Statistical structures on tangent bundles

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Dedicated to professor Hitoshi Furuhata on the occasion of his marriage

Abstract

We study prolongations of statistical structures on manifolds to their tangent bundles. We show that tangent bundles over flat statistical manifolds admit natural almost complex statistical structure with Norden metric.

M.S.C. 2000: 53B05, 53B30, 53A15.
Key words: statistical manifold, Norden metric, tangent bundle, vertical lift, complete lift, horizontal lift.

§0. Introduction

A statistical structure on a manifold M is a pair (h, ∇) such that h is a semi-Riemannian metric and ∇ is a torsion free linear connection with property ∇h is totally symmetric. A manifold M with a statistical structure is called a *statistical manifold*. A semi-Riemannian manifold (M, h) together with Levi-Civita connection ∇^0 of h is a typical example of statistical manifold. In other words, statistical manifolds can be regarded as generalisations of semi-Riemannian manifolds.

Statistical manifolds provide geometric model of probability distributions. Geometry of statistical manifolds has been applied to various fields of information science, *e.g.*, information theory, neural networks, statistical mechanics etc.

We refer to the readers Amari and Nagaoka's textbook [2] for general theory of statistical manifolds. See also [6] and [20].

The study of statistical structures has another motivation. On equiaffinely immersed hypersurface M in unimodular affine space $\mathbf{A}^{n+1}$, a statistical structure (h, ∇) is naturally induced. More precisely the metric h is the affine fundamental form and ∇ is the connection induced from the natural flat connection of $\mathbf{A}^{n+1}$ by the equiaffine immersion (See Section 2.1 in [24]).

For realisation problem of statistical manifolds in unimodular affine space (as equiaffinely immersed hypersurfaces), we refer to [16], [19].

In this paper we shall study prolongation of statistical structures on manifolds to their tangent bundles.

It turned out that in the study of complex-affine differential geometry, complex statistical manifolds with Norden metric would play an important role [22].

We shall show that tangent bundles over flat statistical manifolds admit natural almost complex statistical structure with Norden metric.

Applied Sciences, Vol.5, No.1, 2003, pp. 55-75.

§1. Statistical manifolds

We start with recalling the notion of statistical structure reformulated by Kurose.

Definition 1.1. ([17]) Let M be a manifold, h a semi-Riemannian metric and ∇ a torsion free linear connection. Then ∇ is said to be *compatible* to h if the covariant derivative $C := \nabla h$ is symmetric.

A pair (h, ∇) of a semi-Riemannian metric with a compatible linear connection is called a *statistical structure* on M.

A manifold M together with a statistical structure is called a *statistical manifold*.

In particular statistical manifolds with flat connection are traditionally called *Hessian manifolds* [30].

On a statistical manifold (M, h, ∇), the symmetric tensor field $C = \nabla h$ is called the *cubic form* or *skewness field* of M.

Next, on a statistical manifold (M, h, ∇), the *conjugate connection* $\nabla^\dagger$ of ∇ with respect to h is introduced by the following formula:

$$X \cdot h(Y, Z) = h(\nabla_X Y, Z) + (Y, \nabla^\dagger_X Z), \ X, Y, Z \in \mathcal{X}(M). \tag{1.1}$$

Obviously $\nabla = \nabla^\dagger$ if and only if ∇ coincides with the Levi-Civita connection ∇^0. Let us denote the musical (metrical) isomorphism from TM to T^*M by $\flat$:

$$\flat : TM \to T^*M; \ X^\flat(Y) = h(X, Y), X, Y \in \mathcal{X}(M).$$

Denote by ∇^* the *dual connection* of T^*M induced by ∇, *i.e.*,

$$(\nabla^*_X \omega)Y := X(\omega(Y)) - \omega(\nabla_X Y), \quad \omega \in \mathcal{X}^*(M), \ X, \ Y \in \mathcal{X}(M).$$

Then we get $\flat \circ \nabla^\dagger = \nabla^* \circ \flat$. Namely, if we identify TM with T^*M via $\flat$, then $\nabla^\dagger$ is identified with ∇^*. On this reason, we often denote the conjugate connection by ∇^* and call it *dual connection*.

Define the tensor field K of type $(1, 2)$ by

$$h(K(X)Y, Z) = C(X, Y, Z) = (\nabla_X h)(Y, Z)$$

for all X, Y, Z. Since C is symmetric, $K(X)$ is symmetric with respect to h. We call this tensor field K the *skewness operator* of (M, h, ∇). The difference of ∇ and ∇^0 is given by

$$\nabla - \nabla^0 = -\frac{1}{2}K.$$

This formula implies that ∇^0 is the "mean" of ∇ and ∇^*:

$$\nabla^0 = \frac{1}{2}(\nabla + \nabla^*).$$

More generally for any real number α,

$$\nabla^\alpha_X Y = \nabla^0_X Y - \frac{\alpha}{2}K(X)Y \tag{1.2}$$

defines a torsion free linear connection ∇^α. The linear connection ∇^α is called the *α-connection*. Note that $\nabla^1 = \nabla$ and $\nabla^{-1} = \nabla^*$.

Proposition 1.2 ([20]) *The covariant derivative of h relative to ∇^α is*

$$(\nabla^\alpha_X h)(Y,Z) = \alpha C(X,Y,Z).$$

Thus (M,h,∇^α) is statistical for all $\alpha \in \boldsymbol{R}$.

For every statistical manifold (M,h,∇), there exists a naturally associated symmetric trilinear form C.

Conversely let (M,h,C) be a semi-Riemannian manifold with symmetric trilinear form C. Then define the tensor field A by

$$h(K(X)Y,Z) = C(X,Y,Z).$$

and a linear connection ∇ by $\nabla = \nabla^0 - K/2$. Then ∇ is of torsion free and satisfies $\nabla h = C$. Hence the triplet (M,h,∇) becomes a statistical manifold.

Thus to equip a statistical structure (h,∇) is equivalent to equip a pair of structue (h,C) consisting of a semi-Riemannian metric h and a trilinear form C. Lauritzen [20] introduced the notion of statistical manifold as a semi-Riemannian manifold (M,h) together with a trilinear form C.

Next we consider almost complex structures on statistical manifolds.

Definition 1.3. Let (M,J) be an almost complex manifold with almost complex structure J, and (h,∇) a statistical structure on M. Then the structure (J,h,∇) is said to be an *almost complex statistical structure with Hermitian metric* if h is Hermitian, *i.e.*,

$$h(JX,JY) = h(X,Y).$$

Similarly almost complex statistical manifolds with Norden metrics are defined in the following way:

Definition 1.4. Let (M,J) be an almost complex manifold with almost complex structure J, and (h,∇) a statistical structure on M. Then the structure (J,h,∇) is said to be an *almost complex statistical structure with Norden metric* if h is Norden, *i.e.*,

$$h(JX,JY) = -h(X,Y).$$

It is obvious that every Norden metric is a neutral semi-Riemannian metric. In addition, an almost statistical manifold (M,h,∇,J) with Hermitian or Norden metric is said to be a *complex statistical manifold* with Hermition or Norden metric if $\nabla J = 0$. Note that the requirement $\nabla J = 0$ implies the integrability of J.

§2. Tangent bundles

Let M be a smooth n-manifold and $x \in M$. The tangent space T_xM itself is a real analytic n-manifold. We start with how to describe the *tangent space* $T_u(T_xM)$ of T_xM at $u \in T_xM$. Recall that, by definition, the tangent space $T_u(T_xM)$ is the set of all derivations acting on the space of smooth functions $C^\infty(T_xM)$.

Let $u,\ w \in T_xM$ then the directional derivative w_u

$$w_u(f) := \left.\frac{d}{dt}\right|_{t=0} f(u+tw) \tag{2.1}$$

is a tangent vector of T_xM at u. The correspondence $w \mapsto w_u$ defines a canonical linear isomorphism $T_xM \to T_u(T_xM)$.

This canonical isomorphism can be viewed as a *lifting operation* of tangent vectors from M to its tangent bundle TM. To explain this observation, we recall the vertical lift operation to TM for tangent vectors.

We denote by π the natural projection of TM onto M. Taking a local coordinate system $(U; x^1, \cdots, x^n)$, we denote the induced coordinate system on $\pi^{-1}(U)$ by $(x^1, \cdots, x^n; u^1, \cdots, u^n)$. For a point $u = (x; u)$ of TM, we denote the kernel of π_{*u} by $\mathcal{V}_u$ and call it the *vertical subspace* of $T_u(TM)$ at u. The correspondence $u \mapsto \mathcal{V}_u$ defines a distribution on TM. This distribution is called the *vertical distribution* of TM. Tangent vectors belong to vertical subspaces are called *vertical vectors*. By definition, the vertical subspace $\mathcal{V}_u$ is spanned by $\{\partial/\partial u^1, \cdots, \partial/\partial u^n\}$. It is easy to see that the vertical distribution $\mathcal{V}$ is integrable and hence becomes a foliation and a vector bundle over TM. The sections of vertical subbundle is called *vertical vector fields* on TM.

The two linear spaces T_xM and $\mathcal{V}_u$ have the same dimension. Moreover, there exists a canonical linear isomorphism $V : T_xM \to \mathcal{V}_u$, called the *vertical lift*.

In fact, for any tangent vector w on M with local expression:

$$w = w^i \frac{\partial}{\partial x^i}\bigg|_x,$$

the vertical lift w^{v} to TM is defined by

$$w^{\mathrm{v}} = w^i \frac{\partial}{\partial u^i}\bigg|_u. \tag{2.2}$$

This definition is independent of the choice of local coordinate system.

We can easily see that

$$w_u = (w^{\mathrm{v}})_u, \quad u,\ w \in T_xM. \tag{2.3}$$

Namely the canonical isomorphism $T_xM \to T_u(T_xM)$ is a vertical lift operation $T_xM \to \mathcal{V}_u \subset T_u(TM)$. Thus we conclude that

$$\mathcal{V}_u = T_u(T_xM), \quad x \in M,\ u \in T_xM. \tag{2.4}$$

This fundamental fact implies the following isomorphism of vector bundles:

$$\pi^*TN \cong \mathcal{V}.$$

Here π^*TM is the pulled back bundle of TM by $\pi : TM \to M$.

It is easy to check that

$$\mathbf{U} = u^i \frac{\partial}{\partial u^i} \tag{2.5}$$

is a vertical vector field globally defined on TM. This definition of $\mathbf{U}$ is independent of the choice of local coordinate system. The vertical vector field $\mathbf{U}$ is called the *canonical vertical vector field* of TM. Since the directional derivative w_w is expressed as

$$\mathbf{U}_w = w_w = (w^{\mathrm{v}})_w,\ w \in TM,$$

we may call $\mathbf{U}$ the *vertical position vector field* of TM.

§3. Vertical lifts

In this section we shall collect formulae of vertical lift operations developed by Kobayashi and Yano [36].

For a function f on M, the function $f \circ \pi$ on TM is denoted by f^{v} and call it the *vertical lift* of f to TM.

Next the vertical lift operation for tangent vectors is extended naturally to vector fields. For any vector field X on M with local expression:

$$X = X^i \frac{\partial}{\partial x^i}, \tag{3.1}$$

the vertical lift X^{v} to TM is defined by

$$X^{\mathrm{v}} = (X^i)^{\mathrm{v}} \frac{\partial}{\partial u^i}. \tag{3.2}$$

This definition is independent of the choice of local coordinate system. Note that for any $X \in \mathcal{X}(M)$ its vertical lift X^{v} is complete.

Direct calculation shows that for any vector field $X,\ Y \in \mathcal{X}(M)$,

$$[X^{\mathrm{v}}, Y^{\mathrm{v}}] = 0. \tag{3.3}$$

To define the vertical lift operation of 1-forms, we shall prepare the following notational convention.

Let ω be a 1-form on M. Then ω is naturally regarded as a smooth function on TM. More precisely we shall denote the (induced) function on TM by $\iota\omega$. Note that the vertical lift X^{v} of a vector field X is characterised as

$$X^{\mathrm{v}}(\iota\omega) = \{\omega(X)\}^{\mathrm{v}}.$$

Let f be a smooth function on M. Then the vertical lift of df is defined by

$$(df)^{\mathrm{v}} = d(f^{\mathrm{v}}).$$

In particular for local coordinate function x_i, $(dx^i)^{\mathrm{v}} = dx^i$.

The vertical lift of an arbitrary 1-form ω with local expression:

$$\omega = \omega_i dx^i,$$

is defined by the formula:

$$\omega^{\mathrm{v}} = (\omega_i)^{\mathrm{v}} (dx^i)^{\mathrm{v}}. \tag{3.4}$$

The definition of ω^{v} is independent of the choice of local coordinate system.

We can extend the vertical lift operation on the full tensor algebra $\mathcal{T}(M)$ by the rule:

$$(P \otimes Q)^{\mathrm{v}} = P^{\mathrm{v}} \otimes Q^{\mathrm{v}}, \tag{3.5}$$

for any tensor fields P and Q on M.

§4. Complete lifts

In this section we shall recall the complete lift operation to TM introduced by [36].

For a function f on M, the complete lift f^{c} of f to TM is a function

$$f^{\mathrm{c}} = \iota df = u^i \frac{\partial f}{\partial x_i}.$$

The complete lift of a vector field X to TM is a vector field X^{c} on TM characterised by the formula $X^{\mathrm{c}} f^{\mathrm{c}} = (Xf)^{\mathrm{c}}$ for all function f. Note that $X^{\mathrm{c}} f^{\mathrm{v}} = (Xf)^{\mathrm{v}}$.

Let X be a vector field on M with local expression:

$$X = X^i \frac{\partial}{\partial x^i}.$$

Then the complete lift X^{c} has the following local expression:

$$X^{\mathrm{c}} = (X^i)^{\mathrm{v}} \frac{\partial}{\partial x^i} + u^j \frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial u^i}. \tag{4.1}$$

This formula implies the following

Proposition 4.1 *Let $v = (x; v)$ be a point of TM which is not the zero section. Then the set $\{X^{\mathrm{c}}_v \mid X \in \mathcal{X}(M)\}$ is the whole tangent space of TM at v.*

The complete lift of a 1-form ω is a 1-form ω^{c} on TM defined by $\omega^{\mathrm{c}}(X^{\mathrm{c}}) = (\omega(X))^{\mathrm{c}}$. By definition, we have $d(\omega^{\mathrm{c}}) = (d\omega)^{\mathrm{c}}$ for any 1-form ω on M.

The complete lift operation is extended to the full tensor algebra $\mathcal{T}(M)$ by the rule:

$$(P \otimes Q)^{\mathrm{c}} = P^{\mathrm{c}} \otimes Q^{\mathrm{v}} + P^{\mathrm{v}} \otimes Q^{\mathrm{c}}, \tag{4.2}$$

for any tensor fields P and Q on M.

Here we shall collect some formulae for our use.

Proposition 4.2 *Let P be a tensor field of type $(r, s), r = 0, 1$ on M. Then*

$$P^{\mathrm{c}}(X_1^{\mathrm{c}}, \cdots, X_s^{\mathrm{c}}) = (P(X_1, \cdots, X_s))^{\mathrm{c}}, \quad P^{\mathrm{c}}(X_1^{\mathrm{v}}, \cdots, X_s^{\mathrm{v}}) = 0. \tag{4.3}$$

Corollary 4.3 *Let h be tensor field of type (0,2) on M. Then*

$$h^{\mathrm{c}}(X^{\mathrm{c}}, Y^{\mathrm{c}}) = h(X, Y)^{\mathrm{c}}, \quad h^{\mathrm{c}}(X^{\mathrm{c}}, Y^{\mathrm{v}}) = h(X, Y)^{\mathrm{v}}, \quad h^{\mathrm{c}}(X^{\mathrm{v}}, Y^{\mathrm{v}}) = 0. \tag{4.4}$$

In particular if h is a semi-Riemannian metric on M. Then h^{c} is a *neutral metric* on TM, *i.e.,* a semi-Riemannian metric on TM of signature (n, n). The metric h^{c} is called the *complete lift metric* on TM.

Remark 4.4. (Vertical lifts of metrics) Let h be a tensor field of type $(0, 2)$. Then the vertical lift h^{v} to TM is given by

$$h^{\mathrm{v}}(X^{\mathrm{c}}, Y^{\mathrm{c}}) = h(X, Y)^{\mathrm{v}}, \quad h^{\mathrm{v}}(X^{\mathrm{c}}, Y^{\mathrm{v}}) = 0, \quad h^{\mathrm{v}}(X^{\mathrm{v}}, Y^{\mathrm{v}}) = 0.$$

These formulae imply that h^{v} is degenerate even if h is nondegenerate.

Corollary 4.5 *Let P be a tensor field of type* (1,1)*. Then*

$$P^{\mathrm{c}}X^{\mathrm{c}} = (PX)^{\mathrm{c}}, P^{\mathrm{c}}X^{\mathrm{v}} = (PX)^{\mathrm{v}}, P^{\mathrm{v}}X^{\mathrm{c}} = (PX)^{\mathrm{v}}, P^{\mathrm{v}}X^{\mathrm{v}} = 0.$$

Proposition 4.6 *The Lie derivatives of P are given by*

$$\pounds_{X^{\mathrm{c}}}P^{\mathrm{c}} = (\pounds_X T)^{\mathrm{c}}, \quad \pounds_{X^{\mathrm{c}}}P^{\mathrm{v}} = (\pounds_X T)^{\mathrm{v}}, \quad \pounds_{X^{\mathrm{v}}}P^{\mathrm{c}} = (\pounds_X T)^{\mathrm{v}}, \quad \pounds_{X^{\mathrm{v}}}P^{\mathrm{v}} = 0.$$

In particular for any vector fields X and Y on M,

$$[X^{\mathrm{c}}, Y^{\mathrm{c}}] = [X, Y]^{\mathrm{c}}, \quad [X^{\mathrm{c}}, Y^{\mathrm{v}}] = [X, Y]^{\mathrm{v}}. \tag{4.5}$$

Finally we shall recall the complete lift operation of linear connection. Let ∇ be a linear connection on M. Then the formula

$$\nabla^{\mathrm{c}}_{X^{\mathrm{c}}}Y^{\mathrm{c}} = (\nabla_X Y)^{\mathrm{c}} \tag{4.6}$$

defines a linear connection ∇^{c} on TM. This connection ∇^{c} is called the complete lift of ∇ to TM.

Proposition 4.7 *For any tensor field P on M. Then the covariant derivative of P by $X \in \mathcal{X}(M)$ is described by*

$$\nabla^{\mathrm{c}}_{X^{\mathrm{c}}}P^{\mathrm{c}} = (\nabla_X P)^{\mathrm{c}}, \nabla^{\mathrm{c}}_{X^{\mathrm{c}}}P^{\mathrm{v}} = (\nabla_X P)^{\mathrm{v}}, \nabla^{\mathrm{c}}_{X^{\mathrm{v}}}P^{\mathrm{c}} = (\nabla_X P)^{\mathrm{v}}, \nabla^{\mathrm{c}}_{X^{\mathrm{v}}}P^{\mathrm{v}} = 0.$$

In particular for $X,\ Y \in \mathcal{X}(M)$,

$$\nabla^{\mathrm{c}}_{X^{\mathrm{c}}}Y^{\mathrm{v}} = \nabla^{\mathrm{c}}_{X^{\mathrm{v}}}Y^{\mathrm{c}} = (\nabla_X Y)^{\mathrm{v}}, \quad \nabla^{\mathrm{c}}_{X^{\mathrm{v}}}Y^{\mathrm{v}} = 0.$$

Corollary 4.8 *Let (M, ∇) be a manifold with a linear connection ∇. And let T and R be the torsion and curvature tensor field of ∇ respectively. Then the torsion and curvature of ∇^{c} are T^{c} and R^{c} respectively.*

Corollary 4.9 *Let (M, g, ∇^0) be a semi-Riemanian manifold with Levi-Civita connection ∇^0. Then the Levi-Civita connection of g^{c} is $(\nabla^0)^{\mathrm{c}}$.*

Remark 4.10. On a semi-Riemannian manifold (M, h), we may identify TM with cotangent bundle T^*M via h. Then the complete lift metric h^{c} coincides with so-called *Riemann extension* of the Levi-Civita connection ∇^0 of h to T^*M introduced by Patterson and Walker *[26]*. Note that although the complete lift metric on TM is derived from metric on the base manifold, the Riemann extension on T^*M is derived from a linear connection on M. (See Appendix A1.)

Now let (M, h, ∇) be a statistical manifold. Then we can construct complete lifts of h and ∇. Murathan and Güney [23] studied the complete lift of statistical structure. They showed that a complete lift operation induces a statistical structure on the tangent bundle TM. They lifted cubic form C and α-connections to TM. Here we shall give a simple proof of their result.

Proposition 4.11 *Let (M, h, ∇) be a statistical manifold. Then $(TM, h^{\mathrm{c}}, \nabla^{\mathrm{c}})$ is a statistical manifold with neutral metric. The cubic form of $(TM, h^{\mathrm{c}}, \nabla^{\mathrm{c}})$ is the complete lift C^{c} of C. The conjugate connection of ∇^{c} is $(\nabla^{\mathrm{c}})^* = (\nabla^*)^{\mathrm{c}}$.*

Proof. Since ∇ is torsion free, ∇^{c} is also torsion free by Corollary 4.8. Next by definition of the complete lift,

$$(\nabla^{\mathrm{c}}_{X^{\mathrm{c}}}h^{\mathrm{c}})(Y^{\mathrm{c}},Z^{\mathrm{c}}) = \{(\nabla_X h)(Y,Z)\}^{\mathrm{c}} = \{(\nabla_Y h)(X,Z)\}^{\mathrm{c}} = (\nabla^{\mathrm{c}}_{Y^{\mathrm{c}}}h^{\mathrm{c}})(X^{\mathrm{c}},Z^{\mathrm{c}}).$$

Hence $\nabla^{\mathrm{c}}h^{\mathrm{c}}$ is symmetric. The cubic form of $(TM, h^{\mathrm{c}}, \nabla^{\mathrm{c}})$ is $\nabla^{\mathrm{c}}h^{\mathrm{c}} = (\nabla h)^{\mathrm{c}} = C^{\mathrm{c}}$. Next

$$\begin{aligned}
h^{\mathrm{c}}(Y^{\mathrm{c}}, \nabla^{\mathrm{c}*}_{X^{\mathrm{c}}}Z^{\mathrm{c}}) &= X^{\mathrm{c}}h^{\mathrm{c}}(Y^{\mathrm{c}},Z^{\mathrm{c}}) - h^{\mathrm{c}}(\nabla^{\mathrm{c}}_{X^{\mathrm{c}}}Y^{\mathrm{c}},Z^{\mathrm{c}}) \\
&= X^{\mathrm{c}}\{h(Y,Z)\}^{\mathrm{c}} - \{h(\nabla_X Y, Z)\}^{\mathrm{c}} \\
&= \{Xh(Y,Z)\}^{\mathrm{c}} - \{h(\nabla_X Y, Z)\}^{\mathrm{c}} \\
&= \{Xh(Y,Z) - h(\nabla_X Y, Z)\}^{\mathrm{c}} = \{h(Y, \nabla^*_X Z)\}^{\mathrm{c}} \\
&= h^{\mathrm{c}}(Y^{\mathrm{c}}, (\nabla^*)^{\mathrm{c}}_{X^{\mathrm{c}}}Z^{\mathrm{c}}).
\end{aligned}$$

Hence $\nabla^{c*} = \nabla^{*c}$. Since the cubic form of $(TM, h^{\mathrm{c}}, \nabla^{\mathrm{c}})$ is C^{c}, this statistical structure $(h^{\mathrm{c}}, \nabla^{\mathrm{c}})$ coincides with that of [23]. □

Corollary 4.12 *The α-connection $\nabla^{\mathrm{c}\,\alpha}$ of $(TM, h^{\mathrm{c}}, \nabla^{\mathrm{c}})$ is given by $\nabla^{\mathrm{c}\,\alpha} = (\nabla^{\alpha})^{\mathrm{c}}$ and satisfies the following formula*:

$$\nabla^{\mathrm{c}\alpha}_{X^{\mathrm{c}}}Y^{\mathrm{c}} = \nabla^{h^{\mathrm{c}}}_{X^{\mathrm{c}}}Y^{\mathrm{c}} - \frac{\alpha}{2}K^{\mathrm{c}}(X^{\mathrm{c}})Y^{\mathrm{c}}.$$

Here K^{c} is the complete lift of A and $\nabla^{h^{\mathrm{c}}}$ is the Levi-Civita connection of the complete lift metric h^{c}.

Finally we recall how the complete lift operation is used for study of Jacobi fields.

Theorem 4.13 ([9], [33], [36]) *Let (M, ∇) be a manifold with a linear connection and $\psi : I \subset \boldsymbol{R} \to (TM, \nabla^{\mathrm{c}})$ a smooth curve in TM. Then ψ is a geodesic in TM with respect to ∇^{c} if and only if $\gamma := \pi \circ \psi$ is a geodesic in (M, ∇) and ψ is a Jacobi field along γ.*

Note that this theorem is generalised for harmonic maps. See [10] and [8], p. 391.

Theorem 4.14 ([10]) *Let (M, g_M) and (N, g_N) be* (semi) *Riemannian manifolds and $\psi : M \to TN$ a smooth map. Then the tension field $\tau(\psi)$ of ψ with respect to the complete lift metric g^{c}_N is given by*

$$\tau(\psi) = \{\tau(\varphi)\}^{\mathrm{h}} + \{\mathcal{J}_{\varphi}(\psi)\}^{\mathrm{v}}.$$

Here $\varphi := \pi \circ \psi$ and $\mathcal{J}_{\varphi}$ denotes the Jacobi operator of φ. Thus $\psi : (M, g_M) \to (TN, g^{\mathrm{c}}_N)$ is harmonic if and only if φ is harmonic and ψ is the Jacobi field of φ.

Proposition 4.15 ([32]) *Let (M, g, ∇^0) be a Riemannian manifold and γ be a geodesic in M. Denote by ϕ the geodesic flow of (M, g). For a vector field X along γ, X is a Jacobi field along γ if and only if X^{c} is ϕ-invariant.*

§5. Horizontal lifts

The vertical lift operation and the complete lift operation depend on the smooth structure of the base manifold M. In this section, we shall recall the horizontal lift operation which depends on linear connection on M.

Let M be a smooth n-manifold. A linear connection ∇ on M corresponds to a splitting of the tangent bundle $T(TM)$ of TM into the vertical distribution $\mathcal{V}$ and its complementary distribution $\mathcal{H}$:

$$T(TM) = \mathcal{H} \oplus \mathcal{V}. \tag{5.1}$$

The complementary distribution $\mathcal{H} = \mathcal{H}_\nabla$ corresponds to ∇ is called the *horizontal distribution* of $T(TM)$ with respect to ∇.

Let X_x be a tangent vector of M at x. Then the horizontal lift of X_x to $\mathcal{H}_{(x;u)}$ is the unique tangent vector X_u^{h} such that $\pi_* X_u^{\mathrm{h}} = X_x$. The horizontal lift operation $\mathrm{h} : T_xM \to \mathcal{H}_u$ can be extended for linear map $\mathrm{h} : \mathcal{X}(M) \to \Gamma(\mathcal{H})$.

Let X be a vector field on M with local expression

$$X = X^i \frac{\partial}{\partial x^i}.$$

Then the horizontal lift X^{h} of X to TM has local expression:

$$X^{\mathrm{h}} = (X^i)^{\mathrm{v}} \frac{\partial}{\partial x^i} - (X^j)^{\mathrm{v}} u^k \Gamma^i_{jk} \frac{\partial}{\partial u^i}. \tag{5.2}$$

Here $\{\Gamma^i_{jk}\}$ are connection coefficients of ∇.

To describe commutation relations, we prepare the following notational conventions introduced by Sekizawa [29]:

Let P be a tensor field of type $(1, s)$ on M and $X_1, \cdots, X_{s-1}, u = u^i\partial/\partial x^i \in T_xM$. Then $\mathrm{h}\{P(X_1, \cdots, u, \cdots, X_{s-1})\}$ is a horizontal vector at $(x;u)$ defined by

$$\mathrm{h}\{P(X_1, \cdots, u, \cdots, X_{s-1})\} = u^a (P(X_1, \cdots, \frac{\partial}{\partial x^a}, \cdots, X_{s-1}))^{\mathrm{h}}. \tag{5.3}$$

Similarly we shall define $\mathrm{v}\{P(X_1, \cdots, u, \cdots, X_{s-1})\}$.

Remark 5.1. Let P be a tensor field of type $(1, s)$ on M. Yano and Ishihara [35] defined a tensor field γP of type $(1, s-1)$. In our notation, γP is given by the following formula:

$$(\gamma P)(X_1, \cdots, X_{s-1}) = \mathrm{v}\{P(u, X_1, \cdots, X_{s-1})\}.$$

For a vector field $X \in \mathcal{X}(M)$, its horizontal lift and complete lift are related by

$$X_u^{\mathrm{c}} = X_u^{\mathrm{h}} + \mathrm{v}\{(\nabla X)u\}. \tag{5.4}$$

Thus $X^{\mathrm{c}} = X^{\mathrm{h}}$ if and only if $\nabla X = 0$.

Proposition 5.2 *Let P be a tensor field of type (r, s) on M. Then*

$$P^{\mathrm{c}}(X_1^{\mathrm{v}}, \cdots, X_{j-1}^{\mathrm{v}}, X_j^{\mathrm{h}}, X_{j+1}^{\mathrm{v}}, \cdot, X_s^{\mathrm{v}}) = (P(X_1, \cdots, X_s))^{\mathrm{v}}. \tag{5.5}$$

Proposition 5.3 ([35], p. 106) *Let (M, ∇) be a manifold with a linear connection. Then the covariant derivatives of horizontal and vertical vector field with respect to the complete lift connection ∇^{c} are given by*

$$(\nabla^{\mathrm{c}}_{X^{\mathrm{h}}} Y^{\mathrm{h}})_{(x;u)} = (\nabla_X Y)^{\mathrm{h}}_{(x;u)} + \mathrm{v}\{R(u, X)Y\}, \tag{5.6}$$

$$(\nabla^{\mathrm{c}}_{X^{\mathrm{h}}} Y^{\mathrm{v}})_{(x;u)} = (\nabla_X Y)^{\mathrm{v}}_{(x;u)}, \tag{5.7}$$

$$(\nabla^{\mathrm{c}}_{X^{\mathrm{v}}} Y^{\mathrm{h}})_{(x;u)} = (\nabla^{\mathrm{c}}_{X^{\mathrm{v}}} Y^{\mathrm{v}})_{(x;u)} = 0. \tag{5.8}$$

$$(\nabla^{\mathrm{c}}_{X^{\mathrm{c}}} Y^{\mathrm{h}})_{(x;u)} = (\nabla_X Y)^{\mathrm{h}}_{(x;u)} + \mathrm{v}\{R(u, X)Y\}. \tag{5.9}$$

Proposition 5.4 ([7])

$$[X^{\mathrm{h}}, Y^{\mathrm{h}}]_{(x;u)} = [X, Y]^{\mathrm{h}}_{(x;u)} - \mathrm{v}\{R(X, Y)u\},$$

$$[X^{\mathrm{h}}, Y^{\mathrm{v}}]_{(x;u)} = (\nabla_X Y)^{\mathrm{v}}_{(x;u)} - T(X, Y)^{\mathrm{v}}_{(x;u)}, \quad X, Y \in \mathcal{X}(M).$$

Corollary 5.5 ([7]) *The horizontal distribution $\mathcal{H}$ is integrable if and only if (M, ∇) is flat, i.e., $R = 0$.*

Since a linear connection ∇ induces a splitting of $T(TM)$, ∇ induces an almost complex structure $J = J_\nabla$ on TM:

$$JX^{\mathrm{h}} = X^{\mathrm{v}}, JX^{\mathrm{v}} = -X^{\mathrm{h}}$$

for all $X \in \mathcal{X}(M)$. This almost complex structure J is called the *canonical almost complex structure* of TM induced by ∇. By computing the Nijenhuis torsion of J, Dombrowski obtained the following.

Proposition 5.6 ([7]) *Let (M, ∇) be a manifold with a linear connection. Then the canonical almost complex structure on TM induced by ∇ is integrable if and only if the connection ∇ is flat and of torsion free.*

Remark 5.7. (The geodesic flows)
Let $\gamma(t) = (x^i(t))$ be a curve parametrised by the affine parameter in (M, ∇). Then γ is a geodesic with respect to ∇ if and only if $\nabla_{\gamma'}\gamma' = 0$. In a local coordinate system, this equation is written in the following form:

$$\frac{d^2x^k}{dt^2} + \Gamma^k_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0. \tag{5.10}$$

This system of second order ordinary differential equations on M is equivalent to the following system of first order ordinary differential equation on TM:

$$\frac{dx^k}{dt} = u^k, \quad \frac{du^k}{dt} = -\Gamma^k_{ij}u^iu^j. \tag{5.11}$$

This equation can be understood as an equation of a flow on TM. In fact,

$$\xi = u^k\frac{\partial}{\partial x^k} - \Gamma^k_{ij}u^iu^j\frac{\partial}{\partial u^k} \tag{5.12}$$

is globally defined on TM. In particular ξ is horizontal. The vector field ξ is called the *geodesic flow vector field* or *spray* of TM. The spray ξ is related to $\mathbf{U}$ by

$$\xi = -J\,\mathbf{U}. \tag{5.13}$$

Proposition 5.8 *A manifold (M, ∇) is geodesically complete if and only if ξ is a complete vector field on TM.*

Proposition 5.9 *Let (M, h, ∇) be a statistical manifold. Let us denote by ξ and ξ^* the geodesic sprays of ∇ and ∇^* respectively. Then $\xi^0 = (\xi + \xi^*)/2$ is the spray of ∇^0. More generally the spray of ∇^α is given by*

$$\xi^\alpha = \xi^0 + \frac{\alpha}{2}(\xi - \xi^*) = \frac{1}{2}\{(1+\alpha)\xi + (1-\alpha)\xi^*\}.$$

Take a semi-Riemannian metric h on (M, ∇). Here we do *not* assume that ∇ is metrical (*i.e.,* $\nabla h = 0$). By using the splitting (5.1), we can define a semi-Riemannian metric h^{S} on TM:

$$h^{\mathrm{S}}_{(x;u)}(X^{\mathrm{h}}, Y^{\mathrm{h}}) = h_x(X, Y), \quad h^{\mathrm{S}}_{(x;u)}(X^{\mathrm{h}}, Y^{\mathrm{h}}) = 0, \quad h^{\mathrm{S}}_{(x;u)}(X^{\mathrm{v}}, Y^{\mathrm{v}}) = h_x(X, Y).$$

This semi-Riemannian metric h^{S} is called the *diagonal lift* of h or *Sasaki lift* of h [27]. The Sasaki lift h^{S} has the following coordinate free definition:

$$h^{\mathrm{S}}_{(x;u)}(\tilde{X}, \tilde{Y}) = h_x(\pi_*\tilde{X}, \pi_*\tilde{Y}) + h_x(\mathcal{K}\tilde{X}, \mathcal{K}\tilde{Y})$$

Here the map $\mathcal{K} : T(TM) \to TM$ is defined by

$$\mathcal{K}_u X^{\mathrm{h}}_u = 0, \quad \mathcal{K}_u X^{\mathrm{v}}_u = X_x$$

and called the *connection map* of TM derived from ∇ [7]. Note that

$$d\pi \circ J = -\mathcal{K}, \quad \mathcal{K} \circ J = d\pi.$$

For a manifold (M, h, ∇) with a semi-Riemannian metric h and a linear connection ∇, the canonical almost complex structure J with respect to ∇ and the Sasaki lift h^{S} satisfy the following

$$h^{\mathrm{S}}(J\tilde{X}, J\tilde{Y}) = h^{\mathrm{S}}(\tilde{X}, \tilde{Y}), \ \tilde{X}, \ \tilde{Y} \in \mathcal{X}(TM).$$

Namely h^{S} is a Hermitian metric of (TM, J). In particular, for semi-Riemannian manifold (M, g, ∇^0) with Levi-Civita connection, each horizontal subspace $\mathcal{H}_u$ is the orthogonal complement of $\mathcal{V}_u$ with respect to g^{S}. Hence the natural projection $\pi : (TM, g^{\mathrm{S}}) \to (M, g)$ is a Riemannian submersion with totally geodesic fibres. Moreover we have the following (*cf.* [31], [4])

Proposition 5.10 *Let (M, g, ∇^0) be a semi-Riemannian manifold with the Levi-Civita connection. Then the tangent bundle (TM, g^{S}, J) together with the Sasaki lift metric and the canonical almost complex structure J with respect to ∇^0 is an* (indefinite) *almost Kähler manifold.*

Remark 5.11. Let us denote by Ω^{S} the fundamental 2-form of (TM, J, g^{S}) over a semi-Riemannian manifold (M, g, ∇^0).

$$\Omega^{\mathrm{S}}(\tilde{X}, \tilde{Y}) = g^{\mathrm{S}}(J\tilde{X}, \tilde{Y}).$$

Under the musical isomorphism $\flat : TM \to T^*M$ with respect to g, the fundamental 2-form Ω^{S} of (TM, J, g^{S}) corresponds to the canonical 2-form Ω of T^*M. Note that on a general statistical manifold (M, h, ∇), $\Omega^{\mathrm{S}} \neq \Omega$. See [11].

Corollary 5.12 *The* (indefinite) *almost Kähler manifold* (TM, g^{S}, J) *is* (indefinite) *Kähler if and only if* (M, g) *is flat.*

Problem 5.13. Let (M, h, ∇) be a semi-Riemannian manifold with a torsion free linear connection. We equip TM the Sasaki lift metric h^S and the canonical almost complex structure J with respect to ∇.
(1) When is the fundamental 2-form

$$\Omega^{\mathrm{S}}(\tilde{X}, \tilde{Y}) := h^{\mathrm{S}}(J\tilde{X}, \tilde{Y})$$

closed ?
(2) Let Ω be the canonical symplectic 2-form of T^*M. Then the pulled-back 2-form $\Omega_h := \flat^*\Omega$ is *compatible* to the canonical almost complex structure $J = J_\nabla$ on TM with respect to ∇, *i.e.*,

$$\Omega_h(J\tilde{X}, \tilde{X}) > 0$$

for all non zero vector field $\tilde{X}$ on TM and

$$\Omega_h(J\tilde{X}, J\tilde{Y}) = \Omega_h(\tilde{X}, \tilde{Y})$$

for all vector fields $\tilde{X}, \tilde{Y}$ on TM if and only if ∇ and h are compatible. Characterise symplectic manifolds with compatible almost complex structure which are locally obtained in this way. This problem was proposed by Kurose in [11].

Here we describe relations between the complete lift metric h^{c} and the canonical almost complex structure $J = J_\nabla$. Direct computations show the following formula:

$$h^{\mathrm{c}}_{(x;u)}(J\tilde{X}, J\tilde{Y}) = -h^{\mathrm{c}}_{(x;u)}(\tilde{X}, \tilde{Y}) + 2\{(\nabla_u h)_x(\pi_*\tilde{X}, \pi_*\tilde{Y}) + (\nabla_u h)_x(\mathcal{K}\tilde{X}, \mathcal{K}\tilde{Y})\}$$

for all vector fields $\tilde{X}, \tilde{Y}$ on TM.

Next, direct computations using Proposition 5.3 yield the following formula:

$$(\nabla^{\mathrm{c}}_{\tilde{X}} J)\tilde{Y} = \mathrm{h}\{R(u, \pi_*\tilde{X})\pi_*\tilde{Y}\} - \mathrm{v}\{R(u, \pi_*\tilde{X})\mathcal{K}\tilde{Y}\}.$$

Thus we obtain

Proposition 5.14 *Let* (M, h, ∇) *be a statistical manifold and* J *the canonical almost complex structure on* TM *with respect to* ∇. *Then*
(1) (TM, J, h^{c}) *is an almost complex manifold with Norden metric if and only if* $C = 0$.
(2) J *is parallel with respect to* ∇^{c} *if and only if* (M, ∇) *is flat.*

Now we shall recall the horizontal lift operation of tensor fields with respect to ∇. For any function f on M its horizontal lift f^{h} to TM is $f^{\mathrm{h}} = 0$. Next for a 1-form $\omega = \omega_i dx^i$, its horizontal lift ω^{h} is defined by

$$\omega^{\mathrm{h}} = \Gamma_{ij}^{k} u^j \omega_k dx^i + \omega_k du^k.$$

The horizontal lift ω^{h} is related to the complete lift by

$$\omega^{\mathrm{h}} = \omega^{\mathrm{c}} - \mathrm{v}\{\nabla_u \omega\}.$$

The horizontal lift operation is extended to $\mathcal{T}(M)$ by the rule:

$$(P \otimes Q)^{\mathrm{h}} = P^{\mathrm{v}} \otimes Q^{\mathrm{h}} + P^{\mathrm{h}} \otimes Q^{\mathrm{v}}.$$

In particular, for any tensor field $P \in \mathcal{T}_s^r(M)$ with $r = 0, 1$ (See [35], Proposition 3.5, p. 97):

$$\begin{gathered} P^{\mathrm{h}}(X_1^{\mathrm{h}}, \cdots, X_s^{\mathrm{h}}) = (P(X_1, \cdot, X_s))^{\mathrm{h}}, \\ P^{\mathrm{h}}(X_1^{\mathrm{h}}, \cdot, X_{j-1}^{\mathrm{h}}, X_j^{\mathrm{v}}, X_{j+1}^{\mathrm{h}}, \cdot, X_s^{\mathrm{h}}) = (P(X_1, \cdot, X_s))^{\mathrm{v}}, \\ P^{\mathrm{h}}(X_1^{\mathrm{v}}, \cdot, X_s^{\mathrm{v}}) = 0. \end{gathered} \tag{5.14}$$

Here we can compare two lifting operations, the complete lift and the horizontal lift:

Proposition 5.15 *Let (M, ∇) be a manifold with a linear connection and P be a tensor field on M. Then the horizontal lift P^{h} coincides with P^{c} if and only if $\nabla P = 0$.*

Now let (M, ∇) be a manifold with linear connection and h a semi-Riemannian metric on M.

Then the horizontal lift metric h^{h} with respect to ∇ is described by the following formulae:

$$h^{\mathrm{h}}_{(x;u)}(X^{\mathrm{h}}, Y^{\mathrm{h}}) = h^{\mathrm{h}}_{(x;u)}(X^{\mathrm{v}}, Y^{\mathrm{v}}) = 0, \quad h^{\mathrm{h}}_{(x;u)}(X^{\mathrm{h}}, Y^{\mathrm{v}}) = h_x(X, Y). \tag{5.15}$$

Note that h^{h} is characterized by the following equation:

$$h^{\mathrm{h}}_{(x;u)}(\tilde{X}, \tilde{Y}) = h_x(\pi_* \tilde{X}, \mathcal{K}\tilde{Y}) + h_x(\mathcal{K}\tilde{X}, \pi_* \tilde{Y}).$$

The horizontal lift h^{h} coincides with the complete lift h^{c} if and only if $\nabla h = 0$. In particular for semi-Riemannian manifold (M, g, ∇^0) with Levi-Civita connection, its horizontal lift metric g^{h} and complete lift metric g^{c} coincide.

Both of Sasaki lift metric and horizontal lift metric are *natural metrics* on TM in the sense of Kowalski and Sekizawa [15].

The horizontal lift ∇^{h} of the connection ∇ is defined by the formulae:

$$\nabla^{\mathrm{h}}_{X^{\mathrm{v}}} Y^{\mathrm{v}} = 0, \ \nabla^{\mathrm{h}}_{X^{\mathrm{v}}} Y^{\mathrm{h}} = 0,$$

$$\nabla^{\mathrm{h}}_{X^{\mathrm{h}}} Y^{\mathrm{v}} = (\nabla_X Y)^{\mathrm{v}}, \ \nabla^{\mathrm{h}}_{X^{\mathrm{h}}} Y^{\mathrm{h}} = (\nabla_X Y)^{\mathrm{h}}. \tag{5.16}$$

Even if ∇ is of torsion free, its horizontal lift ∇^{h} has non trivial torsion.

Proposition 5.16 ([35], pp. 109–110) *Let ∇ be a torsion free linear connection on M. Then the horizontal lift ∇^{h} is of torsion free if and only if ∇ is flat. In this case ∇^{h} is flat and $\nabla^{\mathrm{h}} = \nabla^{\mathrm{c}}$.*

Using (5.16) and (5.3), we get the following result.

Proposition 5.17 *Let (M, h, ∇) be a semi-Riemannian manifold with a linear connection. Then the natural projection $\pi : (TM, \nabla^{\mathrm{h}}) \to (M, \nabla)$ is a totally geodesic map, i.e., the second fundamental form $\nabla d\pi$ of π defined by*

$$(\nabla d\pi)(\tilde{X}; \tilde{Y}) = \nabla^{\pi}_{\tilde{Y}} d\pi(\tilde{X}) - d\pi(\nabla^{\mathrm{h}}_{\tilde{Y}} \tilde{X}), \quad \tilde{X},\ \tilde{Y} \in \mathcal{X}(TM)$$

vanishes. Similarly, π is also a totally geodesic map of $(TM, \nabla^{\mathrm{c}})$ onto (M, ∇).

This is a generalisation of the following result due to Ferreira.

Corollary 5.18 ([10]) *Let (M, g, ∇^0) be a semi-Riemannian manifold with Levi-Civita connection. Then the natural projection $\pi : (TM, g^{\mathrm{c}}) \to (M, g)$ is a totally geodesic map.*

Note that in case $\pi : (TM, \nabla^{\mathrm{h}}) \to (M, \nabla)$, the second fundamental form $\nabla d\pi$ is symmetric if and only if ∇ is flat.

Proposition 5.19 *Let P be a tensor field on (M, ∇). Then*

$$\nabla^{\mathrm{h}}_{X^{\mathrm{c}}} P^{\mathrm{v}} = (\nabla_X P)^{\mathrm{v}},\ \nabla^{\mathrm{h}}_{X^{\mathrm{c}}} P^{\mathrm{h}} = (\nabla_X P)^{\mathrm{h}},$$
$$\nabla^{\mathrm{h}}_{X^{\mathrm{v}}} P^{\mathrm{v}} = 0,\ \nabla^{\mathrm{h}}_{X^{\mathrm{v}}} P^{\mathrm{h}} = 0$$

for all $X \in \mathcal{X}(M)$.

These formulae imply the following result.

Proposition 5.20 *Let (M, h, ∇) be a statistical manifold. Then $(TM, h^{\mathrm{S}}, \nabla^{\mathrm{h}})$ or $(TM, h^{\mathrm{h}}, \nabla^{\mathrm{h}})$ is a statistical manifold if and only if $\nabla h = 0$. In such cases, resulting structures on TM are the Sasaki lift metric or the horizontal lift metric with their Levi-Civita connection.*

§6. Statistical structures compatible to the Sasaki lift metric

Let (M, h, ∇) be a statistical manifold with skewness operator K. First we consider the horizontal lift of K. By definition, K^{h} is give by

$$K^{\mathrm{h}}(X^{\mathrm{h}})Y^{\mathrm{h}} = (K(X)Y)^{\mathrm{h}},$$

$$K^{\mathrm{h}}(X^{\mathrm{h}})Y^{\mathrm{v}} = K^{\mathrm{h}}(X^{\mathrm{v}})Y^{\mathrm{h}} = (K(X)Y)^{\mathrm{v}}, \tag{6.1}$$

$$K^{\mathrm{h}}(X^{\mathrm{v}})Y^{\mathrm{v}} = 0$$

for all $X, Y \in \mathcal{X}(M)$. One can check that the trilinear form $h^{\mathrm{S}}(K^{\mathrm{h}}(\tilde{X})\tilde{Y}, \tilde{Z})$ is totally symmetric. Thus $(h^{\mathrm{S}}, K^{\mathrm{h}})$ is a statistical structure on TM.

Theorem 6.1 *Let (M, h, ∇) be a statistical manifold. Define a linear connection $\hat{\nabla}$ by*

$$\tilde{\nabla} = \nabla^{h^{\mathrm{S}}} - \frac{1}{2} A^{\mathrm{h}},$$

where $\nabla^{h^{\mathrm{S}}}$ is the Levi-Civita connection of the Sasaki lift metric h^{S}. Then the triplet $(TM, h^{\mathrm{S}}, \tilde{\nabla})$ is a statistical manifold.

Remark 6.2. Ianus [12] studied statistical structures on TM by using Sasaki lift metric. However her structure is *diffferent* from ours. In fact, let (M, h, ∇) be a statistical manifold with skewness operator K. Ianus considered the horizontal lift and the Sasaki lift with respect to the *Levi-Civita connection* ∇^0 of (M, h). Let $\nabla^{h_0^{\mathrm{S}}}$ be the Levi-Civita connection of the Sasaki lift metric h_0^{S} and the horizontal lift K_0^{h} of K with respect to ∇^0. Then the connection $\nabla^{h_0^{\mathrm{S}}} - K_0^{\mathrm{h}}/2$ is compatible to h_0^{S}. From the viewpoint of information geometry, our sturcture $(h^{\mathrm{S}}, \tilde{\nabla})$ is more natural than Ianus' structure.

Next we equip another compatible linear connection for (TM, h^{S}). Let (M, h, ∇) be a statistical manifold. Since C is symmetric, its horizontal lift C^{h} is also symmetric.

Proposition 6.3 *Let (M, h, ∇) be a statistical manifold. Then $(TM, h^{\mathrm{S}}, C^{\mathrm{h}})$ is a statistical manifold.*

Define a tensor field $\hat{A}$ by

$$h^{\mathrm{S}}(\hat{A}(\tilde{X})\tilde{Y}, \tilde{Z}) = C^{\mathrm{h}}(\tilde{X}, \tilde{Y}, \tilde{Z}), \quad \tilde{X},\ \tilde{Y},\ \tilde{Z} \in \mathcal{X}(TM).$$

Namely $\hat{A}$ is metrically equivalent to C^{h} relative to h.

Note that, in general, this $\hat{A}$ is different from the horizontal lift of A^{h}. In fact, for our structure $\hat{A}$,

$$h^{\mathrm{S}}(\hat{A}(X^{\mathrm{h}})Y^{\mathrm{h}}, Z^{\mathrm{h}}) := C^{\mathrm{h}}(X^{\mathrm{h}}, Y^{\mathrm{h}}, Z^{\mathrm{h}}) = (C(X, Y, Z))^{\mathrm{h}} = 0.$$

On the other hand, A^{h} satisfies

$$h^{\mathrm{S}}(A^{\mathrm{h}}(X^{\mathrm{h}})Y^{\mathrm{h}}, Z^{\mathrm{h}}) = h^{\mathrm{S}}((A(X)Y)^{\mathrm{h}}, Z^{\mathrm{h}}) = C(X, Y, Z).$$

Obviously if $\nabla = \nabla^0$, then $\hat{\nabla}$ coincides with $\tilde{\nabla}$.

§7. Statistical structures compatible to the horizontal lift metric

In this section, we shall study horizontal lift metric h^{h} on TM. The tensor field K^{h} is also compatible to the horizontal lift metric h^{h}.

Theorem 7.1 *Let (M, h, ∇) be a statistical manifold. Then $(TM, h^{\mathrm{h}}, K^{\mathrm{h}})$ is a statistical manifold.*

Proof. Let $\tilde{C}$ be the corresponding trilinear form of K^{h} with respect to h^{h}:

$$\tilde{C}(\tilde{X},\tilde{Y},\tilde{Z}) := h^{\mathrm{h}}(K^{\mathrm{h}}(\tilde{X})\tilde{Y},\tilde{Z}), \quad \tilde{X},\tilde{Y},\tilde{Z} \in \mathcal{X}(TM).$$

For any vector fields $\tilde{X},\tilde{Y},\tilde{Z} \in \mathcal{X}(TM)$, we have

$$\tilde{C}(\tilde{X},\tilde{Y},\tilde{Z}) = \{C(\pi_*\tilde{X},\pi_*\tilde{Y},\mathcal{K}\tilde{Z}) + C(\pi_*\tilde{X},\mathcal{K}\tilde{Y},\pi_*\tilde{Z}) + C(\mathcal{K}\tilde{X},\pi_*\tilde{Y},\pi*\tilde{Z})\}^{\mathrm{v}}.$$

This formula implies that $\tilde{C}$ is totally symmetric. □

As we saw before h^{h} is a neutral metric on TM. With respect to the connection ∇, we shall induce the canonical almost complex structure J on TM. Since, in general, ∇ is not necessarily the metric connection of h, $h^{\mathrm{h}} \neq h^{\mathrm{c}}$. The following is easily verified.

Proposition 7.2 *(TM, h^{h}, J) is an almost complex manifold with a Norden metric. Namely (h^{h}, J) satisfies*

$$h^{\mathrm{h}}(J\tilde{X}, J\tilde{Y}) = -h^{\mathrm{h}}(\tilde{X},\tilde{Y})$$

for all vector fields $\tilde{X},\tilde{Y}$ on TM. In particular if (M,∇) is flat, then (TM, h^{h}, J) is a complex manifold with a Norden metric.

Since the horizontal lift C^{h} of C is symmetric, we have the following result.

Proposition 7.3 *The tangent bundle $(TM, h^{\mathrm{h}}, C^{\mathrm{h}})$ is a statistical manifold.*

To close this article, we would like to propose the following two problems arising from complex-affine differential geometry.

Problem 7.4. (Affine Kähler manifolds) Let (N, J, ∇) be a complex manifold with a torsion free *complex* linear connection. If the curvature R of ∇ is J-invariant, *i.e.,* $R(JX, JY) = R(X, Y)$. Then (N, J, ∇) is said to be an *affine Kähler manifold* [24].

The tangent bundle (TM, J) over a (torsion free) flat manifold (M, ∇) is a complex manifold. Can we construct any affine Kähler structure compatible to J ?

Note that there exists a torsion free linear connection $\hat{\nabla}$ on TM such that $\hat{\nabla}J = 0$. In fact, take any torsion free linear connection $\tilde{\nabla}$ on TM. Define a tensor field Q on TM by

$$4Q(E,F) = (\tilde{\nabla}_{JF}J)E + J(\tilde{\nabla}_F E) + 2J(\tilde{\nabla}_E J)F.$$

Then the linear connection $\hat{\nabla} = \tilde{\nabla} - Q$ is the desired one (See pp. 143-145 of [14]).

Such a connection satisfies

$$\hat{R}(\tilde{X},\tilde{Y}) \circ J = J \circ \hat{R}(\tilde{X},\tilde{Y}).$$

But in general $(\hat{\nabla}, J)$ is not affine Kähler.

Problem 7.5. (Real holomorphic structure) *Let (N, J) be a complex manifold. A torsion free linear connection ∇ is said to be real holomorphic if the curvature R of ∇ satisfies*

$$R(JX, Y) = JR(X, Y).$$

Can we construct such a connection on (TM, J) ?

Acknowledgement
The first author is partially supported by Inamori Foundation and Grant-in-Aid for Encouragement of Young Scientists No. 12740045, Japan Society for Promotion of Science. The second named author is partially supported by Grant-in-Aid for Encouragement of Young Scientists No. 14740053, Japan Society for Promotion of Science.

Appendix

A.1 Cotangent bundles

In this Appendix, we collect fundamental formulae of the cotangent bundles. Let M be a smooth n-manifold and T^*M its cotangent bundle. Denote by π the natural projection of T^*M onto M. Taking a local coordinate system $(U; x^1, \cdots, x^n)$ and denote the induced local coordinate system on $\pi^{-1}(U)$ by $(x^1, \cdots, x^n; p_1, \cdots, p_n)$. For a point $p = (x; p)$ of T^*M, we denote the kernel of π_* by $\mathcal{V}_p^*$ and call it the vertical subspace of $T_p(T^*M)$ at $p \in T^*M$. The 1-form $\vartheta = p_i dx^i$ is globally defined on T^*M. The 1-form ϑ is called the *canonical* 1-*form* or *Liouville form* of T^*M. The canonical 1-form has the following coordinate free definition:

$$\vartheta(\tilde{X}) = p(\pi_* \tilde{X}),\ p \in T^*M,\ \tilde{X} \in \mathcal{X}(T^*M).$$

Next $\Omega := -d\vartheta$ is a symplectic form on T^*M. Namely Ω is a nondegenerate closed 2-form on T^*M. In fact, from the local expression of Ω,

$$\Omega = dx^i \wedge dp_i,$$

it is clear that $\Omega^n \neq 0$ on T^*M. This 2-form Ω is called the *canonical* 2-*form* or *canonical symplectic structure* of T^*M.

Next let ∇ be a linear connection on M. Then ∇ induces a connection ∇^* on T^*M:

$$\nabla^* : \mathcal{X}(M) \times \Gamma(T^*M) \to \Gamma(T^*M);$$

$$(\nabla^*_X \omega)Y := X(\omega(Y)) - \omega(\nabla_X Y), \quad X,\ Y \in \mathcal{X}(M),\ \omega \in \Gamma(T^*M).$$

The connection ∇^* is called the *dual connection* of ∇.

The linear connection ∇ defines a splitting of $T(T^*M)$:

$$T_p(T^*M) = \mathcal{H}_p^* \oplus \mathcal{V}_p^*, \quad p \in T^*M.$$

Here $\mathcal{V}_p = \text{Ker}\ \pi_{*p}$ (vertical subspace at p) and $\mathcal{H}_p$ is the horizontal subspace at p.

By using this splitting, Patterson and Walker [26] introduced a neutral metric– called the *Riemann extension* of ∇. The Riemann extension g^R is defined by

$$g^R(\tilde{X}, \tilde{Y}) = \Omega(\Pi_{\mathcal{H}}(\tilde{Y}),\ \Pi_{\mathcal{V}}(\tilde{X})) + \Omega(\Pi_{\mathcal{H}}(\tilde{X}), \Pi_{\mathcal{V}}(\tilde{Y})).$$

Here $\Pi_{\mathcal{H}}$ and $\Pi_{\mathcal{V}}$ denotes the projections $\mathcal{X}(T^*M) \to \Gamma(\mathcal{H}^*)$ and $\mathcal{X}(T^*M) \to \Gamma(\mathcal{V}^*)$ respectively. See also [34]. In [34], Willmore showed that by making use of Riemann extension it is possible to define an affine immersion of manifolds in affine differential geometry without making a suitable choice of normal planes. By definition, the Riemann extension depends only on the connection ∇.

As in the geometry of TM, vertical and horizontal lift operations $V : \mathcal{X}(M) \to \Gamma(\mathcal{V}^*)$, $H : \mathcal{X}(M) \to \Gamma(\mathcal{H}^*)$ for vector fields to T^*M can be introduced. Now let us define an almost complex structure J on T^*M by

$$JX^{\mathrm{h}} = X^{v}, \quad JX^{\mathrm{v}} = -X^{\mathrm{h}}, \quad X \in \mathcal{X}(M).$$

Then one can prove the following

Proposition A.1 *Let (M, ∇) be a manifold with a linear connection. Then (T^*M, g^R, J) is an almost complex manifold with Norden metric g^R. In particular (T^*M, g^R, J) is a complex manifold with Norden metric if and only if (M, ∇) is flat.*

Let h be a semi-Riemannian metric on (M, ∇). Then the pulled back one form ϑ_h of ϑ by $\flat$ is

$$\vartheta_h := \flat^*\vartheta = h_{ij}u^i dx^j.$$

The pulled-back 2-form Ω_h of Ω is

$$\Omega_h = dx^i \wedge d(h^{ij}p_j).$$

On a semi-Riemannian manifold (M, g, ∇^0) with its Levi-Civita connection, its tangent bundle (TM, g^{c}) with complete lift metric is isometric to (T^*M, g^R). Under this identification, we notice the following fact (*cf.* [13], [34]):

Proposition A.2 *Let (M, g, ∇^0) be a semi-Riemannian manifold with its Levi-Civita connection. Then (TM, g^{c}) is conformally flat if and only if (M, g) is projectively flat.*

In information geometry, α-conformal flatness [16], [25] and conformal-projective flatness are studied extensively [18], [21], [28].

Problem A.3. Let (M, h, ∇) be a statistical manifold.
When is the tangent bundle $(TM, h^{\mathrm{c}}, \nabla^{\mathrm{c}})$ conformal-projectively flat ?
When is $(TM, h^{\mathrm{c}}, \nabla^{\mathrm{c}})$ α-conformally flat ?

A.2 Remarks on Norden metrics

In [5], the authors proved that on an almost Hermitian manifold (M, J, g), its tangent bundle TM together with complete lift metric g^{c} and *natural lift* $\tilde{J}$ of J is an almost complex manifold with Norden metric. Here the natural lift $\tilde{J}$ is defined by

$$\tilde{J}X^{\mathrm{h}} = -(JX)^{\mathrm{h}}, \ \tilde{J}X^{\mathrm{v}} = (JX)^{\mathrm{v}}, \ X \in \mathcal{X}(M).$$

Note that, $\tilde{J}$ is different from the horizontal lift J^{h}. In fact, the horizontal lift J^{h} with respect to ∇^0 is defined by

$$J^{\mathrm{h}}X^{\mathrm{h}} = (JX)^{\mathrm{h}}, \ J^{\mathrm{h}}X^{\mathrm{v}} = (JX)^{\mathrm{v}}, \ X \in \mathcal{X}(M).$$

On the other hand, according to our observation, we have gotten the following fact:

Proposition A.4 *Let (M, g, ∇^0) be a semi-Riemannian manifold with Levi-Civita connection. Then (TM, g^c) with canonical almost complex structure J with respect to ∇^0 is an almost complex manifold with a Norden metric.*

Moreover we give here another construction of almost complex manifolds with Norden metric.

Proposition A.5 ([4]) *Let (M, g, J) be a Kähler manifold. Then its tangent bundle TM admits a Norden-Kähler structure. Namely a complex structure (J, g) together with a Norden metric g such that J is parallel with respect to the Levi-Civita connection of g.*

Proof. Let us denote the Nijenhuis tensor of J by N_J. Since the Nijenhuis tensor of J^c is $(N_J)^c$, (TM, J^c) is complex. Moreover $\nabla^c J^c = 0$ by Proposition 6.7 of [36]-Part I. Thus (TM, g^c, J^c) is a Norden-Kähler manifold. □

Remark A.6. Let (M, g, J) be a flat Kähler manifold. Then its tangent bundle TM together with Sasaki lift metric g^S and the horizontal lift J^h is a Kähler manifold (see [4]).

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