K-arcs in finite Benz planes

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Abstract

In this paper, it is shown that certain combinatorial properties about $k-$arc in finite Benz planes by using the concept of $k-$arc and $(k;n)-$ arc in finite affine planes.

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§1. Introduction

The subject of arcs has been studied in detail in projective planes ([4], [5], [6]) but it is rather new in Benz planes. Arcs have been partially studied in Benz planes [2]. In this paper, we introduce some combinatorial properties about $k-$arcs in Benz planes.

An *incidence structure* is a pair of sets $(\mathcal{P}, B)$ with a binary relation $I \subseteq \mathcal{P} \times B$ called *incidence*. The elements of $\mathcal{P}$ are called *points*, and the elements of B are called blocks.

Two points are *dependent* if there is no block incident to both of them, otherwise they are said to be *independent*.

Given a point $P \in \mathcal{P}$, the set of all points of $\mathcal{P}$ which are dependent with P is denoted by $P^{\perp}$.

The *nucleus* of an incidence structure is the set of points of $\mathcal{P}$ which are incidence with no block of B.

A *singular line* is a maximal set of pairwise dependent points of $\mathcal{P}$.

Given an incidence structure $(\mathcal{P}, B)$, the *derivate structure* at one of its points, say X, is an incidence structure $(\mathcal{P}_X, B_X)$ with the set of points $\mathcal{P}_X = \mathcal{P} \backslash (\{X\} \cup X^{\perp})$, and the set of blocks B_X consisting of the restrictions to $\mathcal{P}_X$ of all the blocks in B that are incident with X, plus the singular lines containing at least two points of $\mathcal{P}_X$.

A *finite Benz plane* is an incidence structure $\mathcal{B} = (\mathcal{P}, B)$ whose blocks are called *circles*, such that:

B1 Any independent three points are on exactly one circle and any circle has at least three points.

B2 Given $X \in \mathcal{P}$ and $c \in B$, if $\left|X^{\perp} \cap c\right| \geq 3$ then $c \subset X^{\perp}$.

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B3 Any circle has precisely one point in common with each singular line.

B4 Given a point $X \in \mathcal{P}$ and a singular line l , then $X^{\perp}$ either contains one point of l or all the points of l.

B5 Given a point $X \in \mathcal{P}$, if X is not in the nucleus then the derivate structure at X is a finite affine plane.

If q is the order of the derived affine plane, then q is said to be the *order* of the Benz plane.

Theorem 1. ([1],[3]) *There are only three types of Benz planes.*

I The nucleus is empty and there are no singular lines ($M\ddot{o}bius\ plane$, $or\ inversive\ plane$).

II The nucleus contains precisely one point. Any point, except the one in the nucleus, is contained in precisely one singular line (*Laguerre plane*).

III The nucleus is empty. Any point is contained in precisely two singular lines. The set of singular lines is partitioned in two families: Two lines belonging to the same family are disjoint and any line has precisely one point in common with each line of the opposite family ($Minkowski\ plane$).

A Möbius plane of order q contains $q^2 + 1$ points, a Laguerre plane of order q contains $q^2 + q$ points, and a Minkowski plane of order q contains $(q + 1)^2$ points.

The set of all circles through two distinct points P, Q is called bundle and denoted by $[PQ]$. These points P, Q are called the carriers of the bundle $[PQ]$. In a Benz plane of order q the bundle $[PQ]$ contains $q+\lambda$ circles such that Benz plane is Möbius plane, Laguerre plane, Minkowski plane, for $\lambda = 1, 0, -1$, respectively.

A $(k; 3)-$arc $\mathcal{K}$ in an affine plane of order q, is a set of k points such that some line of the plane meets $\mathcal{K}$ in three points but such that no line meets $\mathcal{K}$ in more than three points [7].

$\mathcal{A}$ will denote a finite affine plane of order q. A line l of $\mathcal{A}$ is an $i-$secant of a $(k; 3)-$arc $\mathcal{K}$ if $|l \cap \mathcal{K}| = i$, $i = 0, 1, 2, 3$. Let τ'_i denotes total number of $i-$secants to $\mathcal{K}$, δ'_i denotes the number of $i-$secants to $\mathcal{K}$ through a point P of $\mathcal{K}$, σ'_i denotes the number of $i-$secants to $\mathcal{K}$ through a point Q of $\mathcal{B} \setminus \mathcal{K}$.

Lemma 1. ([6]) *For a* $(k; 3)-$*arc* $\mathcal{K}$*, the following equations hold:*

$$i\tau'_i = \sum_P \delta'_i, \qquad (q - i)\tau'_i = \sum_Q \sigma'_i.$$

§2. $k-$arcs

Definition. ([1]) A $k-$ arc in a finite Benz plane is a set of k points, none two of which are on the same singular line, such that there is no circle containing more than three of them.

Throughout this section, $\mathcal{B}$ will denote a finite Benz plane of order q. A circle c of $\mathcal{B}$ is an $i-$secant of a $k-$arc $\mathcal{K}$ if $|c \cap \mathcal{K}| = i$, $i = 0, 1, 2, 3$. Let τ_i denotes total number of $i-$secants to $\mathcal{K}$, $\delta_i = \delta_i(P)$ denotes total number of $i-$secants to $\mathcal{K}$ through a point P of $\mathcal{K}$, $\sigma_i = \sigma_i(Q)$ denotes total number of $i-$secants to $\mathcal{K}$ through a point Q of $\mathcal{B} \setminus \mathcal{K}$, $\gamma_i = \gamma_i(R,S)$ denotes total number of $i-$secants to $\mathcal{K}$ through two points R, S of $\mathcal{K}$, $\eta_i = \eta_i(M,N)$ denotes total number of $i-$secants to $\mathcal{K}$ through two points R, S of $\mathcal{B} \setminus \mathcal{K}$.

A $k-$arc in $\mathcal{B}$ is complete if there is no $k+1-$arc containing it.

Proposition 1. *For a $k-$arc $\mathcal{K}$, the following equations hold:*

(i) $\tau_0 + \tau_1 + \tau_2 + \tau_3 = q(q^2 + \lambda)$

(ii) $\tau_1 + 2\tau_2 + 3\tau_3 = kq(q + \lambda)$

(iii) $\delta_1 + \delta_2 + \delta_3 = q(q + \lambda)$

(iv) $\gamma_2 + \gamma_3 = q + \lambda$

(v) $\gamma_3 = k - 2$

(vi) $\sigma_0 + \sigma_1 + \sigma_2 + \sigma_3 = q(q + \lambda)$

(vii) $\eta_0 + \eta_1 + \eta_2 + \eta_3 = q + \lambda$

(viii) $\eta_1 + 2\eta_2 + 3\eta_3 = k$

(ix) $\begin{pmatrix} 2 \\ i \end{pmatrix} \tau_i = \sum_{[P,Q]} \gamma_i$, $i = 2, 3$

(x) $i\tau_i = \sum_{P} \delta_i$, $i = 1, 2, 3$

(xi) $(q + 1 - i)\tau_i = \sum_{Q} \sigma_i$, $i = 0, 1, 2, 3$

Proof. The proof is finished each equation in the proposition expresses in a different way the cardinality of the following sets, respectively.

(i) $\{c \mid c \in B\}$.

(ii) $\{(P,c) \mid P \in \mathcal{K} \cap c,\ c \text{ is an } i-\text{secant of } \mathcal{K}\}$.

(iii) $\{c \mid P \in \mathcal{K},\ cIP,\ c \text{ is an } i-\text{secant of } \mathcal{K}\}$.

(iv) $\{c \mid c \in [RS] \text{ with } R,S \in \mathcal{K},\ c \text{ is an } i-\text{secant of } \mathcal{K}\}$.

(v) $\{(P,c) \mid P \in \mathcal{K},\ cIP,\ c \in [RS] \text{ with } R,S \in \mathcal{K},\ c \text{ is an } i-\text{secant of } \mathcal{K}\}$.

(vi) $\{c \mid Q \in \mathcal{B}\backslash\mathcal{K},\ cIQ,\ c \text{ is an } i-\text{secant of } \mathcal{K}\}$.

(vii) $\{c \mid c \in [RS] \text{ with } R,S \in \mathcal{B}\backslash\mathcal{K},\ c \text{ is an } i-\text{secant of } \mathcal{K}\}$.

(viii) $\{(P,c) \mid P \in \mathcal{K} \cap c,\ c \in [RS] \text{ with } R,S \in \mathcal{B}\backslash\mathcal{K},\ c \text{ is an } i-\text{secant of } \mathcal{K}\}$.

(ix) $\{(\{P,Q\},c) \mid c \in [PQ] \text{ with } P,Q \in \mathcal{K},\ c \text{ is an } i-\text{secant of } \mathcal{K}\}$.

(x) $\{(P, c) \mid P \in \mathcal{K},\ cIP,\ c \text{ is an } i-\text{secant of } \mathcal{K}\}$.

(xi) $\{(Q, c) \mid Q \in \mathcal{B}\backslash\mathcal{K},\ QIc,\ c \text{ is an } i-\text{secant of } \mathcal{K}\}$.

□

For a $k-$arc $\mathcal{K}$, let $(c_j)_i$ be the number of points of $\mathcal{B}\backslash\mathcal{K}$ such that through each of them exactly i, $j-$secants of $\mathcal{K}$ pass.

Proposition 2. *The constants $(c_j)_i$ of a $k-$arc $\mathcal{K}$ in $\mathcal{B}$ satisfy the following equations with the summation taken from* 0 *to β_j,*

$$\begin{array}{lll} \textit{If } \mathcal{B} \textit{ is Möbius plane,} & \textit{then} & \sum (c_j)_i = q^2+1-k \\ \textit{If } \mathcal{B} \textit{ is Laguerre plane,} & \textit{then} & \sum (c_j)_i = q^2+q-k \\ \textit{If } \mathcal{B} \textit{ is Minkowski plane,} & \textit{then} & \sum (c_j)_i = (q+1)^2-k \end{array}$$

Proof. The proof is clear that the equations express in different ways the cardinality of $\{Q \mid Q \in \mathcal{B}\backslash\mathcal{K}\}$ set in $\mathcal{B}$. □

Proposition 3. *The constants $(c_j)_i$ of a $k-$arc $\mathcal{K}$ in $\mathcal{B}$ ($\mathcal{B}$ is Möbius, Laguerre, Minkowski plane, for $\lambda = 1, 0, -1$,* respectively*), satisfy the following equations ;*

$$\begin{aligned} \sum_{i=0}^{\beta_0} i(c_0)_i &= (q+1)\left[q(q^2+\lambda) - (\tau_1+\tau_2+\tau_3)\right] \\ \sum_{i=0}^{\beta_1} i(c_1)_i &= kq\left[q(q+\lambda) - \begin{pmatrix} 2 \\ k-1 \end{pmatrix} - (k-1)(q-k+\lambda+2)\right] \\ \sum_{i=0}^{\beta_2} i(c_2)_i &= \begin{pmatrix} 2 \\ k \end{pmatrix}(q-1)(q-k+\lambda+2) \\ \sum_{i=0}^{\beta_3} i(c_3)_i &= \begin{pmatrix} 3 \\ k \end{pmatrix}(q-2) \end{aligned}$$

Proof. The proof is easily obtained by considering the equations express in different ways the cardinality of the following sets;

$$\{(Q, c) \mid Q \in c\backslash\mathcal{K},\ c \text{ a } i-\text{secant of } \mathcal{K}\}$$

when c is $0-$secant, $1-$secant, $2-$secant, $3-$secant for i, ii, iii, iv, respectively. □

Let $\mathcal{K}$ be a $k-$arc in $\mathcal{B}$, $Q \in \mathcal{B}\backslash\mathcal{K}$, if Q is not contained by singular lines through $\mathcal{K}$, then $\vartheta = 0$, if Q is contained by one singular line through $\mathcal{K}$, then $\vartheta = 1$, if Q is contained by two singular lines through $\mathcal{K}$, then $\vartheta = 2$, and a $k-$arc in $\mathcal{B}$ is $(k-\vartheta; 3)-$arc in $\mathcal{B}_Q$. Hence, while circles through Q are $0, 1, 2, 3-$secant circles to $\mathcal{K}$ in $\mathcal{B}$, this circles are $0, 1, 2, 3-$secant lines to $(k; 3)-$arc in $\mathcal{B}_Q$.

Proposition 4. *Let $P \in \mathcal{B}\backslash\mathcal{K}$ be any point and l is any singular line through $\mathcal{K}$ with $P \notin l$. If T_i denotes total number of $i-$secants to $\mathcal{K}$ through P and $\mathcal{B}$ is Möbius,* *Laguerre, Minkowski plane, for $\mu = 0, 1, 2,$ respectively, then the following equations hold:*

$$T_0 = \tau_0' - \mu(q-k) = \frac{1}{q}\sum_{Q}\sigma_0' - \mu(q-k)$$

$$T_1 = \tau_1' - \mu k = \sum_{P}\delta_1' - \mu k = \frac{1}{q-1}\sum_{Q}\sigma_1' - \mu k$$

$$T_2 = \tau_2' = \frac{1}{2}\sum_{P}\delta_2' = \frac{1}{q-2}\sum_{Q}\sigma_2'$$

$$T_3 = \tau_3' = \frac{1}{3}\sum_{P}\delta_3' = \frac{1}{q-3}\sum_{Q}\sigma_3'.$$

Proof. For $\vartheta = 0$, $k-$arc in $\mathcal{B}$ is $(k;3)-$arc in $\mathcal{B}_P$. Therefore, it is obtained upper equations from Lemma 1. □

Proposition 5. *Let $P \in \mathcal{B}\backslash\mathcal{K}$ be any point and l is any singular line through $\mathcal{K}$ with $P \in l$. If T_i denotes total number of $i-$secants to $\mathcal{K}$ through P and $\mathcal{B}$ is Möbius, Laguerre, Minkowski plane, for $\mu = 0, 1, 2,$ respectively, then the following equations hold:*

$$T_0 = \tau_0' - \mu(q-k+1) = \frac{1}{q}\sum_{Q}\sigma_0' - \mu(q-k+1)$$

$$T_1 = \tau_1' - \mu(k-1) = \sum_{P}\delta_1' - \mu(k-1) = \frac{1}{q-1}\sum_{Q}\sigma_1' - \mu(k-1)$$

$$T_2 = \tau_2' = \frac{1}{2}\sum_{P}\delta_2' = \frac{1}{q-2}\sum_{Q}\sigma_2'$$

$$T_3 = \tau_3' = \frac{1}{3}\sum_{P}\delta_3' = \frac{1}{q-3}\sum_{Q}\sigma_3'.$$

Proof. For $\vartheta = 1$, $k-$arc in $\mathcal{B}$ is $(k-1;3)-$arc in $\mathcal{B}_P$. Hence, it is obtained upper equations from Lemma 1. □

Proposition 6. *Let l_1 and l_2 be any two singular lines meet $\mathcal{K}$ and $P \in \mathcal{B}\backslash\mathcal{K}$ with $P = l_1 \cap l_2$. If T_i denotes total number of $i-$secants to $\mathcal{K}$ through P and $\mathcal{B}$ is Möbius, Laguerre, Minkowski plane, for $\mu = 0, 1, 2,$ respectively, then the following equations hold:*

$$T_0 = \tau_0' - \mu(q-k+2) = \frac{1}{q}\sum_{Q}\sigma_0' - \mu(q-k+2)$$

$$T_1 = \tau_1' - \mu(k-2) = \sum_{P}\delta_1' - \mu(k-2) = \frac{1}{q-1}\sum_{Q}\sigma_1' - \mu(k-2)$$

$$T_2 = \tau_2' = \frac{1}{2}\sum_{P}\delta_2' = \frac{1}{q-2}\sum_{Q}\sigma_2'$$

$$T_3 = \tau_3' = \frac{1}{3}\sum_{P}\delta_3' = \frac{1}{q-3}\sum_{Q}\sigma_3'.$$

Proof. For $\vartheta = 2$, $k-$arc in $\mathcal{B}$ is $(k-2;3)-$arc in $\mathcal{B}_P$. Hence, it is obtained upper equations from Lemma 1. □

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