Regular conditional linear spaces with two consecutive line degrees

Pınar Anapa

Abstract

In this paper, we show that an $(n+1)$-regular conditional linear space with two consecutive line degrees is a projective plane of order n less two lines and all their points, or is a linear space with 12 points, 19 lines, every point of degree 5 and each point lying on precisely one 4-line and four 3-lines.

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§1. Introduction

One of the most natural strictly numerical questions to ask is what can be said if all the line degrees of a linear space S are known. Clearly, this problem will have a reasonable answer only if the set of allowable line degrees is quite small. If there is only one line degree, then S is a design, and in a sense, S is "known". We therefore turn to the case of two line degrees. A non-trivial linear spaces with two consecutive line degrees has been examined by several authors ([2],[3],[5],[9]).

A conditional linear space was firstly defined by İ.Günaltılı in [6]. In this paper, first of all, we examined the relation between an $(n+1)-$regular linear space with two consecutive line degrees and conditional linear spaces. Then, we classified an $(n+1)-$regular conditional linear space with two consecutive line degrees.

According to our determination, an $(n+1)-$regular conditional linear space with two consecutive line degrees is a projective plane of order n less two lines and all their points, or is a linear space with 12 points, 19 lines, every point of degree 5 and each point lying on precisely one 4-line and four 3-lines.

Definition 1.1 ([2]). A finite linear space is a pair $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ consisting of a set $\mathcal{P}$ of elements called points and a set $\mathcal{L}$ of distinguished subsets of points, called lines satisfying the following axioms:

(**L1**) Any two distinct points of $\mathcal{S}$ belong to exactly one line of $\mathcal{S}$.

(**L2**) Any line of $\mathcal{S}$ has at least two points of $\mathcal{S}$.

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In talking about finite linear spaces we shall use a rather easy-going terminology borrowed from classical geometry; for example, we shall use words such as "collinear," "concurrent," "meeting," "joining," and expressions such as " a line (passing) through a point" or " a point
(lying) on a line."

In a finite linear space $\mathcal{S}$, v and b denote respectively the number of points and of lines of $\mathcal{S}$. The number $v(l)$ of points on a line l is called the degree of l and the number $b(P)$ of lines passing through a point P is called the degree of P. The terms $i-$point or $i-$line may also used to refer respectively to a point or a line of degree i. In addition; b_k denotes the total number of $k-$lines, v_k denotes the total number of $k-$points and $b_k(P)$ denotes the total number of $k-$lines passed through a point P.

An finite linear space is non-trivial if $b > 1$.

A matrix $R = [r_{ij}]_{vxb}$ is called an incidence matrix of $\mathcal{S}$ if there are orders $P_1, P_2, ..., P_v$ and $l_1, l_2, ..., l_b$ of the points and lines such that $r_{ij} = 1$ if P_i is a point of l_j and $r_{ij} = 0$ if not.

Definition 1.2 ([4]). Let $\mathcal{S}$ be a non-trivial linear space. If every point of $\mathcal{S}$ lies on exactly t lines of $\mathcal{S}$ then $\mathcal{S}$ is called a $t-$regular linear space. $(t \geq 1, t \in \mathbf{Z})$

Definition 1.3 Let $\mathcal{S}$ be a non-trivial linear space. $\mathcal{S}$ is called a linear space with $A-$line ranges, if A is complete set of degrees of all lines of $\mathcal{S}$.

Definition 1.4 ([4]). The order of a non-trivial finite linear space is defined one less than the highest degree of both points and lines.

Definition 1.5 ([4]). Let $\mathcal{S}$ be a non-trivial finite linear space of order n,
$P \in \mathcal{P}, l \in \mathcal{L}$. (P, l) is called a flag if P lies on l.

A finite affine plane of order $n \geq 2$ is an finite linear space with n^2 points in which $v(l) = n$, $b(P) = n + 1$ for every line l and point P. A finite projective plane of order $n \geq 2$ is an finite linear space with $n^2 + n + 1$ points in which $v(l) = b(P) = n + 1$ for every line l and point P.

Definition 1.6 Let $\mathcal{S}$ be a non-trivial linear space of order n. $\mathcal{S}$ is called a conditional linear space if the following condition is valid for every $(P, l)-$flag of $\mathcal{S}$

$$b_n(P) \leq b - v - (n + 1 - v(l))$$

A conditional linear space of order n with $b_n = 0$ is called a trivial conditional linear space. It is clear that, $\max_{l \in \mathcal{L}}(n + 1 - v(l)) \leq b - v$ for any line l of a trivial conditional linear space.

In this paper, we will prove the following theorem.

Theorem (A). *Let $\mathcal{S}$ be an $(n+1)-$regular non-trivial conditional linear space with two consecutive line degrees. Then, $\mathcal{S}$ is a projective plane of order n less two lines and all their points, or is a linear space with 12 points, 19 lines, every point of degree 5 and each point lying on precisely one 4-line and four 3-lines.*

The following results are well known and are listed here for easy reference ([2],[4]).

T1

$$\sum_{j=1}^{b} v(l_j) = \sum_{i=1}^{v} b(P_i)$$

T2 At each point P_i we have

$$v-1=\sum_{j=1}^{b}(v(l_j)-1)r_{ij}$$

Hence

$$v(v-1)=\sum_{j=1}^{b}v(l_j)(v(l_j)-1)$$

T3 If P does not lie on l, then $b(P)\geq v(l)$. Equality holds iff all lines through P meet l.

T4 If $\pi(l_j)$ is the number of lines that miss l_j then

$$b-1=\pi(l_j)+\sum_{i=1}^{v}(b(P_i)-1)r_{ij}$$

Proposition 1.1 *[2] Let $\mathcal{S}$ be a non-trivial linear space in which each line has k or $k+1$ points.* $(2\leq k\leq n, k\in Z)$. *For any point P we have*

$$\frac{v-1}{k}\leq b(P)\leq\frac{v-1}{k-1}$$

Proof. Lines on P have at most $k+1$ points and at least k points. Hence $kb(P)\geq v-1$ and $(k-1)b(P)\leq v-1$. The inequality follows. □

Proposition 1.2 *[2] Let $\mathcal{S}$ be a non-trivial linear space in which each line has k or $k+1$ points.* $(2\leq k\leq n, k\in \boldsymbol{Z})$. *Any point P is on $kb(P)-v+1$ $k-$lines and $(1-k)b(P)+v-1$ $(k+1)-$lines.*

Proof. Let a and c be the number of $k-$ and $(k+1)-$lines respectively on P. Then $a+c=b(P)$. Also,

$$a(k-1)+ck=v-1$$

So $(a+c)k-a=kb(P)-a=v-1$ implying $a=kb(P)-v+1$. Then

$$c=b(P)-a=(1-k)b(P)+v-1$$

□

Proposition 1.3 *[2] Let $\mathcal{S}$ be a non-trivial $t-$regular linear space with $\{k,k+1\}-$line ranges. Then;*

$kb_k=v(kt-v+1)$ *and* $(k+1)b_{k+1}=v((1-k)t+v-1)$ $\quad (2\leq k,t\leq n, k,n\in\boldsymbol{Z})$

Proof. Every point of $\mathcal{S}$ is on exactly t lines, since $\mathcal{S}$ is a non-trivial $t-$ regular linear space with $\{k,k+1\}-$line ranges. Also, from the Proposition 1.8, every point P of $\mathcal{S}$, $b_k(P)=kt-v+1$ and $b_{k+1}(P)=(1-k)t+v-1$. Thus,

$$kb_k=v(kt-v+1) \text{ and } (k+1)b_{k+1}=v((1-k)t+v-1).$$

□

Theorem 1.1 (De Bruijn and Erdös, [2]). *Let $\mathcal{S}$ be a finite non-trivial linear space. Then $b \geq v$. Moreover, equality holds if and only if $\mathcal{S}$ is a generalized projective plane, i.e projective plane or a near-pencil.*

The following result can be obtained similarly as Corollary 2.3.3, [2].

Proposition 1.4 *[2] If $\mathcal{S}$ is a non-trivial $(n+1)-$regular linear space with n^2-n points, n^2+n-1 lines and $\{n, n-1\}-$lines ranges, $n \geq 3$, then $\mathcal{S}$ is a projective plane of order n less two lines and all their points, or is a linear space with 12 points, 19 lines, every point of degree 5 and each point lying on precisely one $4-$line and four $3-$lines.*

§2. Main Results

We suppose that $\mathcal{S}$ is an $(n+1)$-regular finite non-trivial linear space with $\{k, k+1\}-$line ranges. $(2 \leq k \leq n,\ k, n \in \mathbf{Z})$. If $k \neq n, n-1$ then $\mathcal{S}$ does not contain $n-$lines. Thus; from the Definition 1.6, if $n+1-k \leq b-v$ then $\mathcal{S}$ is a trivial conditional linear space.

Proposition 2.1 *An $(n+1)-$regular non-trivial conditional linear space with two consecutive line degrees, $n \geq 3$, is a linear space with $\{n, n-1\}$ line ranges.*

Proof. Let $\mathcal{S}$ be an $(n+1)-$regular non-trivial conditional linear space with $\{k, k+1\}-$line ranges. $\mathcal{S}$ contains at least one $n-$line, since $\mathcal{S}$ is an $(n+1)-$regular non-trivial conditional linear space. Therefore; $k \in \{n, n-1\}$. We must show that $k = n-1$.

We suppose that $k = n$. In this case, $\mathcal{S}$ is an $(n+1)-$regular non-trivial linear space with $\{n, n+1\}-$line ranges. Thus; $\mathcal{S}$ contains at least one $n-$line and $(n+1)-$line.

Firstly, we show that $b_{n+1} \geq 2$. We assume that $\mathcal{S}$ contains exactly one $(n+1)-$line. $\mathcal{S}$ contains at least one point P not on $(n+1)-$line, since $\mathcal{S}$ is an $(n+1)-$regular non-trivial linear space. Since every line to be passed on P has degree n, from $T2$, the total number of points of S is $v = n^2$. In addition; since $b_n(P) = n$ and $b_{n+1}(P) = 1$, for every point P which is on $(n+1)-$line, again using $T2$, the total number of points of $\mathcal{S}$ is calculated $v = n^2+1$. Thus, we obtain $v = n^2 = n^2+1$. This is a contradiction and $b_{n+1} \geq 2$. Since $\mathcal{S}$ is an $(n+1)-$regular, any two $(n+1)-$lines intersect. From T4, $b = n^2+n+1$. Using T2, we obtain $b_n(P) = b - v$, for each point P. Since $\mathcal{S}$ is an $(n+1)-$regular non-trivial conditional linear space, from the Definition 1.6 we obtain $0 \leq -1$. This is a contradiction. Thus, $k = n-1$. □

Proposition 2.1 showed that the line degrees of an $(n+1)-$regular non-trivial conditional linear space with two consecutive line degrees are $\{n, n-1\}$. By the way, it is trivial that an $(n+1)-$regular non-trivial conditional linear space is also an $(n+1)-$regular non-trivial linear space. Thus; any $(n+1)-$regular non-trivial linear space with $\{n, n+1\}-$line ranges is not a conditional linear space. Since we examined the relation between an $(n+1)-$regular linear space with two consecutive line degrees and an $(n+1)-$regular non-trivial conditional linear space; we assume that $\mathcal{S}$ is an $(n+1)-$regular linear space with $\{n-1, n\}-$line ranges.

Proposition 2.2 *Let $\mathcal{S}$ be an $(n+1)-$regular linear space with $\{n-1, n\}-$line ranges. The total number of $\mathcal{S}$ is either $n^2 - n$ or $n^2 - 1$.*

Proof. We assume that $\mathcal{S}$ is an $(n+1)-$regular linear space with $\{n-1, n\}-$line ranges. By the Proposition 1.7

$$\frac{v-1}{n-1} \leq n+1 \leq \frac{v-1}{n-2}$$

Then we obtain from the above inequality; $n^2 - n - 1 \leq v \leq n^2$. On the other hand, from the Proposition 1.8, we obtain $b_n(P) = (n-1)(n+1) + 1 - v$ and $b_{n-1}(P) = (1-(n-1))(n+1) + v - 1$ for every point P of $\mathcal{S}$. Since the line degrees are $\{n-1, n\}$, $b_n(P) > 0$ and $b_{n-1}(P) > 0$. Thus, $v \neq n^2 - n - 1$, $v \neq n^2$.

Now we show that the total number of points of $\mathcal{S}$ is either $n^2 - 1$ or $n^2 - n$. We assume that $n^2 - n + 1 \leq v \leq n^2 - 2$. Thus, we can write $v = n^2 - m$, $2 \leq m \leq n-1, n, m \in \mathbf{Z}$. Since S is an $(n+1)-$regular linear space, $b_n(P) = n + 1 - m$ and $b_{n-1}(P) = m$ every point P of S. Using Proposition 1.9, gives $nb_n = (n^2 - m)(n + 1 - m)$ and $(n-1)b_{n-1} = (n^2 - m)m$. Thus,

$$b = b_n + b_{n-1} = n^2 + n - \frac{m^2 - m}{n(n-1)}$$

Since $0 < \dfrac{m(m-1)}{n(n-1)} < 1$, $b = n^2 + n - \dfrac{m(m-1)}{n(n-1)} \notin \mathbf{Z}$. This contradicts $b \in \mathbf{Z}$. Thus our assumption is false. Therefore; the total number of points of is either $n^2 - 1$ or n^2. □

Theorem (A). *Let S be an $(n+1)-$regular non-trivial conditional linear space with two consecutive line degrees. Thus, S is a projective plane order n less two lines and all their points, or is a linear space with 12 points, 19 lines, every point of degree 5 and each point lying on precisely one 4-line and four 3-lines.*

Proof. $\mathcal{S}$ is an $(n+1)-$regular non-trivial linear space with line ranges $\{n-1, n\}$, from the Proposition 2.1, since $\mathcal{S}$ is an $(n+1)-$regular non-trivial conditional linear space with two consecutive line degrees. Also, from the Proposition 2.2, the total number of points of $\mathcal{S}$ is either $n^2 - n$ or $n^2 - 1$.

Now we must show that the total number of points of $\mathcal{S}$ is $v = n^2 - n$. We assume that the total number of points of $\mathcal{S}$ is $v = n^2 - 1$. Since $\mathcal{S}$ is an $(n+1)-$regular linear space with $n^2 - 1$ points, from the Proposition 1.8, $b_n(P) = n$ and $b_{n-1}(P) = 1$, every point P of $\mathcal{S}$. From the Proposition 1.9, the number of $n-$lines is $n^2 - 1$ and the number of $(n-1)-$lines is $n+1$. Thus the total number of lines of S is

$b = b_n + b_{n-1} = n^2 + n$. In addition, $n + 1 - v(l) \in \{1, 2\}$ for every $l \in L$, since the line range is $\{n, n-1\}$. In this case,

$$b - v - \max_{l \in \mathcal{L}}\{n + 1 - v(l)\} = n^2 + n - n^2 + 1 - 2 = n - 1$$

Thus, from the Definition 1.6, we obtain $n \leq n - 1$. This is a contradiction. Thus, the total number of points of $\mathcal{S}$ is $v = n^2 - n$.

From the Proposition 1.10, $\mathcal{S}$ is a projective plane of order n less two lines and all their points, or a finite linear space with 12 points, 19 lines, every point of degree 5 and each point lying on precisely one 4-line and four 3-lines. □

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Author's address:

Pınar Anapa
Mathematics Department, Science and Art Faculty,
Osmangazi University, 26480, Eskisehir, Turkey,
E-mail: panapa@ogu.edu.tr