S-linear operators in quantum mechanics and in economy

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Abstract

In this paper we study the concept of $\mathcal{S}$-linear operator and we show some of its properties and applications to the foundations of Quantum Mechanics. This new operators are a generalization of the linear operators defined between two finite dimensional vector spaces. A generalization to the infinite dimensional case of the space of tempered distributions, endowed with the new operation of superposition introduced by the author in 1998.

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Introduction

Let $V = (X, +, \cdot)$ and $W = (Y, +, \cdot)$ be two finite-dimensional vector spaces on $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), we know that a function $f : X \to Y$ is called *linear* if, for each $x, y \in X$ and for each $\lambda \in \mathbb{K}$, one has $f(\lambda x + y) = \lambda f(x) + f(y)$. Moreover, we recall that $f \in \mathrm{Hom}(V, W)$ if and only if for every $\forall k \in \mathbb{N}$, $\forall x = (x_i)_{i=1}^k \in X^k$ and $\lambda = (\lambda_i)_{i=1}^k \in \mathbb{K}^k$, setting $\sum_k \lambda x = \sum_{i=1}^k \lambda_i x_i$ and $f(x) = (f(x_i))_{i=1}^k$, one has $f(\sum_k \lambda x) = \sum_k \lambda f(x)$ i.e., the image of the λ-linear combination of a family is the λ-linear combination of the image of the family under f; in indexed notation,

$$f\left(\sum_{i=1}^k \lambda_i x_i\right) = \sum_{i=1}^k \lambda_i f(x_i).$$

The intention of this paper is to give a similar definition for a certain class of families of vectors indexed by $\mathbb{R}^k$, using, as coefficients system, certain maps from $\mathbb{R}^k$ to $\mathbb{K}$ and, more generally, using the Schwartz tempered distributions from $\mathbb{R}^k$ to $\mathbb{K}$ (that are regarded as "non-locally defined" families in $\mathbb{K}$ indexed by $\mathbb{R}^k$). This requires the definition of linear superposition of the author: if $v = (v_i)_{i \in \mathbb{R}^k}$ is an $\mathcal{S}$-family in $\mathcal{S}'(\mathbb{R}^n, \mathbb{K})$ i.e., if, for every $\phi \in \mathcal{S}(\mathbb{R}^n, \mathbb{K})$, the function $v(\phi) : \mathbb{R}^k \to \mathbb{K} : i \mapsto v_i(\phi)$, belongs to $\mathcal{S}(\mathbb{R}^k, \mathbb{K})$ (see section 0) and if $\lambda \in \mathcal{S}'(\mathbb{R}^k, \mathbb{K})$ is a tempered distribution we set

$$\int_{\mathbb{R}^k} \lambda v := \lambda \circ \widehat{v} = {}^t(\widehat{v})(\lambda),$$

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where we have used the following operator

$$\widehat{v} : \mathcal{S}(\mathbb{R}^n, \mathbb{K}) \to \mathcal{S}(\mathbb{R}^k, \mathbb{K}) : \phi \mapsto v(\phi).$$

The idea is very natural: an operator $L : \mathcal{S}'_n \to \mathcal{S}'_m$ is calledd $\mathcal{S}$-ultralinear if, for every $k \in \mathbb{N}$, $\lambda \in \mathcal{S}'_k$ and $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_n)$, one has

$$L\left(\int_{\mathbb{R}^k} \lambda v\right) = \int_{\mathbb{R}^k} \lambda L(v).$$

History

The paper finds its origin in the Rigged Hilbert space formulation of quantum mechanics, the formulation developed by Bohm and Gadella in the booklet published in the series "Lecture notes in Physics" of Springer Verlag. The aim of that report was to present a rigorous mathematics framework for the Dirac formalism. But the theory presented did not accommodate all the features of Dirac's formulation of Quantum Mechanics.

It is known that Von Neumann's Hilbert space formulation does not fulfill the following feature of Dirac calculus:

1) Some state of physical system can't be normalizable in the Hilbert sense;

2) Some state that is not normalizable in the Hilbert sense can be normalized in the sense of Dirac;

3) There are some continuous families of vector states for which is reasonable to write:

$$\int_{\mathbb{R}} a_x v_x dx.$$

4) The functions representing vector states are always smooth, i.e. of class C^∞;

5) It's possible to calculate a kind of scalar product among the non normalizable state;

6) It is possible to decompose a vector in the following way:

$$u = \int_{\mathbb{R}} \langle u, v_x \rangle v_x dx.$$

1') The observables are defined in the whole space of vector states;

2') the operation among them are always possible,in particular, the commutation relation are identities and not inclusion;

3') The observables could be treated as continuous operator;

4') The Hermitian operators have a complete system of eigenkets, with a system of eigenvalues that can be also continuous;

5') It is possible to decompose an operator as follows

$$A(u) = \int_{\mathbb{R}} a(x) \langle u, v_x \rangle v_x dx.$$

6') The observables are linear with respect to the superpositions of a continuous family of states:

$$A(u) = A(\int_{\mathbb{R}} \langle u, v_x \rangle v_x dx) = \int_{\mathbb{R}} \langle u, v_x \rangle A(v_x) dx.$$

1") Finally, it's possible to superpose certain continuous families of solutions of linear differential equations obtaining a new state that is still a solution of the equation.

The Rigged Hilbert space formulation of quantum mechanics gives an answer to the Dirac's requisitions 1), 4), partially to 6), 1'), 2'), 3'), 4'), partially to 5').

The $\mathcal{S}$-algebra in the space of tempered distributions of the author of the present paper gives a unitary answer to all the requests, as we shall show in the paper.

Preliminaries and notations on tempered distributions

In this paper we shall use the following notations:

1) n, m, h, k are natural numbers, $\mathbb{N}(\leq k) = \{i\}_{i=1}^{k}$;

2) μ_n is the Lebesgue measure on $\mathbb{R}^n$; $(\cdot) = \mathbb{I}_{(\mathbb{R},\mathbb{C})}$ is the immersion of $\mathbb{R}$ in $\mathbb{C}$ and, if X is a non-empty set, $\mathbb{I}_X = (\cdot)_X$ is the identity map on X;

3) if X and Y are two topological vector spaces on $\mathbb{K}$, $\mathrm{Hom}(X, Y)$ is the set of all the linear operators from X to Y, $\mathcal{L}(X, Y)$ is the set of all the linear and continuous operators from X to Y, $X' = \mathrm{Hom}(X, \mathbb{K})$ is the algebraic dual of X and $X^* = \mathcal{L}(X, \mathbb{K})$ is the topological dual of X;

4) $\mathcal{S}_n = \mathcal{S}_n(\mathbb{K}) := \mathcal{S}(\mathbb{R}^n, \mathbb{K})$ is the $(n, \mathbb{K})$-*Schwartz space*, that is to say the set of all the smooth functions (i.e., of class C^∞) of $\mathbb{R}^n$ in $\mathbb{K}$ *rapidly decreasing at infinity* (the functions and all their derivatives tend to 0 at $\pm\infty$ faster than the reciprocal of any polynomial):
$\mathcal{S}(\mathbb{R}^n, \mathbb{K}) = \{f \in C^\infty(\mathbb{R}^n, \mathbb{K}) : \forall \alpha, \beta \in \mathbb{N}_0^n \lim_{|x| \to \infty} |x^\beta D^\alpha f(x)| = 0\}$.

5) $\mathcal{S}_{(n)}$ is the *standard Schwartz topology* on $\mathcal{S}_n$, it is a topology generated by a metric: in fact, $\mathcal{S}_n$ is closed under differentiation and multiplication by polynomials, for each nonnegative integer k, define p_k on $\mathcal{S}_n$ by $p_k(f) = \sup_{x \in \mathbb{R}^n} \max_{\substack{\alpha,\beta \in \mathbb{N}_0^n \\ 0 \leq |\alpha|, |\beta| \leq k}} |x^\beta D^\alpha f(x)|$. Each p_k is a norm on $\mathcal{S}_n$, and $p_k(f) \leq p_{k+1}(f)$ for all $f \in \mathcal{S}_n$, the pair $(\mathcal{S}_n, (p_k)_{k \in \mathbb{N}_0})$ is a countably complete normed space and so a Frèchet space (see also [5] and [1]);

6) $\mathcal{S}'_n := \mathcal{S}'(\mathbb{R}^n, \mathbb{K})$ is the space of tempered distributions from $\mathbb{R}^n$ to $\mathbb{K}$, that is, the topological dual of the topological vector space $(\mathcal{S}_n, \mathcal{S}_{(n)})$ i.e., $\mathcal{S}'_n = (\mathcal{S}_n, \mathcal{S}_{(n)})^*$;

7) if $x \in \mathbb{R}^n$, δ_x is the *distribution of Dirac on* $\mathcal{S}_n$ *centered at* x, i.e., the functional: $\delta_x : \mathcal{S}_n \to \mathbb{K} : \phi \mapsto \phi(x)$;

8) if $f \in \mathcal{O}_M(\mathbb{R}^n, \mathbb{K}) = \{g \in C^\infty(\mathbb{R}^n, \mathbb{K}) : \forall \phi \in \mathcal{S}_n(\mathbb{K}), \phi g \in \mathcal{S}_n(\mathbb{K})\}$, then the functional $[f] = [f]_n : \mathcal{S}_n \to \mathbb{K} : \phi \mapsto \int_{\mathbb{R}^n} f \phi d\mu_n$ is a tempered distribution, called the *regular distribution generated by* f (see [1, p. 110]);

9) Let $a, b \in \mathbb{R}^{\neq} = \mathbb{R}\backslash\{0\}$, $\underset{(a,b)}{\mathcal{S}}$ is the (a,b) *-Fourier-Schwartz transformation*, i.e., the operator $\underset{(a,b)}{\mathcal{S}} : \mathcal{S}_n \to \mathcal{S}_n$, such that, for all $f \in \mathcal{S}_n$ and $\xi \in \mathbb{R}^n$, one has

$$\underset{(a,b)}{\mathcal{S}}(f)(\xi) = \left(\frac{1}{a}\right)^n \int_{\mathbb{R}^n} f e^{-ib(\cdot|\xi)} d\mu_n = \left[\left(\frac{1}{a}\right)^n e^{-ib(\cdot|\xi)}\right](f),$$

where $(\cdot \mid \cdot)$ is the standard scalar product on $\mathbb{R}^n$. Moreover, we recall that $\underset{(a,b)}{\mathcal{S}}$ is a homeomorphism with respect to the standard topology $\mathcal{S}_{(n)}$ and, concerning its inverse, for every $x \in \mathbb{R}^n$ and $g \in \mathcal{S}_n$,one has

$$\underset{(a,b)}{\mathcal{S}}^{-}(g)(x) = \left(\frac{|b|a}{2\pi}\right)^n \int_{\mathbb{R}^n} g e^{ib(x|\cdot)} d\mu_n = \underset{(2\pi/(|b|a),-b)}{\mathcal{S}}(g)(x);$$

10) Let $a, b \in \mathbb{R}^{\neq}$, $\underset{(a,b)}{\mathcal{F}}$ is the (a,b)-*Fourier transformation on the space of tempered distributions*, i.e., the operator $\underset{(a,b)}{\mathcal{F}} : \mathcal{S}'_n \to \mathcal{S}'_n$, such that, for all $u \in \mathcal{S}'_n$ and for every $\phi \in \mathcal{S}_n$, one has $\underset{(a,b)}{\mathcal{F}}(u)(\phi) = u(\underset{(a,b)}{\mathcal{S}}(\phi))$, i.e., the transpose of $\underset{(a,b)}{\mathcal{S}}$: $\underset{(a,b)}{\mathcal{F}} = {}^t(\underset{(a,b)}{\mathcal{S}})$. Moreover, we recall that $\underset{(a,b)}{\mathcal{F}}$ is a homeomorphism in the weak* topology $\sigma_n^* = \sigma(\mathcal{S}'_n, \mathcal{S}_n)$ and that one has $\underset{(a,b)}{\mathcal{F}}^{-} = \underset{(2\pi/(|b|a),-b)}{\mathcal{F}}$ and, for all $\alpha \in \mathbb{N}_0^n$, $\underset{(a,b)}{\mathcal{F}}(u^{(\alpha)}) = (bi)^\alpha \mathbb{I}^\alpha_{\mathbb{R}^n} \underset{(a,b)}{\mathcal{F}}(u)$ and $\underset{(a,b)}{\mathcal{F}}(\mathbb{I}^\alpha_{\mathbb{R}^n} u) = (\frac{i}{b})^\alpha (\underset{(a,b)}{\mathcal{F}}(u))^{(\alpha)}$.

§0. Some concepts of $\mathcal{S}-$ultralinear algebra

Let I be a non-empty set, we denote by $s(I, \mathcal{S}'_n)$ the space of all the families in $\mathcal{S}'_n$ indexed by I, i.e., the set of all the surjective maps from I onto a subset of $\mathcal{S}'_n$. Moreover, if v is one of these families, for each $p \in I$, the distribution $v(p)$ is denoted by v_p, and v also by $(v_p)_{p\in I}$. The set $s(I, \mathcal{S}'_n)$ is a vector space with respect to the following standard operations: the addition $+ : s(I, \mathcal{S}'_n)^2 \to s(I, \mathcal{S}'_n) : (v, w) \mapsto v + w$, where $v + w = (v_p + w_p)_{p\in I}$, i.e., $(v + w)_p = v_p + w_p$; the multiplication by scalars $\cdot : \mathbb{K} \times s(I, \mathcal{S}'_n) \to s(I, \mathcal{S}'_n) : (\lambda, v) \mapsto \lambda v$ where $\lambda v = (\lambda v_p)_{p\in I}$, i.e., $(\lambda v)(p) = (\lambda v)_p = \lambda v_p$. Moreover, we shall use the following definitions of D. Carfì:

Definition 0.1 (family of tempered distributions of class $\mathcal{S}$). Let $v \in s(\mathbb{R}^m, \mathcal{S}'_n)$ be a family of distributions. The family v is called **family of class $\mathcal{S}$** or **$\mathcal{S}$-family** if, for each $\phi \in \mathcal{S}_n$, the function $v(\phi) : \mathbb{R}^m \to \mathbb{K}$, defined by $v(\phi)(p) = v_p(\phi)$, for each $p \in \mathbb{R}^m$, belongs to the space $\mathcal{S}_m$. The set of all these families is denoted by $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$.

Definition 0.2 (operator generated by an $\mathcal{S}$-family). Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be a family of class $\mathcal{S}$. The **operator generated by the family** v (or **associated with** v) is the operator $\widehat{v} : \mathcal{S}_n \to \mathcal{S}_m : \phi \mapsto v(\phi)$.

The set $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ is a subspace of the vector space $(s(\mathbb{R}^m, \mathcal{S}'_n), +, \cdot)$ and for each $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ the operator $\widehat{v}$ is linear and the map $(\cdot)^\wedge : \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) \to \mathrm{Hom}(\mathcal{S}_n, \mathcal{S}_m) : v \mapsto \widehat{v}$ is an injective linear operator.

Theorem 0.1 (basic lemma for the superpositions of an $\mathcal{S}$-family). *Let $a \in \mathcal{S}'_m$ and $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $\mathcal{S}$-family. Then, the composition $u = a \circ \widehat{v}$, i.e., the functional $u : \mathcal{S}_n \to \mathbb{K} : \phi \mapsto a(\widehat{v}(\phi))$, is a tempered distribution.*

Proof. Let $a \in \mathcal{S}'_m$. Because the subspace $\operatorname{span}(\{\delta_y\}_{y\in\mathbb{R}^m})$ is sequentially dense in $\mathcal{S}'_m$ (see [2, p. 205]), there is a sequence of distributions $(\alpha_k)_{k\in\mathbb{N}}$ in $\operatorname{span}(\{\delta_y\}_{y\in\mathbb{R}^m})$ such that $\sigma^*_n \lim_{k\to+\infty} \alpha_k = a$. Now, since $\alpha_k \in \operatorname{span}(\{\delta_y\}_{y\in\mathbb{R}^m})$ there exist a finite set $\{y_i\}_{i=1}^h$ in $\mathbb{R}^m$ and $\{\lambda_i\}_{i=1}^h$ in $\mathbb{K}$ such that $\alpha_k = \sum_{i=1}^h \lambda_i \delta_{y_i}$, thus $\alpha_k \circ \widehat{v} = \sum_{i=1}^h \lambda_i v_{y_i}$, hence, for every $k \in \mathbb{N}$, the composition $\alpha_k \circ \widehat{v}$ belongs to $\mathcal{S}'_n$; moreover, let τ be the topology of the pointwise convergence in $\operatorname{Hom}(\mathcal{S}_n, \mathbb{K})$, one has $\tau \lim_{k\to+\infty} \alpha_k \circ \widehat{v} = a \circ \widehat{v}$, in fact $\lim_{k\to+\infty} (\alpha_k \circ \widehat{v})(\phi) = \lim_{k\to+\infty} \alpha_k(\widehat{v}(\phi)) = a(\widehat{v}(\phi))$, so we have that $(\alpha_k \circ \widehat{v})_{k\in\mathbb{N}} \xrightarrow{\tau} a \circ \widehat{v}$ and that $\{\alpha_k \circ \widehat{v}\}_{k\in\mathbb{N}} \subset \mathcal{S}'_n$, then, by the completeness theorem of $\mathcal{S}'_n$ (see [4, p. 602]), one has $a \circ \widehat{v} \in \mathcal{S}'_n$. □

Definition 0.3 (linear superpositions of an $\mathcal{S}$-family). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ and $a \in \mathcal{S}'_m$. The distribution $a \circ \widehat{v} = {}^t(\widehat{v})(a)$ is called* **the (linear) superposition of v with respect to (the system of coefficients) a** *or the* **ultralinear combination of v with respect to (the system of coefficients) a** *and it is denoted by $\int_{\mathbb{R}^m} av$. Moreover, if $u \in \mathcal{S}'_n$ and there exists an $a \in \mathcal{S}'_m$ such that $u = \int_{\mathbb{R}^m} av$, u is said to be an* **$\mathcal{S}'$-linear superposition of v**. *Finally, we define* **linear superposition of v** *the distribution $\int_{\mathbb{R}^m} v := \int_{\mathbb{R}^m} 1_{\mathcal{S}'_m} v$, where $1_{\mathcal{S}'_m} := [1_{(\mathbb{R}^m,\mathbb{K})}]$ is the distribution generated by the $\mathbb{K}$-constant functional on $\mathbb{R}^m$ of value 1.*

Definition 0.4 (of $\mathcal{S}$-ultralinear independence). Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ v is said $\mathcal{S}$-ultralinearly independent, if one has $(u \in \mathcal{S}'_m \wedge \int_{\mathbb{R}^m} uv = 0_{\mathcal{S}'_n}) \Rightarrow u = 0_{\mathcal{S}'_m}$.

Definition 0.5 (of $\mathcal{S}$-linear hull). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. The* **$\mathcal{S}$-ultralinear hull** *of v is the set $\mathcal{S}\operatorname{uspan}(v) = \{u \in \mathcal{S}'_n : \exists a \in \mathcal{S}'_m : u = \int_{\mathbb{R}^m} av\}$.*

Definition 0.6 (system of $\mathcal{S}$-generators). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. v is called system of* **$\mathcal{S}$-generators** *for $V \subseteq \mathcal{S}'_n$ if and only if $\mathcal{S}\operatorname{uspan}(v) = V$.*

Definition 0.7 (of $\mathcal{S}$-basis). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ and let $V \subseteq \mathcal{S}'_n$. v is an* **$\mathcal{S}$-basis** *of V if it is $\mathcal{S}$-ultralinearly independent, and one has $\mathcal{S}\operatorname{uspan}(v) = V$.*

It's possible to prove that if $u \in \mathcal{S}\operatorname{uspan}(v)$ and v is ultralinearly independent then there exists a unique $a \in \mathcal{S}'_m$ such that $u = \int_{\mathbb{R}^m} av$. So we can give the following

Definition 0.8 (system of coordinates). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $\mathcal{S}$ -ultralinearly independent family and $w \in \mathcal{S}\operatorname{uspan}(v)$. The only tempered distribution $a \in \mathcal{S}'_m$ such that $w = \int_{\mathbb{R}^m} av$ is denoted by $[w|v]$ and is called* **the system of coordinates of w in v**.

§1. $\mathcal{S}-$operators and $\mathcal{S}-$ultralinear operators

Definition 1.1 (image of a family of distributions). *Let $W \subseteq \mathcal{S}'_n$, $A : W \to \mathcal{S}'_m$ be an operator and $v = (v_p)_{p\in\mathbb{R}^k}$ be a family of tempered distributions in W,* *i.e., such that* $\{v_p\}_{p\in\mathbb{R}^k} \subseteq W$. *The* **image of** v **under** A *is the family in* $\mathcal{S}'_m$ $A(v) = (A(v_p))_{p\in\mathbb{R}^k}$, *i.e., the family such that, for all* $p \in \mathbb{R}^k$, *one has* $A(v)_p = A(v_p)$.

We can read the above definition saying that "the image of a family of vectors is the family of the images of vectors".

Definition 1.2 (operator of class $\mathcal{S}$). *Let* $W \subseteq \mathcal{S}'_n$ *and* $L : W \to \mathcal{S}'_m$ *be an operator.* L *is an* $\mathcal{S}$-**operator** *or* **operator of class** $\mathcal{S}$ *if, for each natural* k *and for each* $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_n)$, *such that* $\{v_p\}_{p\in\mathbb{R}^k} \subseteq W$, *one has* $L(v) \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_m)$.

We can read the above definition as follows: "L is of class $\mathcal{S}$ if the image of an $\mathcal{S}$-family is an $\mathcal{S}$-family". In the following we put $\sigma_n = \sigma(\mathcal{S}_n, \mathcal{S}'_n)$.

Example 1.3 (the transpose). Let $A : \mathcal{S}_n \to \mathcal{S}_m$ be a (σ_n, σ_m)-continuous operator. A is transposable (i.e., for every $a \in \mathcal{S}'_m$, $a \circ A$ is in $\mathcal{S}'_n$) and its transpose is ${}^tA : \mathcal{S}'_m \to \mathcal{S}'_n : a \mapsto a \circ A$. Let $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_n)$, one has, by definition, ${}^tA(v)_p = {}^tA(v_p)$, and hence one infers

$$ {}^tA(v)(\phi)(p) = {}^tA(v)_p(\phi) = {}^tA(v_p)(\phi) = v_p(A(\phi)) = v(A(\phi))(p), $$

so, taking into account that v is an $\mathcal{S}$-family, one has ${}^tA(v)(\phi) = \widehat{v}(A(\phi)) \in \mathcal{S}_k$. Concluding one has ${}^tA(v) \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_n)$, and thus the operator ${}^tA$, sending $\mathcal{S}$-family in $\mathcal{S}$-family, is an $\mathcal{S}$-operator.

Application 1.1. Let $L : \mathcal{S}'_n \to \mathcal{S}'_n$ be a differential operator with constant coefficients and v be an $\mathcal{S}$-family in $\mathcal{S}'_n$. Then $L(v)$ is an $\mathcal{S}$-family, in fact L is the transpose of a certain operator. For instance, the family $(\delta_x)_{x\in\mathbb{R}^n}$ is obviously an $\mathcal{S}$-family, and so the families of derivatives $(\delta_x^{(i)})_{x\in\mathbb{R}^n}$ are $\mathcal{S}$-families for every multi-index i.

Definition 1.3 ($\mathcal{S}$-ultralinear operator). Let $L : \mathcal{S}'_n \to \mathcal{S}'_m$ be an $\mathcal{S}$-operator. L is called $\mathcal{S}$-**ultralinear operator** if, for each natural k, for each $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_n)$ and for every $a \in \mathcal{S}'_k$, one has $L(\int_{\mathbb{R}^k} av) = \int_{\mathbb{R}^k} aL(v)$. The set of all the $\mathcal{S}$-ultralinear operators from $\mathcal{S}'_n$ to $\mathcal{S}'_m$ is denoted by $\mathcal{S}\,\mathrm{uHom}(\mathcal{S}'_n, \mathcal{S}'_m)$.

In the following we denote by $\mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ the set of all the linear and continuous operator among the two topological vector spaces $(\mathcal{S}_n, \mathcal{S}_{(n)})$ and $(\mathcal{S}_m, \mathcal{S}_{(m)})$, since these spaces are complete and metrizable, $\mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ is also the set of all the linear and (σ_n, σ_m)-continuous operator (see [5, p. 258, Corollary]), i.e., the set of all the transposable linear operators (see [5, p. 254, § 12, Proposition 1]) among those spaces. It's, at this point, obvious that the two vector spaces $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ and $\mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ are isomorphic, being the map $(\cdot)^\wedge : \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) \to \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m) : v \mapsto \widehat{v}$ an isomorphism, moreover, its inverse is the map $(\cdot)^\vee : \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m) \to \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) : A \mapsto A^\vee := (\delta_x \circ A)_{x\in\mathbb{R}^m}$. Now, we show the intimate essence of the $\mathcal{S}$ -ultralinear operator defined on $\mathcal{S}'_n$.

Definition 1.4 (superposition of a family with respect to a family). Let $v \in s(\mathbb{R}^k, \mathcal{S}'_m)$ and $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. The family in $\mathcal{S}'_n$ $\displaystyle\int_{\mathbb{R}^m} vw := \left(\int_{\mathbb{R}^m} v_p w\right)_{p\in\mathbb{R}^k}$ is called **the superposition of** w **with respect to** v.

If $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_m)$ then $\displaystyle\int_{\mathbb{R}^m} vw \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_n)$ and $\displaystyle\left(\int_{\mathbb{R}^m} vw\right)^\wedge = \widehat{v} \circ \widehat{w}$. In this case, $\displaystyle\int_{\mathbb{R}^m} vw$ is denoted by vw and it's called product of v by w.

Lemma 1.1 (the image under a transpose operator). *Let $B \in \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ and $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_m)$. Then, ${}^tB(v) = \displaystyle\int_{\mathbb{R}^k} vB^\vee$, so in particular, ${}^tB$ is an $\mathcal{S}$-operator.*

Proof. For each $p \in \mathbb{R}^k$, one has

$$\left(\int_{\mathbb{R}^k} vB^\vee\right)_p = \int_{\mathbb{R}} v_p B^\vee = v_p \circ (B^\vee)^\wedge = v_p \circ B = {}^tB(v_p) = {}^tB(v)(p),$$

and hence $\displaystyle\int_{\mathbb{R}^k} vB^\vee = {}^tB(v)$. □

Theorem 1.1 ($\mathcal{S}$-ultralinearity of a transpose operator). *Let $B \in \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ and $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_m)$. Then, for each $a \in \mathcal{S}'_k$ one has ${}^tB(\int_{\mathbb{R}^k} av) = \int_{\mathbb{R}^k} a\, {}^tB(v)$.*

Proof. One has

$$\begin{aligned} {}^tB\left(\int_{\mathbb{R}^k} av\right) &= \left(\int_{\mathbb{R}^k} av\right) \circ B = (a \circ \widehat{v}) \circ B = a \circ (\widehat{v} \circ B) = \\ &= \int_{\mathbb{R}^k} a\,(\widehat{v} \circ B)^\vee = \int_{\mathbb{R}^k} a\left(\int_{\mathbb{R}^k} vB^\vee\right) = \int_{\mathbb{R}^k} a\, {}^tB(v). \square \end{aligned}$$

Application 1.2. As a simple application, we prove the formula: $u' = \displaystyle\int_{\mathbb{R}} u\delta'$, where δ' is the $\mathcal{S}$-family in $\mathcal{S}'_1$ $(\delta'_p)_{p\in\mathbb{R}}$. Let δ be the Dirac family of $\mathcal{S}'_1$, then for each $u \in \mathcal{S}'_1$, one has $u = \displaystyle\int_{\mathbb{R}} u\delta$, and thus

$$u' = D\left(\int_{\mathbb{R}} u\delta\right) = \int_{\mathbb{R}} uD(\delta) = \int_{\mathbb{R}} u\delta'.$$

Theorem 1.2 (characterization of $\mathcal{S}$-ultralinearity). *Let $L : \mathcal{S}'_n \to \mathcal{S}'_m$. Then, L is $\mathcal{S}$-ultralinear if and only if there exists a $B \in \mathcal{L}(\mathcal{S}_m, \mathcal{S}_n)$ such that $L = {}^t(B)$.*

Proof. Sufficiency. Follows from the above theorem. *Necessity.* Let δ be the Dirac's family in $\mathcal{S}'_n$, one has

$$L(u) = L\left(\int_{\mathbb{R}^n} u\delta\right) = \int_{\mathbb{R}^n} uL(\delta) = {}^t(L(\delta)^\wedge)(u),$$

so $L = {}^t(L(\delta)^\wedge)$. □

Application 1.3 (*the case of a continuous range of fundamental states in quantum mechanics*, see [3, p.66]). A *pure state* σ of a quantum system is a monodimensional subspace of the space $\mathcal{S}'_n$, each $\psi \in \sigma$ is a *vector-state representing* σ. Let $\psi = (\psi_p)_{p\in\mathbb{R}^m}$ be an $\mathcal{S}$-basis of $\mathcal{S}'_n$ and α be an observable of the system, we assume $\alpha \in \mathcal{S}\,\mathrm{uEnd}(\mathcal{S}'_n)$. One has

$$\alpha(\psi_p) = \int_{\mathbb{R}^m} [\alpha(\psi_p) \mid \psi]\psi,$$

the family

$$(\alpha)_\psi = ([\alpha(\psi_p) \mid \psi])_{p\in\mathbb{R}^m}$$

is called *the representation of* α *in* ψ. Let now α, β be two observables; one has

$$\begin{aligned}\alpha(\beta(\psi_p)) &= \alpha\int_{\mathbb{R}^m}[\beta(\psi_p) \mid \psi]\psi = \int_{\mathbb{R}^m}(\beta)_\psi{}^p\alpha(\psi) = \\ &= \int_{\mathbb{R}^m}(\beta)_\psi{}^p\int_{\mathbb{R}^m}(\alpha)_\psi\psi = \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}(\beta\right)_\psi^p(\alpha)_\psi)\psi,\end{aligned}$$

so $(\alpha\beta)_\psi = (\alpha)_\psi(\beta)_\psi$. We have, moreover, $u = \int_{\mathbb{R}^m}[u \mid \psi]\psi$, the non localized-family $(u)_\psi = [u \mid \psi]$ is called *the representation of* α *in* ψ, so one has

$$\alpha(u) = \int_{\mathbb{R}^m}(u)_\psi\alpha(\psi) = \int_{\mathbb{R}^m}(u)_\psi\int_{\mathbb{R}^m}(\alpha)_\psi\psi = \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}(u)_\psi(\alpha)_\psi\right)\psi,$$

thus $(\alpha(u))_\psi = \int_{\mathbb{R}^m}(u)_\psi(\alpha)_\psi$. If we regard the multiplication by a number as an observable: $M_c(u) = cu$, we have

$$M_c(u) = cu = c\int_{\mathbb{R}^m}(u)_\psi\psi = \int_{\mathbb{R}^m}(u)_\psi(c\psi) = \int_{\mathbb{R}^m}c(u)_\psi\psi,$$

hence $M_c(\psi_p) = \int_{\mathbb{R}^m}c(\psi_p)_\psi\psi = \int_{\mathbb{R}^m}c\delta_p\psi$, so the $\mathcal{S}$-family representing the observable M_c is the family $(c\delta_p)_{p\in\mathbb{R}^m}$, i.e., the family $c\delta$. The correspondence

$$(\cdot)_\psi : \mathcal{S}\,\mathrm{uEnd}(\mathcal{S}'_n) \to \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$$

is bijective, in fact, $(\alpha)_\psi = (\beta)_\psi$ implies

$$(\alpha(u))_\psi = \int_{\mathbb{R}^m}(u)_\psi(\alpha)_\psi = \int_{\mathbb{R}^m}(u)_\psi(\beta)_\psi = (\beta(u))_\psi,$$

and thus $\alpha u = \beta u$ i.e., $\alpha = \beta$, so it's injective. It's also surjective, in fact, if $(v_p)_{p\in\mathbb{R}^m}$ is an $\mathcal{S}$-family and we put

$$\alpha(u) = \int_{\mathbb{R}^m}(u)_\psi(\int_{\mathbb{R}^m}v\psi)$$

one has

$$\alpha(\psi_p) = \int_{\mathbb{R}^m}(\psi_p)_\psi\left(\int_{\mathbb{R}^m}v\psi\right) = \int_{\mathbb{R}^m}\delta_p\left(\int_{\mathbb{R}^m}v\psi\right) = \left(\int_{\mathbb{R}^m}v\psi\right)_p = \int_{\mathbb{R}^m}v_p\psi,$$

and thus $(\alpha)_\psi = v$. It's simple to prove that ψ is an $\mathcal{S}$-basis of the entire space if and only if ${}^t\widehat{\psi}$ is bijective and in this case one has $u_\psi = ({}^t\widehat{\psi})^-(u)$ and $(\alpha)_\psi{}^p = \int_{\mathbb{R}^n}\alpha\psi_p\psi^-$. Let $X : \mathcal{S}'_1 \to \mathcal{S}'_1 : u \mapsto (\cdot)u$ be the position operator and let f be the $(1/, -1/\hbar)$-Fourier family, then one has

$$(X)_f{}^p = \int_{\mathbb{R}}Xf_pf^- = \mathop{\mathcal{F}}_{(1,1/\hbar)}((\cdot)f_p) = \left(\frac{i}{1/\hbar}\right)^1(2\pi\mathop{\mathcal{F}}_{(1,-1/\hbar)}{}^-(f_p))' = i\hbar(f_p)'_f = i\hbar\delta'_p.$$

Let $P : \mathcal{S}'_1 \to \mathcal{S}'_1 : u \mapsto -i\hbar u'$ be the momentum operator of a particle. One has

$$(P)_f{}^p = \int_{\mathbb{R}} P f_p f^- = \int_{\mathbb{R}} p f_p f^- = p(f_p)_f = p\delta_p.$$

and hence $(P)_f = (\cdot)\delta$. Let

$$T : \mathcal{S}'_1 \to \mathcal{S}'_1 : u \mapsto \frac{\hbar^2}{2m} u'' = \frac{1}{2m} P^2$$

be the kinetic energy operator of a nonrelativistic particle, one has

$$\begin{aligned}(T)_f{}^p &= \int_{\mathbb{R}} T f_p f^- = \int_{\mathbb{R}} \frac{1}{2m} P^2 f_p f^- = \frac{1}{2m} \int_{\mathbb{R}} p^2 f_p f^- = \\ &= \frac{1}{2m} p^2 \int_{\mathbb{R}} f_p f = \frac{1}{2m} p^2 (f_p)_f = \frac{1}{2m} p^2 \delta_p.\end{aligned}$$

§2. Spectral theorems

First of all we recall, for convenience of the reader, some basic notions from theory of distributions.

Definition 2.1. We denote by $\mathcal{O}_M(\mathbb{R}^n, \mathbb{K})$ the space of all $f \in C^\infty(\mathbb{R}^n, \mathbb{K})$ such that for every $\phi \in \mathcal{S}_n$ one has $\phi f \in \mathcal{S}_n$. The set $\mathcal{O}_M(\mathbb{R}^n, \mathbb{K})$ is said to be the **space of C^∞ functions from $\mathbb{R}^n$** to $\mathbb{K}$ **slowly increasing at infinity**.

Proposition 2.1. *Let $f \in C^\infty(\mathbb{R}^n, \mathbb{K})$. The following are equivalent conditions:*

1. *For all $p \in \mathbb{N}_0^n$ there is a polynomial P_p such that $\forall x \in \mathbb{R}^n$, $|\partial^p f(x)| \le |P_p(x)|$.*
2. *For all $\phi \in \mathcal{S}_n$ one has $\phi f \in \mathcal{S}_n$.*
3. *For every $p \in \mathbb{N}_0^n$ and for every $\phi \in \mathcal{S}_n$ the function $(\partial^p f)\phi$is bounded in $\mathbb{R}^n$.*

The **standard topology of** $\mathcal{O}_M(\mathbb{R}^n, \mathbb{K})$ *is the locally convex topology defined by the family of seminorms* $\gamma_{\phi,p}(\phi) = \sup_{x \in \mathbb{R}^n} |\phi(x)\partial^p f(x)|$ *where* $\phi \in \mathcal{S}(\mathbb{R}^n, \mathbb{K})$ *and* $p \in \mathbb{N}_0^n$. This topology does not have a countable basis. Also, it can be shown that $\mathcal{O}_M(\mathbb{R}^n, \mathbb{K})$ is a complete space. A sequence (or filter) $(f_j)_{j \in \mathbb{N}}$ converges to zero in $\mathcal{O}_M(\mathbb{R}^n, \mathbb{K})$ if and only if for every $\phi \in \mathcal{S}_n$ and for every $p \in \mathbb{N}_0^n$, the sequence $(\phi \partial^p f_j)_{j \in \mathbb{N}}$ converges to zero uniformly on $\mathbb{R}^n$. Or, equivalently, for every $\phi \in \mathcal{S}_n$, $(\phi f_j)_{j \in \mathbb{N}}$ converges to zero in $\mathcal{S}_n$. A set B is bounded in $\mathcal{O}_M(\mathbb{R}^n, \mathbb{K})$ if and only if for all $p \in \mathbb{N}_0^n$ there is a polynomial P_p such that $\forall x \in \mathbb{R}^n$, $\forall f \in B$, $|\partial^p f(x)| \le P_p(x)$. Moreover, the bilinear map $\Phi : \mathcal{O}_M(\mathbb{R}^n, \mathbb{K}) \times \mathcal{S}_n \to \mathcal{S}_n : (\phi, f) \mapsto \phi f$ is separately continuous.

Proposition 2.2. *Let $A \in \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ and $f \in \mathcal{O}_M(\mathbb{R}^m, \mathbb{K})$. Then, the mapping $fA : \mathcal{S}_n \to \mathcal{S}_m : \phi \mapsto fA(\phi)$ is a linear and c ontinuous operator.*

Proof. First of all we note that fA is well defined in fact $(fA)(\phi) = fA(\phi) \in \mathcal{S}_m$ because $f \in \mathcal{O}_M(\mathbb{R}^m, \mathbb{K})$ and $A(\phi) \in \mathcal{S}_m$. Moreover, the bilinear application $\Phi : \mathcal{O}_M(\mathbb{R}^m, \mathbb{K}) \times \mathcal{S}_m \to \mathcal{S}_m : (f, \psi) \mapsto f\psi$ is separately continuous and because $(fA)(\phi) = fA(\phi) = \Phi(f, A(\phi))$ i.e. $fA = \Phi(f, A) := \Phi(f, \cdot) \circ A$ the operator fA is the composition of two linear continuous maps and then is a linear and continuous operator. □

Let $A \in \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ and $f \in \mathcal{O}_M(\mathbb{R}^m, \mathbb{K})$. The operator $fA : \mathcal{S}_n \to \mathcal{S}_m : \phi \mapsto fA(\phi)$ is called the *product of A by f*.

Proposition 2.3. *Let $A, B \in \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ and $f, g \in \mathcal{O}_M(\mathbb{R}^m, \mathbb{K})$. Then, one has*

1) $(f+g)A = fA + gA$; $f(A+B) = fA + fB$; $1_{\mathcal{O}_M} A = A$;

2) the map $\Phi : \mathcal{O}_M(\mathbb{R}^m, \mathbb{K}) \times \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m) \to \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m) : (f, A) \mapsto fA$ is a bilinear map.

Proof. It's a straightforward computation. □

The above bilinear application is called *multiplication of operators by $\mathcal{O}_M$ functions.*

Remark 2.1. It's easy to see that the algebraic structure $(\mathcal{O}_M(\mathbb{R}^n, \mathbb{K}), +, \cdot)$ is a commutative ring with identity, where: $\cdot : \mathcal{O}_M(\mathbb{R}^n, \mathbb{K}) \times \mathcal{O}_M(\mathbb{R}^n, \mathbb{K}) \to \mathcal{O}_M(\mathbb{R}^n, \mathbb{K}) : (f, g) \mapsto fg$ (obviously if $f, g \in \mathcal{O}_M(\mathbb{R}^n, \mathbb{K})$ one has $fg \in \mathcal{O}_M(\mathbb{R}^n, \mathbb{K})$) and $1_{(\mathcal{O}_M, +, \cdot)} = 1_{(\mathbb{R}^n, \mathbb{K})}$). Moreover, one has that $\mathcal{S}_n$ is an ideal of $\mathcal{O}_M(\mathbb{R}^n, \mathbb{K})$.

Proposition 2.4. *Let $\cdot$ the operation defined in the above theorem. Then, the algebraic structure $(\mathcal{L}(\mathcal{S}_n, \mathcal{S}_m), +, \cdot)$ is a left module over the ring $(\mathcal{O}_M(\mathbb{R}^m, \mathbb{K}), +, \cdot)$.*

Proof. Recall the preceding theorem, we have to prove only the pseudo-associative law, i.e. we have to prove that $\forall f, g \in \mathcal{O}_M(\mathbb{R}^m, \mathbb{K}), \forall A \in \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m), (fg)A = f(gA)$. In fact, for each $\phi \in \mathcal{S}_n$, one has $[(fg)A](\phi) = (fg)A(\phi) = f(gA(\phi)) = f(gA)(\phi)) = [f(gA)](\phi)$. □

Definition 2.2 (product of a family by an $\mathcal{O}_M$ function). Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ and $f \in C^\infty(\mathbb{R}^m, \mathbb{K})$. The product of v by f is the family $fv = (f(p)v_p)_{p \in \mathbb{R}^m}$.

Theorem 2.1. *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ and $f \in \mathcal{O}_M(\mathbb{R}^m, \mathbb{K})$. Then, the family fv lies in $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. Moreover, one has $(fv)^\wedge = f\widehat{v}$.*

Proof. Let $\phi \in \mathcal{S}_n$, one has

$$(fv)(\phi)(p) = (fv)_p(\phi) = (f(p)v_p)(\phi) = f(p)v_p(\phi) = f(p)\widehat{v}(\phi)(p)$$

and hence $(fv)(\phi) = f\widehat{v}(\phi) \in \mathcal{S}_m$. Thus, one has $fv \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$, $\forall \phi \in \mathcal{S}_n$, $(fv)^\wedge(\phi) = f\widehat{v}(\phi)$, i.e. $(fv)^\wedge = f\widehat{v}$, where $f\widehat{v}$, is the product of $\widehat{v}$ by f and $f\widehat{v} \in \mathcal{L}(\mathcal{S}_m, \mathcal{S}_n)$. □

Theorem 2.2. *Let $f, g \in \mathcal{O}_M(\mathbb{R}^m, \mathbb{K})$, $v, w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. Then, one has*

1) $(f+g)v = fv + gv$; $f(v+w) = fv + fw$; $1_{\mathcal{O}_M} v = v$.

2) The map $\Phi : \mathcal{O}_M(\mathbb{R}^m, \mathbb{K}) \times \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) \to \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) : (f, v) \mapsto fv$ is a bilinear map.

Proof. 1) For all $p \in \mathbb{R}^m$, one has

$$[(f+g)v](p) = (f+g)(p)v_p = (f(p) + g(p))v_p = f(p)v_p + g(p)v_p = (fv)_p + (gv)_p,$$

i.e. $(f+g)v = fv + gv$; For all $p \in \mathbb{R}^m$, one has

$$[f(v+w)](p) = f(p)(v+w)_p = f(p)(v_p + w_p) = f(p)v_p + f(p)w_p = (fv)_p + (fw)_p,$$

i.e. $f(v+w) = fv + fw$. For all $p \in \mathbb{R}^m$, one has $(1_{(\mathbb{R}^m,\mathbb{K})}\, v)(p) = 1_{(\mathbb{R}^m,\mathbb{K})}\,(p)v_p = v_p$; i.e. $1_{\mathcal{O}_M} v = v$. 2) is straightforward. □

The bilinear application of the point 2) of the preceding theorem is called *multiplication of families by $\mathcal{O}_M$ functions.*

Theorem 2.3 (of structure). *Let $\cdot$ the operation defined above. Then, the algebraic structure $(\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n), +, \cdot)$ is a left module over the ring $(\mathcal{O}_M(\mathbb{R}^m, \mathbb{K}), +, \cdot)$.*

Proof. It's analogous to the proof of the proposition 2.4. □

Theorem 2.4 (of isomorphism). *The application $(\cdot)^\wedge : \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) \to \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ is a module isomorphism.*

Proof. It follows easily from theorem 2.1. □

In the following we shall use the notation $\mathcal{S}\,\mathrm{uEnd}(\mathcal{S}'_n) = \mathcal{S}\,\mathrm{uHom}(\mathcal{S}'_n, \mathcal{S}'_n)$. Let D, C be two vector spaces and $A \in \mathrm{Hom}(D, C)$. The set of all the eigenvectors of the operator A is denoted by $\mathrm{EV}(A)$ and is called the family of the eigenvectors of A. The set of all the eigenvalues of the operator A is denoted by $\mathrm{ES}(A)$; moreover the eigenspace relative to an eigenvalue $a \in \mathbb{K}$ is denoted by $|a\rangle_A$.

Theorem 2.5 (spectral theorem). *Let $A \in \mathcal{S}\,\mathrm{uEnd}(\mathcal{S}'_n)$, $f \in \mathcal{O}_M(\mathbb{R}^m, \mathbb{K})$ and $v \in \mathcal{S}_{ind}(\mathbb{R}^m, \mathcal{S}'_n)$ such that, for each $p \in \mathbb{R}^m$, one has $A(v_p) = f(p)v_p$,* i.e. *$A(v) = fv$.* Then, for each *$u \in \mathcal{S}\,\mathrm{uspan}(v)$, one has $A(u) = \int_{\mathbb{R}^m} f[u \mid v]v$.*

Proof. For each $u \in \mathcal{S}\,\mathrm{uspan}(v)$, one has

$$A(u) = A(\int_{\mathbb{R}^m} [u \mid v]v) = \int_{\mathbb{R}^m} [u \mid v]A(v) = \int_{\mathbb{R}^m} [u \mid v](fv) = \int_{\mathbb{R}^m} (f[u \mid v])v.$$

In fact, the third equality holds because, $A(v)_p = A(v_p) = f(p)v_p = (fv)(p)$, and the forth because

$$\int_{\mathbb{R}^m} [u|v](fv)(\phi) = [u|v]((fv)^\wedge(\phi)) = [u|v](f\widehat{v}(\phi)) = (f[u|v])(\widehat{v}(\phi)) = \int_{\mathbb{R}^m} (f[u|v])v,$$

this concludes the proof. □

Let $X \subseteq \mathcal{S}'_n$ be a subspace of $\mathcal{S}'_n$, $A \in \mathrm{Hom}(X, \mathcal{S}'_m)$ and $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be a family of distributions. The superposition of v with respect to A, is the operator

$$\int_{\mathbb{R}^m} Av : X \to \mathcal{S}'_n : u \mapsto \int_{\mathbb{R}^m} A(u)v.$$

In the condition of the above theorem one has: $A_{|X} = \int_{\mathbb{R}^n} f[\cdot \mid v]v$.

Applications (the building of the basic observables of quantum mechanics). A particle moving in the real line can be in a state in which its position is $x \in \mathbb{R}$. It's natural to assume that this state can be represented by the distribution δ_x, so if we denote by Q the observable "position" we have $Q\delta_x = x\delta_x$, i.e., $Q\delta = \mathbb{I}_{\mathbb{R}}\delta$, applying the above theorem one has $Q(u) = \int_{\mathbb{R}^n} \mathbb{I}_{\mathbb{R}}[u \mid \delta]\delta = \int_{\mathbb{R}^n} (\mathbb{I}_{\mathbb{R}} u)\delta = \mathbb{I}_{\mathbb{R}} u$. This justifies the definition of the position operator, which is now possible to define, more naturally, the only observable that in the state δ_x assume the value x. Analogously, following De Broglie, we assume that the state of a particle moving in one dimension with momentum $p \in \mathbb{R}$ be represented by the regular distribution $[e^{\frac{i(p|\cdot)}{\hbar}}]$, so, if we denote by P and T the observables "momentum" and "Hamiltonian of a classic free particle in $\mathbb{R}$", respectively, we have $P[e^{\frac{i(p|\cdot)}{\hbar}}] = p[e^{\frac{i(p|\cdot)}{\hbar}}]$ and

$$T[e^{\frac{i(p|\cdot)}{\hbar}}] = \frac{p^2}{2m}[e^{\frac{i(p|\cdot)}{\hbar}}],$$

i.e., putting $f = ([e^{\frac{i(p|\cdot)}{\hbar}}])_{p\in\mathbb{R}}$, $Pf = \mathbb{I}_{\mathbb{R}} f$ and $Tf = \frac{p^2}{2m}f$, applying the above theorem, one has

$$\begin{aligned} P(u) &= \int_{\mathbb{R}} \mathbb{I}_{\mathbb{R}}[u \mid f]f = (\mathbb{I}_{\mathbb{R}} \underset{(1,-1/\hbar)}{\mathcal{F}}{}^{-}(u)) \circ \mathcal{S} = \underset{(1,-1/\hbar)}{\mathcal{F}}(\mathbb{I}_{\mathbb{R}} \underset{(1,-1/\hbar)}{\mathcal{F}}{}^{-}(u)) = \\ &= \left(\frac{i}{-1/\hbar}\right)^1 (\underset{(1,-1/\hbar)}{\mathcal{F}}(\underset{(1,-1/\hbar)}{\mathcal{F}}{}^{-}(u)))' = -i\hbar u', \end{aligned}$$

$$\begin{aligned} T(u) &= \int_{\mathbb{R}} \frac{\mathbb{I}_{\mathbb{R}}^2}{2m}[u \mid f]f = (\frac{\mathbb{I}_{\mathbb{R}}^2}{2m}\mathcal{F}^{-}(u)) \circ \mathcal{S} = \frac{1}{2m}\underset{(1,-1/\hbar)}{\mathcal{F}}(\mathbb{I}_{\mathbb{R}}^2 \mathcal{F}^{-}(u)) = \\ &= \frac{1}{2m}\left(\frac{i}{-1/\hbar}\right)^2 (\mathcal{F}(\mathcal{F}^{-}(u)))'' = -\frac{\hbar^2}{2m}u''. \end{aligned}$$

§3. The functional "price" as an $S-$linear functional

We present here an original model for the one consumer model with a continuous set of choices. In our model, if we have a market with infinitely many commodities, more precisely, with an ordered set $g = (g_i)_{i\in\mathbb{R}}$ of commodities, we have a "distribution of prices" p, this is a function $p \in C^\infty(\mathbb{R}, \mathbb{R})$. In our model, a bundle of commodities in this market is represented by a distribution $\beta \in \mathcal{E}'(\mathbb{R}, \mathbb{R})$ with compact support. The price of β is

$$P(\beta) = \int_{\mathbb{R}} p\beta d\mu,$$

where μ is the Lebesgue-measure on $\mathbb{R}$ and $\int_{\mathbb{R}} p\beta d\mu$ is the usual integral of the compact support distribution $p\beta$. We recall that the product of a smooth function by a distribution with compact support is yet a distribution with compact support.

At this point we consider the functional

$$P : \mathcal{E}'(\mathbb{R}, \mathbb{R}) \to \mathbb{R} : \beta \mapsto \int_{\mathbb{R}} p\beta d\mu,$$

and for β a family of distributions with compact support that represents a family of bundles, we put

$$P(\beta) = (P(\beta_i))_{i\in\mathbb{R}} = \left(\int_{\mathbb{R}} p\beta_i d\mu\right)_{i\in\mathbb{R}}.$$

In other words, we say that the image of a family of bundles is the family of the image of the bundles, so $P(\beta)$ is a family of real numbers indexed by the set of real numbers, and hence it can be viewed as a real function defined on the real line.

Further, for a given family of real numbers $f = (f_i)_{i\in\mathbb{R}}$, viewed as a smooth function, we can consider, for every compact support distribution a, the following number

$$\int_{\mathbb{R}} af := \int_{\mathbb{R}} fad\mu,$$

where in the right hand we have the usual integral of the distribution fa, and f is considered as a smooth function; the symbol on the left is, according to us, "the superposition of the family f under the system of coefficients a". We recall that the integral of a compact support distribution is simply its value on the constant function $1_{\mathbb{R}}$ (1 everywhere), so

$$\int_{\mathbb{R}} fad\mu = (fa)\,(1_{\mathbb{R}}),$$

and recalling the definition of the product of a distribution by a smooth function, we have

$$\int_{\mathbb{R}} fad\mu = a(f).$$

We shall prove that the functional P is subject to the following:

Theorem. *Let a be a distribution with compact support a, and β a family of bundles verifying the property that for every smooth function g, the map $\beta(g) : x \in \mathbb{R} \to \beta_x(g) \in \mathbb{R}$ is smooth too. Then*

$$P\left(\int_{\mathbb{R}} a\beta\right) = \int_{\mathbb{R}} aP(\beta)\,.$$

In this case P is called $\mathcal{S}$-ultralinear.

Proof. First we have to prove that the expression $P\left(\int_{\mathbb{R}} a\beta\right) = \int_{\mathbb{R}} aP(\beta)$ is correctly stated. Since $\beta(g)$ is a smooth function for every smooth function g, one has that it is possible to consider the linear functional

$$s : \mathcal{E}(\mathbb{R},\mathbb{R}) \to \mathbb{R} : g \mapsto a(\beta(g));$$

moreover by the completeness theorem of $\mathcal{E}'(\mathbb{R},\mathbb{R})$, s is a distribution with compact support. We have to prove that $P(\beta)$ is a family which is superposable under every compact support distribution. But this is obvious because $P(\beta)$ as a function is a smooth one; more precisely, it is $\beta(p)$:

$$P(\beta) = (P(\beta_i))_{i\in\mathbb{R}} = \left(\int_{\mathbb{R}} p\beta_i d\mu\right)_{i\in\mathbb{R}} = (\beta_i(p))_{i\in\mathbb{R}} = \beta(p),$$

which is smooth, since p is smooth.

If $a \in \mathcal{E}'(\mathbb{R},\mathbb{R})$ we have $\left(\int_{\mathbb{R}} a\beta\right) = a \circ \widehat{\beta}$, so we infer

$$\begin{aligned}
P\left(\int_{\mathbb{R}} a\beta\right) &= \int_{\mathbb{R}} p\left(\int_{\mathbb{R}} a\beta\right) d\mu = \int_{\mathbb{R}} p\left(a \circ \widehat{\beta}\right) d\mu = \left(p\left(a \circ \widehat{\beta}\right)\right)(1_{\mathbb{R}}) = \\
&= \left(a \circ \widehat{\beta}\right)(p1_{\mathbb{R}}) = \left(a \circ \widehat{\beta}\right)(p) = a\left(\widehat{\beta}(p)\right) = \int_{\mathbb{R}} \beta(p) a d\mu = \\
&= \int_{\mathbb{R}} (\beta_i(p))_{i\in\mathbb{R}}\, a d\mu = \int_{\mathbb{R}} a\, (\beta_i(p))_{i\in\mathbb{R}} = \int_{\mathbb{R}} a\left(\int_{\mathbb{R}} p\beta_i d\mu\right)_{i\in\mathbb{R}} = \\
&= \int_{\mathbb{R}} a\left(\int_{\mathbb{R}} p\beta d\mu\right) = \int_{\mathbb{R}} aP(\beta)\,.
\end{aligned}$$

Recall that, by $\int_{\mathbb{R}} p\beta d\mu$, we indicate the family $\left(\int_{\mathbb{R}} p\beta_i d\mu\right)_{i\in\mathbb{R}}$, so a smooth distribution of prices in a market with infinite commodities generates an $\mathcal{S}$-linear functional. □

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