}, B \cap S_{l_0}(p,r)_{b(r)})\right|$$

$$\leq \frac{1}{r^k} f(\rho_2(A \cap S_{l_0}(p,r)_{a(r)}, B \cap S_{l_0}(p,r)_{b(r)})) - \frac{1}{r^k} f(\rho_1(A \cap S_{l_0}(p,r)_{a(r)}, B \cap S_{l_0}(p,r)_{b(r)}))$$

$$\leq \frac{1}{r^k} f(d_\rho(A \cap S_{l_0}(p,r)_{a(r)})) = \frac{1}{r^k} d_{l_0}(A \cap S_{l_0}(p,r)_{a(r)}). \tag{2.26}$$

Because $A, B \in D_p(E, l_0)$, then the pair of sets (A, B) is (a, b)-clustered at the point p of the space (E, l_1) and (E, l_2). From this, from the inequality (2.26), from the assumption (2.25) and from the condition (2.12) it follows that the tangency relations $T_{l_1}(a, b, k, p)$ and $T_{l_2}(a, b, k, p)$ are compatible in the classes of sets $\widetilde{M}_{p,k} \cap D_p(E, l_0)$, when the functions l_1, l_2 satisfy the first of inequalities (2.24).

If the functions l_1, l_2 fulfill the second or third of inequalities (2.24), then using the appropriate inequalities (2.22) analogously we prove the statement of this theorem.

Now we suppose that the function l fulfills one of the inequalities:

$$\begin{gathered} f(\rho_1(A,B)) \leq l(A,B) \leq f(\rho_3(A,B)), \\ f(\rho_2(A,B)) \leq l(A,B) \leq f(\rho_4(A,B)), \\ f(2\rho_2(A,B)) \leq l(A,B) \leq f(\rho_5(A,B)) \end{gathered} \tag{2.27}$$

for any sets $A, B \in E_0$.

Evidently, every function l fulfilling the inequalities (2.27) belongs to the class $\mathcal{F}_{f,\rho}$ and generates on the set E the metric l_0.

Theorem 2.4 *If*

$$a(r) \xrightarrow[r \to 0^+]{} 0 \quad \text{and} \quad \frac{b(r)}{r^k} \xrightarrow[r \to 0^+]{} 0, \tag{2.28}$$

and the functions l_1, l_2 satisfy simultaneously one and only one of the inequalities (2.27) for $A, B \in \widetilde{M}_{p,k} \cap D_p(E, l_0)$, then the tangency relations $T_{l_1}(a, b, k, p)$, $T_{l_2}(a, b, k, p)$ are compatible in the classes of sets $\widetilde{M}_{p,k} \cap D_p(E, l_0)$.

Proof. Let us assume that the functions l_1, l_2 fulfil the first of inequalities (2.27) for $A, B \in \widetilde{M}_{p,k} \cap D_p(E, l_0)$. Hence, from the first of inequalities (2.23) and from the properties of the function f we obtain

$$\left|\frac{1}{r^k} l_1(A \cap S_{l_0}(p,r)_{a(r)}, B \cap S_{l_0}(p,r)_{b(r)}) - \frac{1}{r^k} l_2(A \cap S_{l_0}(p,r)_{a(r)}, B \cap S_{l_0}(p,r)_{b(r)})\right|$$

$$\leq \frac{1}{r^k} f(\rho_3(A \cap S_{l_0}(p,r)_{a(r)}, B \cap S_{l_0}(p,r)_{b(r)})) - \frac{1}{r^k} f(\rho_1(A \cap S_{l_0}(p,r)_{a(r)}, B \cap S_{l_0}(p,r)_{b(r)}))$$

$$\leq \frac{1}{r^k} f(d_\rho(B \cap S_{l_0}(p,r)_{b(r)})) = \frac{1}{r^k} d_{l_0}(B \cap S_{l_0}(p,r)_{b(r)}). \tag{2.29}$$

Because $A, B \in D_p(E, l_0)$, then the pair of sets (A, B) is (a, b)-clustered at the point p of the space (E, l_1) and (E, l_2). Hence, from the inequality (2.29), from the assumption (2.28) of this theorem and from Lemma 2.1 of the paper [10] it follows that the tangency relations $T_{l_1}(a, b, k, p)$ and $T_{l_2}(a, b, k, p)$ are compatible in the classes of sets $\widetilde{M}_{p,k} \cap D_p(E, l_0)$, when the functions l_1, l_2 fulfil the first of inequalities (2.27).

If the functions l_1, l_2 satisfy the second or third of inequalities (2.27), then using the suitable inequalities (2.23) identically we prove the statement of the theorem.

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