On finite near-circular spaces

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Abstract

In this paper, we introduce the concept of a near-circular space, and show that certain propositions holding in finite linear spaces or their similarization hold, also, in finite circular spaces.

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§1. Introduction

The subject of finite near-linear spaces and finite linear spaces have been studied in detail, and some combinatorial results have been obtained in ([1], [2], [3] and [4]) on this subject. In this paper, we define the concept of a near-circular space (circular space) probably as an alternative to a near-linear space (linear space).

Then we give some propositions about finite circular spaces as the propositions about finite near-linear spaces given in [2].

Finally, we use a theorem which characterizes the connection between the finite near-linear spaces and finite linear spaces to determine connection between finite near-circular spaces and finite circular spaces.

The notations used in this paper are the same as those in [2].

When we say there is one line (or there are two points), we always mean that there is precisely one line(or there are precisely two points). Otherwise we shall add "at least" or "at most" or some equivalent expression.

Definition 1.(Batten[2]). *A near-linear space is a space $S = (P, \mathcal{L})$ of points $\mathcal{P}$ and lines $\mathcal{L}$ such that*

NL1. *any line has at least two points and*

NL2. *two points are on at most one line.*

If $\mathcal{P}$ and $\mathcal{L}$ are finite then S is called finite.

If P and Q are distinct points which are on a line, then this line is unique by NL2. This unique line is denoted by PQ. Then it should be clear that if R and S are any distinct points on the line PQ, it must be the case, by NL2, that $PQ = RS$.

Definition 2. *A near-circular space is a space $C = (P, \mathcal{C})$ of points $\mathcal{P}$ and circles $\mathcal{C}$ such that*

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NC1. *every circle contains at least three distinct points and*

NC2. *three distinct points are contained in at most one circle.*

If $\mathcal{P}$ and $\mathcal{C}$ are finite then C is called finite.

If P, Q and R are distinct points which are on a circle, then this circle is unique by NC2 and this unique circle is denoted by PQR. It should be clear that if S, T and U are any distinct points on the circle PQR, it must be the case by NC2, that $PQR = STU$.

§2. Near-circular Spaces

Now, we give two essential properties of a near-circular space by the following propositions.

Proposition 1. *Two distinct circles of a near-circular space intersect in at most two points.*

Proof: Suppose c_1 and c_2 are distinct circles. If $|c_1 \cap c_2| \geq 3$, we contradict NC2. □

Proposition 2. *Let c_1 and c_2 be any circles in C . If $c_1 \subseteq c_2$ then $c_1 = c_{2.}$*

Proof: c_1 contains at least three points by NC1. Then $c_1 = c_2$ by NC2. □

Notation. For the number of points in a near-circular space we use v and for the number of circles, b.

For a circle c_j,the number of points on c_j is denoted by v_j or $|c_j|$,

For a point P_i,the number of circles on P_i is denoted by b_i or $b(P_i)$.

Now, we give an important connection between near-circular spaces and near-linear spaces as the connection between inversive planes and affine planes.

Proposition 3. *Let $C = (\mathcal{P}, \mathcal{C})$ be a near-circular space and$P \in \mathcal{P}$. Consider the set of all circles on P. If $\mathcal{P}' = \mathcal{P} \backslash \{P\}$ and $\mathcal{L} = \{c \mid c \cup \{P\} \in \mathcal{C}\}$ then $C_P = (\mathcal{P}', \mathcal{L})$ is a near-linear space.*

Proof: Let c be any line of $\mathcal{L}$. $|c \cup \{P\}| \geq 3$ since $c \cup \{P\} \in \mathcal{C}$, by NC1. So $|c| \geq 2$, that is , NL1 holds in C_P. Let Q and R be any two distinct points. Q and R are on at most one line in C_P since P, Q and R are on at most one circle in C , that is, NL2 holds. So C_P is a near-linear space. □

§3. New Near-circular Spaces from Old

In this section , we consider the construction of a new near-circular space from the one given.

Let $C = (\mathcal{P}, \mathcal{C})$ be a near-circular space. We define a new near-circular space $R = (\mathcal{P}', \mathcal{C}')$ as the following. $\mathcal{P}'$ is arbitrary subset of $\mathcal{P}$ and $\mathcal{C}'$ is the set of intersections $c \cap \mathcal{P}'$ for any c in $\mathcal{C}$ with at least three points in $\mathcal{P}'$. Then , it is easily shown that R is indeed a near-circular space. R is called a restriction of C.

Let $C = (\mathcal{P}, \mathcal{C})$ be a near-circular space. We define the dual space $R = (\mathcal{P}', \mathcal{C}')$ of C as the follows:

$$\mathcal{P}' = \mathcal{C}$$

and

$\mathcal{C}' = \{c' = \{c_1, c_2, ..., c_m\} \mid m \geq 3, c_i \in \mathcal{C}$ and every c_i is incident with two fixed points in $\mathcal{P}\}$.

Proposition 4. *The dual of a near-circular space is a near-circular space.*

Proof: Since any circle in the dual space has at least three points by definition, NC1 is satisfied.

Consider three points of the dual space and let c_1, c_2 and c_3 be the circles of near-circular space C which correspond these three points. Each circle joining c_1, c_2 and c_3 in the dual space corresponds a point of intersection of c_1, c_2 and c_3 in C and , since there are at most two such points of intersection by Proposition 2, there is at most one circle on c_1, c_2 and c_3 in the dual space. □

A subspace of a near-circular space $(\mathcal{P}, \mathcal{L})$ is a set X of points of $\mathcal{P}$ such that whenever P, Q and R are points of X which are on a circle PQR of $\mathcal{C}$, then the entire circle PQR is in X. The empty set, any point, any circle and the whole space itself are always subspaces of a given space.

It is clear that a subspace is a near-circular space.

Proposition 5. *Intersection of any two subspaces of a near-circular space is a subspace.*

Proof: Let C_1 and C_2 be any subspace and $X = C_1 \cap C_2$. We must show that if P, Q and R are points of X and P, Q and R are on a circle PQR, then $PQR \subseteq X$. Since the circle PQR is in both subspaces, PQR is a subset of X. □

§4. The Connection Number

Let $C = (\mathcal{P}, \mathcal{C})$ be a finite near-circular space and P_i be a point not on the circle c_j. We define the connection number $c(P_i, c_j) = c_{ij}$ as the number of circles on P_i which intersect c_j in exactly two points. If $P_i \in c_j$ then it is defined as $c_{ij} = 1$.

Proposition 6. *For any point P_j and circle c_j , $c_{ij} \leq \left(\begin{array}{c} v_j \\ 2 \end{array}\right)$.*

Proof: This is obtained from NC2 if $P_i \notin c_j$. If $P_i \in c_j$, it is obtained from NC1. □

Proposition 7. *If $P_i \notin c_j$, the number of circles on P_i intersecting c_j in one point is at most $b_i - c_{ij}$.*

Proof: The result is obvious since b_i is the total number of circles on P_i and c_{ij} is the number of these circles which intersect c_j in exactly two points. □

We define a circular space as a *near-circular space* in which any three points are on a circle.

It is trivial that in a circular space $P_i \in c_j$ implies $c_{ij} = \left(\begin{array}{c} v_j \\ 2 \end{array}\right)$

Proposition 8. *If C is a near-circular space with $b \geq 1$ and if $c_{il} = \left(\begin{array}{c} v_l \\ 2 \end{array}\right)$ for every P_i and c_l such that $P_i \notin c_l$, then C is a circular space.*

Proof: Since $b \geq 1$, there is a circle c_l. We must show that any three points are on a circle.

Let P_i, P_j and P_k be three points. If P_i, P_j, $P_k \in c_l$ the proof is finished. If $P_i \notin c_l$ and $P_j, P_k \in c_l$ then , by assumption $c_{il} = \left(\begin{array}{c} v_l \\ 2 \end{array}\right)$ so that P_i is joined to any two points of c_l by a circle. In particular, there is a circle through P_i, P_j and P_k.

Let $P_i \notin c_l$, $P_j \notin c_l$ and $P_k \in c_l$. Then there is a circle through P_i, P_k and any point of c_l distinct from P_k by assumption.

Finally, let none of the P_i, P_j and P_k be on c_l. Consider all circles through P_i and intersecting c_l in exactly two points. If both P_j and P_k are on one of these circles, the proof is done. If one of the P_j and P_k says P_j is on one of mentioned circles, then we have a circle through P_i, P_j and P_k by assumption. □

Now we give our main result.

Theorem 9. *Let $C = (\mathcal{P}, C)$ be a finite near-circular space with v points. Then C is a circular space if and only if*

$$\sum_{j=1}^{b} v_j(v_j-1)(v_j-2) \geq v(v-1)(v-2).$$

Proof: Suppose that C is a circular space. We count the number of triples of points in two different ways. First of all, there are $\left(\begin{array}{c} v \\ 3 \end{array}\right) = \frac{1}{6}v(v-1)(v-2)$ triples of points (counting $\{P_i, P_j, P_k\}$ as the same triple as any permutation of $\{P_i, P_j, P_k\}$). Since any triples of points determine a unique circle, total number of triples of points is total number of triples of points on each circle, summed over all circles, namely, $\sum\limits_{j=1}^{b} v_j(v_j-1)(v_j-2)/6$. So $\sum\limits_{j=1}^{b} v_j(v_j-1)(v_j-2) = v(v-1)(v-2)$. Thus we have equality here. Conversely, suppose that $\sum\limits_{j=1}^{b} v_j(v_j-1)(v_j-2) \geq v(v-1)(v-2)$. Let us prove that C is a circular space by induction on v. We may assume that $v \geq 3$, since empty set, a single point set and only set of two points are trivial circular spaces. If $v = 3$, there are two possibilities: $b = 0$ or $b = 1$. However, the inequality above only holds when $b = 1$ and so C is a circular space. If $v = 4$, b may be $0, 1, 2, 3, 4$. Then the inequality above only holds for the cases $b = 1, v_1 = v = 4$ and $b = 3, v_1 = v_2 = v_3 = 3$. Therefore C is a circular space in these cases. Now suppose that if the inequality above holds for a near circular space C' with fewer than v points then C' is a circular space. Consider the near circular space C with v points. We may assume that the inequality $\sum\limits_{j=1}^{b} v_j(v_j-1)(v_j-2) \geq v(v-1)(v-2)$ holds, for $v \geq 5$, in C. Let P be a point in C. We consider the near circular space $R = (\mathcal{P}', C')$ with $\mathcal{P}' = \mathcal{P} \setminus \{P\}$, where C' consists of all circles not through P and of all circles through P with at least four points(except for P). We will first show that R is a circular space as $|\mathcal{P}'| = v - 1 < v$. For this we must show that the inequality above in C holds for R, that is, $\sum\limits_{c'_j} v'_j(v'_j-1)(v'_j-2) \geq v'(v'-1)(v'-2)$, where $c'_j \in C'$ and $v'_j = \left|c'_j\right|$ the number of points of c'_j.

Its right hand side becomes $6\left(\begin{array}{c} v-1 \\ 3 \end{array}\right)$. We have following combinatorial computation in R,

$$\begin{aligned}
\sum_{c'_j}\binom{v'_j}{3} &= \sum_{c_j\notin P}\binom{v_j}{3}+\sum_{\substack{c_j\in P\\|c_j|\geq 4}}\binom{v'_j}{3}\\
&= \sum_{c_j\notin P}\binom{v_j}{3}+\sum_{\substack{c_j\in P\\|c_j|\geq 4}}\binom{v_j-1}{3}\\
&= \sum_{c_j\notin P}\binom{v_j}{3}+\sum_{\substack{c_j\in P\\|c_j|\geq 4}}\binom{v_j-1}{3}-3(\sum_{\substack{c_j\in P\\|c_j|\geq 4}}\binom{v_j-1}{2}).
\end{aligned}$$

In C,

$$\sum_{c_j}\binom{v_j}{3}= \sum_{c_j\notin P}\binom{v_j}{3}+\sum_{\substack{c_j\in P\\|c_j|\geq 4}}\binom{v_j}{3}+\sum_{\substack{c_j\in P\\|c_j|=3}}\binom{v_j}{3}$$

So,

$$\sum_{c_j\notin P}\binom{v_j}{3}= \sum_{c_j}\binom{v_j}{3}-\sum_{\substack{c_j\in P\\|c_j|\geq 4}}\binom{v_j}{3}-\sum_{\substack{c_j\in P\\|c_j|=3}}\binom{v_j}{3}.$$

Therefore,

$$\begin{aligned}
\sum_{c'_j}\binom{v'_j}{3} &= \sum_{c_j}\binom{v_j}{3}-\sum_{\substack{c_j\in P\\|c_j|\geq 4}}\binom{v_j}{3}-\sum_{\substack{c_j\in P\\|c_j|=3}}\binom{v_j}{3}\\
&\quad+\sum_{\substack{c_j\in P\\|c_j|\geq 4}}\binom{v_j}{3}-2\sum_{\substack{c_j\in P\\|c_j|\geq 4}}\binom{v_j-1}{2}\\
&= \sum_{c_j}\binom{v_j}{3}-\sum_{\substack{c_j\in P\\|c_j|=3}}\binom{v_j}{3}-3\sum_{\substack{c_j\in P\\|c_j|\geq 4}}\binom{v_{j-1}}{2}\\
&= \sum_{c_j}\binom{v_j}{3}-3\sum_{c_j\in P}\binom{v_j-1}{2}.
\end{aligned}$$

By the hypothesis,

$$\sum_{c_j} v_j(v_j-1)(v_j-2)\geq v(v-1)(v-2).$$

On the other hand, by counting pairs of points on the circles on P , without the point P , we can write that

$$\sum_{c_j\in P}(v_j-1)(v_j-2)\leq (v-1)(v-2)$$

and then,

$$-3\sum_{c_j\in P}(v_j-1)(v_j-2)\geq -3(v-1)(v-2).$$

Therefore,

$$\begin{aligned}
\sum_{c'_j} v'_j(v'_j-1)(v'_j-2) &\geq v(v-1)(v-2)-3(v-1)(v-2)\\
&\geq (v-1)(v-2)(v-3)
\end{aligned}$$

is the desired inequality. Thus, R is a circular space by our induction hypothesis.

Now, it is sufficient to show that: P and any two points of R determine a circle. Let Q and R be fixed points in R . We know that $C\setminus\{S\}$ is a circular space while S is an other fixed point in R . So the points P, Q and R determine a circle. □

Corollary 10. *C is a circular space if and only if* $\sum_{j=1}^{b}\binom{v_j}{3}=\binom{v}{3}$.

Proof: It is straightforward. □

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