Isomorphisms of cyclic abelian covers of symmetric digraphs II

Iwao Sato

Abstract

Let D be a connected symmetric digraph, Γ a group of automorphisms of D, and A a finite abelian group with some specified property. We give an algebraic characterization for two A-covers of D to be Γ-isomorphic, for any A. We give the number of isomorphism classes of g-cycic F_2^r-covers of a connected bipartite symmetric digraph D with respect to the trivial group I of automorphisms of D, for $g \in F_2^r$, where F_2^r is the r-dimesional vector space over the finite field F_2 with two elements. Furthermore, we enumerate the number of I-isomorphism classes of g-cyclic Z_{2^m}-covers of a connected bipartite symmetric digraph D for the cyclic group Z_{2^m} of order 2^m.

M.S.C. 2000: 05C10, 05C30.
Key words: voltage assignment, covering digraph, enumeration, isomorphism.

§1. Introduction

Graphs and digraphs treated here are finite and simple.

A graph H is called a *covering* of a graph G with projection $\pi : H \longrightarrow G$ if there is a surjection $\pi : V(H) \longrightarrow V(G)$ such that $\pi|_{N(v')} : N(v') \longrightarrow N(v)$ is a bijection for all vertices $v \in V(G)$ and $v' \in \pi^{-1}(v)$. The projection $\pi : H \longrightarrow G$ is an *n-fold covering* of G if π is n-to-one. A covering $\pi : H \longrightarrow G$ is said to be *regular* if there is a subgroup B of the automorphism group $Aut\ H$ of H acting freely on H such that the quotient graph H/B is isomorphic to G.

Let G be a graph and A a finite group. Let $D(G)$ be the arc set of the symmetric digraph corresponding to G. Then a mapping $\alpha : D(G) \longrightarrow A$ is called an *ordinary voltage assignment* if $\alpha(v,u) = \alpha(u,v)^{-1}$ for each $(u,v) \in D(G)$. The (*ordinary*) *derived graph* G^α derived from an ordinary voltage assignment α is defined as follows:

> $V(G^\alpha) = V(G) \times A$, and $((u,h),(v,k)) \in D(G^\alpha)$ if and only if $(u,v) \in D(G)$ and $k = h\alpha(u,v)$.

The graph G^α is called an *A-covering* of G . The A-covering G^α is an $\mid A \mid$-fold regular covering of G. Every regular covering of G is an A-covering of G for some group A (see [3]).

Applied Sciences, Vol.6, 2004, pp. 53-65.

Let D be a symmetric digraph and A a finite group. A function $\alpha : A(D) \longrightarrow A$ is called *alternating* if $\alpha(y,x) = \alpha(x,y)^{-1}$ for each $(x,y) \in A(D)$. For $g \in A$, a *g-cyclic A-cover* $D_g(\alpha)$ of D is the digraph as follows:

$$V(D_g(\alpha)) = V(D) \times A, \text{ and } ((u,h),(v,k)) \in A(D_g(\alpha)) \text{ if and only if } (u,v) \in A(D) \text{ and } k^{-1}h\alpha(u,v) = g.$$

The *natural projection* $\pi : D_g(\alpha) \longrightarrow D$ is a function from $V(D_g(\alpha))$ onto $V(D)$ which erases the second coordinates. A digraph D' is called a *cyclic A-cover* of D if D' is a g-cyclic A-cover of D for some $g \in A$. In the case that A is abelian, then $D_g(\alpha)$ is called simply a *cyclic abelian cover*. Furthermore the 1-cyclic A-cover $D_1(\alpha)$ of a symmetric digraph D can be considered as the A-covering G^α of the underlying graph G of D.

Let α and β be two alternating functions from $A(D)$ into A, and let Γ be a subgroup of the automorphism group $Aut\ D$ of D, denoted $\Gamma \leq Aut\ D$. Let $g, h \in A$. Then two cyclic A-covers $D_g(\alpha)$ and $D_h(\beta)$ are called *Γ-isomorphic*, denoted $D_g(\alpha) \cong_\Gamma D_h(\beta)$, if there exist an isomorphism $\Phi : D_g(\alpha) \longrightarrow D_h(\beta)$ and a $\gamma \in \Gamma$ such that $\pi\Phi = \gamma\pi$, i.e., the diagram

$$\begin{tikzcd}
D_g(\alpha) \arrow[r, "\Phi"] \arrow[d, "\pi"'] & D_h(\beta) \arrow[d, "\pi"] \\
D \arrow[r, "\gamma"] & D
\end{tikzcd}$$

commutes. Let $I = \{1\}$ be the trivial group of automorphisms.

A general theory of graph coverings is developed in [4]. Z_2-coverings (double coverings) of graphs were dealed in [5] and [17]. Hofmeister [7] and, independently, Kwak and Lee [11] enumerated the I-isomorphism classes of n-fold coverings of a graph, for any $n \in N$. Dresbach [2] obtained a formula for the number of strong isomorphism classes of regular coverings of graphs with voltages in finite fields. The I-isomorphism classes of regular coverings of graphs with voltages in finite dimensional vector spaces over finite fields were enumerated by Hofmeister [6]. Hong, Kwak and Lee [8] gave the number of I-isomorphism classes of Z_n-coverings, $Z_p \bigoplus Z_p$-coverings and D_n-coverings, n:odd, of graphs, respectively. Sato [16] counted the Γ-isomorphism classes of Z_p-coverings of graphs for any prime $p(>2)$.

Cheng and Wells [1] discussed isomorphism classes of cyclic triple covers (1-cyclic Z_3-covers) of a complete symmetric digraph. Furthermore, Mizuno and Sato [13] gave a formula for the characteristic polynomial of a cyclic A-cover of a symmetric digraph, for any finite group A. Mizuno and Sato [15] discussed the number of Γ-isomorphism classes of cyclic V-covers of a connected symmetric digraph for any finite dimensional vector space V over the finite field $GF(p)(p>2)$. For a connected symmetric digraph D, Mizuno and Sato [14] obtained a sufficient condition for two Γ-isomorphism classes of cyclic abelian covers of D to be of the same cardinality, and presented the number of I-isomorphism classes of g-cyclic Z_{p^m}-covers of D for any prime $p(>2)$. For a connected cyclic A-covers, Mizuno, Lee and Sato [12] enumerated the number of I-isomorphism classes of connected g-cyclic A-covers of D, when A is the cyclic group Z_{p^m} and the direct sum of m copies of Z_p for any prime $p(>2)$.

In Section 2, we give a necessary and sufficient condition for two cyclic A-covers of a connected symmetric digraph to be Γ-isomorphic for any finite abelian group A. As a corollary, we present the number of I-isomorphism classes of g-cyclic Z_2^r-covers of a connected bipartite symmetric digraph D. In Section 3, we treat the enumeration and the structure of Γ-isomorphism classes of cyclic F_2^r-covers of D. In Section 4, we count the number of I-isomorphism classes of g-cyclic Z_{2^m}-covers of D.

§2. Isomorphisms of cyclic abelian covers

Let D be a symmetric digraph and A a finite group. The group Γ of automorphisms of D acts on the set $C(D)$ of alternating functions from $A(D)$ into A as follows:

$$\alpha^{\gamma}(x,y) = \alpha(\gamma(x), \gamma(y)) \ for\ all\ (x,y) \in A(D),$$

where $\alpha \in C(D)$ and $\gamma \in \Gamma$. Any voltage $g \in A$ determines a permutation $\rho(g)$ of the symmetric group S_A on A which is given by $\rho(g)(h) = hg,\ h \in A$.

Mizuno and Sato [15] gave a characterization for two cyclic A-covers of D to be Γ-isomorphic.

Theorem 1 *[15, Theorem 3.1] Let D be a symmetric digraph, A a finite group, $g, h \in A$, $\alpha, \beta \in C(D)$ and $\Gamma \leq Aut\ D$.*

1. *$D_g(\alpha) \cong {}_{\Gamma} D_h(\beta)$.*
2. *There exist a family $(\pi_u)_{u \in V(D)} \in S_A^{V(D)}$ and $\gamma \in \Gamma$ such that*

$$\rho(\beta^{\gamma}(u,v)h^{-1}) = \pi_v \rho(\alpha(u,v)g^{-1})\pi_u^{-1} \ for\ each\ (u,v) \in A(D),$$

where the multiplication of permutations is carried out from right to left.

From now on, assume that D is connected and A is abelian. Let G be the underlying graph, T be a spanning tree of G and w a root of T. For any $\alpha \in C(D)$ and any walk W in G, the *net α-voltage* of W, denoted $\alpha(W)$, is the sum of the voltages of the edges of W. Then the *T-voltage* α_T of α is defined as follows:

$$\alpha_T(u,v) = \alpha(P_u) + \alpha(u,v) - \alpha(P_v) \ for\ each\ (u,v) \in D(G) = A(D),$$

where P_u and P_v denote the unique walk from w to u and v in T, respectively. For a function $f : A(D) \longrightarrow A$, the *net f-value* $f(W)$ of any walk W is defined as the net α-voltage of W.

Corollary 1 *Let D be a connected symmetric digraph, G its underlying graph, T be a spanning tree of G and $\alpha \in C(D)$. Furthermore, let A be a finite abelian group and $g \in A$. Then*

$$D_g(\alpha) \cong {}_I D_g(\alpha_T).$$

Moreover, there exists a function $s : V(D) \longrightarrow A$ such that

$$\alpha_T(u,v) = s(v) + \alpha(u,v) - s(u) \ for\ each\ (u,v) \in D(G) = A(D),$$

Proof. Let $s(v) = \rho(-\alpha(P_v))$ for $v \in V(D)$. Then, by Theorem 1, the result follows.

□

For a function $f : C(D) \longrightarrow A$, let $A_f(v)$ denote the subgroup of A generated by all net f-values of the closed walk based at $v \in V(D)$. Let $ord(g)$ be the order of $g \in A$. For a subset B of A, let $< B >$ denote the subgroup of A generated by B.

Theorem 2 *Let D be a connected symmetric digraph, A a finite abelian group, $g, h \in A$ and $\alpha, \beta \in C(D)$. Furthermore, let G be the underlying graph of D, T a spanning tree of G and $\Gamma \leq Aut\ G$. Then the following are equivalent:*

1. $D_g(\alpha) \cong {}_\Gamma D_h(\beta)$.
2. *There exist $\gamma \in \Gamma$ and an isomorphism*

$$\sigma :< A_{\alpha_T - \epsilon g}(w) \cup \{2g\} > \longrightarrow < A_{\beta_{\gamma T} - \epsilon h}(\gamma(w)) \cup \{2h\} >$$

such that

$$\beta^{\gamma}_{\gamma T}(u,v) - \epsilon^{\gamma} h = \sigma(\alpha_T(u,v) - \epsilon g) \ for\ each\ (u,v) \in A(D)$$

and

$$\sigma(2g) = 2h,$$

where $(\alpha_T - g)(u,v) = \alpha_T(u,v) - g,\ (u,v) \in A(D)$, $w \in V(D)$ and

$$\epsilon = \begin{cases} 1 & if\ d_T(u,v)\ is\ even, \\ 0 & otherwise, \end{cases}$$

Proof. At first, suppose that $D_g(\alpha) \cong {}_\Gamma D_h(\beta)$. By Corollary 1, we have $D_g(\alpha_T) \cong {}_\Gamma D_g(\beta_T)$. By Theorem 1, there exist a family $(\pi_u)_{u \in V(D)} \in S_A^{V(D)}$ and $\gamma \in \Gamma$ such that

$$\rho(\beta^{\gamma}_{\gamma T}(u,v) - h) = \pi_v \rho(\alpha_T(u,v) - g)\pi_u^{-1} \ for\ each\ (u,v) \in A(D).$$

Let $(u,v) \in D(T)$. Then we have $\beta^{\gamma}_{\gamma T}(u,v) = \alpha_T(u,v) = 0$. Thus $\rho(-h) = \pi_v \rho(-g)\pi_u^{-1}$. Since $(v,u) \in D(T)$, we have $\rho(-h) = \pi_u \rho(-g)\pi_v^{-1}$. Therefore it follows that

$$\pi_v = \rho(h)\pi_u \rho(-g) = \rho(-h)\pi_u \rho(g).$$

Note that $\rho(2h) = \pi_u \rho(2g)\pi_u^{-1}$.

Let $P : u, v, w$ be any path of length two in D. Then we have

$$\rho(\beta^{\gamma}_{\gamma T}(P) - 2h) = \pi_w \rho(\alpha_T(P) - 2g)\pi_u^{-1}.$$

If $(u,v),(v,w) \in D(T)$, then we have $\beta^{\gamma}_{\gamma T}(P) = \alpha_T(P) = 0$. Thus $\rho(-2h) = \pi_w \rho(-2g)\pi_u^{-1}$. Since $Q : w, v, u$ is a path of length two in T, we have $\rho(-2h) = \pi_u \rho(-2g)\pi_w^{-1}$. Therefore it follows that

$$\pi_w = \rho(2h)\pi_u \rho(-2g) \ and\ \pi_w = \rho(-2h)\pi_u \rho(2g),$$

i.e.,

$$\pi_w(k+2g) = \pi_u(k) + 2h \; and \; \pi_u(k+2g) = \pi_w(k) + 2h \; for k \in A.$$

Let $ord(2g) = t$. Then $\pi_u(k) = \pi_u(k+2tg) = \pi_u(k) + 2th$, i.e., $ord(2h) \mid ord(2g)$. Since the converse is clear, we have $ord(2g) = ord(2h)$. Thus we have

$$\pi_w(k) = \pi_w(k+2tg) = \pi_u(k) + 2th = \pi_u(k),$$

i.e.,

$$\pi_u = \pi_w.$$

Since D is connected, for any $w \in V(D)$, we have

$$\pi_w = \begin{cases} \pi_u & \text{if } d_T(u,w) \text{ is even,} \\ \rho(-h)\pi_u\rho(g) & \text{otherwise,} \end{cases}$$

where $u \in V(D)$ and $d_T(u,w)$ is the distance between u and w in T.

Let $(v,w) \in A(D) \setminus D(T)$. If $d_T(v,w)$ is even, then we have $\pi_v = \pi_w$, and so

$$\rho(\beta^{\gamma}_{\gamma T}(v,w) - h) = \pi_v \rho(\alpha_T(v,w) - g)\pi_v^{-1}.$$

Since $\pi_v = \pi_u, \rho(-h)\pi_u\rho(g)$, we have

$$\rho(\beta^{\gamma}_{\gamma T}(v,w) - h) = \pi_u \rho(\alpha_T(v,w) - g)\pi_u^{-1}.$$

In the case that $d_T(v,w)$ is odd, we have $\pi_w = \rho(-h)\pi_v\rho(g)$, and so

$$\rho(\beta^{\gamma}_{\gamma T}(v,w) - h) = \rho(-h)\pi_v\rho(g)\rho(\alpha_T(v,w) - g)\pi_v^{-1}.$$

i.e.,

$$\rho(\beta^{\gamma}_{\gamma T}(v,w)) = \pi_v \rho(\alpha_T(v,w))\pi_v^{-1}.$$

Since $\pi_v = \pi_u, \rho(-h)\pi_u\rho(g)$, we have

$$\rho(\beta^{\gamma}_{\gamma T}(v,w)) = \pi_u \rho(\alpha_T(v,w))\pi_u^{-1}.$$

Therefore it follows that $\rho(\beta^{\gamma}_{\gamma T}(u,v) - \epsilon^{\gamma} h) = \pi_u \rho(\alpha_T(u,v) - \epsilon g)\pi_u^{-1}$ for each $(u,v) \in A(D)$, where

$$\epsilon = \begin{cases} 1 & \text{if } d_T(u,v) \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\epsilon = 0$ if $(v,w) \in D(T)$. Hence there exists an isomorphism $\sigma :< A_{\alpha_T - \epsilon g}(w) \cup \{2g\} > \longrightarrow < A_{\beta_{\gamma T} - \epsilon h}(\gamma(w)) \cup \{2h\} >$ such that

$$\beta^{\gamma}_{\gamma T}(u,v) - \epsilon^{\gamma} h = \sigma(\alpha_T(x,y) - \epsilon g) \; for \; each \; (u,v) \in A(D).$$

Since $\rho(2h) = \pi_u \rho(2g)\pi_u^{-1}$, we have $\sigma(2g) = 2h$.

Conversely, assume that there exist $\gamma \in \Gamma$ and a group isomorphism $\sigma :< A_{\alpha_T - \epsilon g}(w) \cup \{2g\} > \longrightarrow < A_{\beta_{\gamma T} - \epsilon h}(\gamma(w)) \cup \{2h\} >$ such that

$$\beta^{\gamma}_{\gamma T}(u,v) - \epsilon^{\gamma} h = \sigma(\alpha_T(u,v) - \epsilon g) \; for \; each \; (u,v) \in A(D)$$

and

$$\sigma(2g) = 2h.$$

Set $X =< A_{\alpha_T - \epsilon g}(w) \cup \{2g\} >$ and $Y =< A_{\beta_{\gamma T} - \epsilon h}(\gamma(w)) \cup \{2h\} >$. Let $\{a_1 = 1, \cdots, a_m\}$ and $\{b_1 = 1, \cdots, b_m\}$ be the representatives of A/X and A/Y, respectively. For any $c \in A$, there exist $c_\alpha \in A_{\alpha_T - \epsilon g}$ and $i(c) \in \{1, ..., m\}$ such that

$$c = c_\alpha + a_{i(c)}.$$

For a distiguished vertex $u \in V(D)$, we define a mapping $\pi_u : A \longrightarrow A$ by

$$\pi_u(c) = \sigma(c_\alpha) + b_{i(c)} \ for \ each \ c \in A.$$

Then π_u is well-defined, and π_u is bijective.

Since $\pi_u \mid_X = \sigma$, for any $g' \in X$, we have

$$\begin{aligned} g' + c &= (g' + c)_\alpha + a_{i(g'+c)} \\ &= g' + c_\alpha + a_{i(c)}. \end{aligned}$$

Futhermore, we have

$$\begin{aligned} \pi_u(g' + c) &= \pi_u(g' + c_\alpha + a_{i(c)}) \\ &= \sigma(g' + c_\alpha) + b_{i(c)} \\ &= \sigma(g') + \sigma(c_\alpha) + b_{i(c)} \\ &= \sigma(g') + \pi_u(c) \end{aligned}$$

for each $c \in A$. Since $\sigma(2g) = 2h$,

$$\rho(h)\pi_u\rho(-g) = \rho(-h)\pi_u\rho(g).$$

Now, let

$$\pi_w = \begin{cases} \pi_u & \text{if } d_T(u, w) \text{ is even,} \\ \rho(-h)\pi_u\rho(g) & \text{otherwise,} \end{cases}$$

Let $(v, w) \in D(T)$. Then we have $\beta^\gamma_{\gamma T}(v, w) = \alpha_T(v, w) = 0$. If $\pi_v = \pi_u, \pi_w = \rho(-h)\pi_u\rho(g)$, then we have

$$\begin{aligned} \pi_w\rho(\alpha_T(v, w) - g)\pi_v^{-1} &= \rho(-h)\pi_u\rho(g)\rho(\alpha_T(v, w) - g)\pi_u^{-1} \\ &= \rho(-h)\rho(\beta^\gamma_{\gamma T}(v, w)) \\ &= \rho(\beta^\gamma_{\gamma T}(v, w) - h). \end{aligned}$$

In the case that $\pi_w = \pi_u, \pi_v = \rho(-h)\pi_u\rho(g)$, the same formula also holds.

Let $(v, w) \in A(D) \setminus D(T)$. If $d_T(v, w)$ is even, then we have $\pi_v = \pi_w$. If $\pi_v = \pi_w = \pi_u$, then we have

$$\begin{aligned} \pi_w\rho(\alpha_T(v, w) - g)\pi_v^{-1} &= \pi_u\rho(\alpha_T(v, w) - g)\pi_u^{-1} \\ &= \rho(\beta^\gamma_{\gamma T}(v, w) - h). \end{aligned}$$

In the case that $\pi_v = \pi_w = \rho(-h)\pi_u\rho(g)$, the same formula also holds.

If $d_T(v,w)$ is odd, then we have

$$\pi_w \rho(\alpha_T(v,w) - g)\pi_v^{-1} = \rho(\beta^{\gamma}_{\gamma T}(v,w) - h).$$

Therefore it follows that

$$\rho(\beta^{\gamma}_{\gamma T}(v,w) - h) = \pi_w \rho(\alpha_T(v,w) - g)\pi_v^{-1} \ for\ each\ (v,w) \in A(D).$$

Hence $D_g(\alpha_T) \cong_\Gamma D_g(\beta_T)$, which completes the proof. $\square$

Corollary 2 ([14], Theorem 2) *Let D be a connected symmetric digraph, A a finite abelian group, $g,h \in A$ and $\alpha,\beta \in C(D)$. Furthermore, let G be the underlying graph of D, T a spanning tree of G and $\Gamma \leq Aut\ G$. Assume that both $ord(g)$ and $ord(h)$ are odd. Then the following are equivalent:*

1. *$D_g(\alpha) \cong_\Gamma D_h(\beta)$.*
2. *There exist $\gamma \in \Gamma$ and an isomorphism $\sigma : A_{\alpha_T - g}(w) \longrightarrow A_{\beta_{\gamma T} - h}(\gamma(w))$ such that*

$$\beta^{\gamma}_{\gamma T}(u,v) = \sigma(\alpha_T(u,v)) \ for\ each\ (u,v) \in A(D)$$

and

$$\sigma(g) = h.$$

Corollary 3 *Let D be a connected symmetric digraph, A a finite abelian group, $g,h \in A$ and $\alpha,\beta \in C(D)$. Furthermore, let G be the underlying graph of D, T a spanning tree of G and $\Gamma \leq Aut\ G$. If G is bipartite, then the following are equivalent:*

1. *$D_g(\alpha) \cong_\Gamma D_h(\beta)$.*
2. *There exist $\gamma \in \Gamma$ and an isomorphism $\sigma : < A_{\alpha_T}(w) \cup \{2g\} > \longrightarrow < A_{\beta_{\gamma T}}(\gamma(w)) \cup \{2h\} >$ such that*

$$\beta^{\gamma}_{\gamma T}(u,v) = \sigma(\alpha_T(u,v)) \ for\ each\ (u,v) \in A(D)$$

and

$$\sigma(2g) = 2h.$$

Let D be a connected symmetric digraph, G its underlying graph and A a finite abelian group. The set of ordinary voltage assignments of G with voltages in A is denoted by $C^1(G;A)$. Note that $C(D) = C^1(G;A)$. Furthremore, let $C^0(G;A)$ be the set of functions from $V(G)$ into A. We consider $C^0(G;A)$ and $C^1(G;A)$ as additive groups. The homomorphism $\delta : C^0(G;A) \longrightarrow C^1(G;A)$ is defined by $(\delta s)(x,y) = s(x) - s(y)$ for $s \in C^0(G;A)$ and $(x,y) \in A(D)$. For each $\alpha \in C^1(G;A)$, let $[\alpha]$ be the element of $C^1(G;A)/Im\delta$ which contains α.

The automorphism group $Aut\ A$ acts on $C^0(G;A)$ and $C^1(G;A)$ as follows:

$$(\sigma s)(x) = \sigma(s(x)) \ for\ x \in V(D),$$

$$(\sigma\alpha)(x,y) = \sigma(\alpha(x,y)) \ for\ (x,y) \in A(D),$$

where $s \in C^0(G;A)$, $\alpha \in C^1(G;A)$ and $\sigma \in Aut\ A$. An finite group $\mathcal{B}$ is said to have the *isomorphism extension property(IEP)*, if every isomorphism between any two isomorphic subgroups $\mathcal{E}_1$ and $\mathcal{E}_2$ of $\mathcal{B}$ can be extented to an automorphism of $\mathcal{B}$ (see [8]). For example, the cyclic group Z_n for any $n \in N$, the dihedral group D_n for odd $n \geq 3$, and the direct sum of m copies of Z_p have the IEP.

Corollary 4 *Let D be a connected symmetric digraph, G its underlying graph, A a finite abelian group, $\alpha, \beta \in C(D)$ and $g, h \in A$. Suppose that A has the IEP. If G is bipartite, then $D_g(\alpha) \cong {}_\Gamma D_h(\beta)$ if and only if $\beta = \sigma\alpha^\gamma + \delta s$ and $\sigma(2g) = 2h$ for some $\sigma \in Aut\ A$, some $\gamma \in \Gamma$ and some $s \in C^0(G; A)$.*

Corollary 5 *Let D be a connected symmetric digraph, G its underlying graph, A a finite abelian group, $\alpha, \beta \in C(D)$ and $g \in A$. Suppose that A has the IEP. If G is bipartite, then $D_g(\alpha) \cong {}_\Gamma D_g(\beta)$ if and only if $\beta = \sigma\alpha^\gamma + \delta s$ and $\sigma(2g) = 2g$ for some $\sigma \in Aut\ A$, some $\gamma \in \Gamma$ and some $s \in C^0(G; A)$.*

Corollary 6 *Let D, G, A and Γ be as in Corollary 5. If $ord(2g) = 1$, then the number of Γ-isomorphism classes of g-cyclic A-covers of D is equal to that of Γ-isomorphism classes of A-coverings of G.*

Proof. Since $ord(2g) = 1$, we have $(Aut\ A)_{2g} = Aut\ A$. The rest is by Theorem 4 in [8]. □

Now we consider the number of Γ-isomorphism classes of cyclic A-covers of a connected bipartite symmetric digraph D. Let G be the underlying graph of D, A a finite abelian group with the IEP and $\Pi = Aut\ A$. For any $k \in A$, set

$$\Pi_k = \{\sigma \in \Pi \mid \sigma(k) = k\}.$$

Then Π_k is a subgroup of Π.

Let $\Gamma \leq Aut\ D$ and $g \in A$. Set $H^1(G; A) = C^1(G; A)/Im\delta$. A action of $\Pi_{2g} \times \Gamma$ on $H^1(G; A)$ are defined as follows:

$$(\sigma, \gamma)[\alpha] = [\sigma\alpha^\gamma] = \{\sigma\alpha^\gamma + \delta s \mid s \in C^0(G; A)\},$$

where $\sigma \in \Pi_2 g$, $\gamma \in \Gamma$ and $\alpha \in C^1(G; A)$. By Corollary 5, the number of Γ-isomorphism classes of g-cyclic A-covers of D is equal to that of $\Pi_{2g} \times \Gamma$-orbits on $H^1(G; A)$. Let $Iso(D, A, g, \Gamma)$ be the number of Γ-isomorphism classes of g-cyclic A-covers of D.

Theorem 3 *Let D be a connected bipartite symmetric digraph, G its underlying graph, A a finite abelian group with the IEP, $g, h \in A$ and $\Gamma \leq Aut\ D$. Assume that $\kappa(2g) = 2h$ for some $\kappa \in Aut\ A$. Then*

$$Iso(D, A, g, \Gamma) = Iso(D, A, h, \Gamma).$$

Proof. Similar to the proof of Theorem 3 in [14]. □

Let D be a connected symmetric digraph, p prime and $F_p = GF(p)$ the finite field with p elements. Let F_p^r be the r-dimensional vector space over F_p. Then the additive group F_p^r has the IEP and the general linear group $GL_r(F_p)$ is the automorphism group of F_p^r. Furthermore, $GL_r(F_p)$ acts transitively on $F_p^r \backslash \{0\}$.

Corollary 7 *Let D be a connected bipartite symmetric digraph, G its underlying graph and $\Gamma \leq Aut\ D$. Let g, h be any two elements of $F_2^r \backslash \{0\}$. Then*

$$Iso(D, F_2^r, g, \Gamma) = Iso(D, F_2^r, h, \Gamma).$$

Specially, $isc(D, F_2^r, g, \Gamma)$ is equal to that of Γ-isomorphism classes of F_2^r-coverings of G.

Proof. Note that $2g = 2h = 0$. By Theorem 3 and Corollary 6. □

In the case of $p > 2$, the similar result to Corollary 7 is obtained by [14] and [15].

For a connected symmetric digraph D, let $B(D) = m - n + 1$ be the *Betti-number* of D, where $m = \mid A(D) \mid /2$ and $n = \mid V(D) \mid$. We give the enumeration of I-isomorphism classes of g-cyclic F_2^r-covers of D for any $g \in F_2^r$.

Corollary 8 *Let D be a connected bipartite symmetric digraph and $g \in F_2^r$. Then the number of I-isomorphism classes of g-cyclic F_2^r-covers of D is*

$$Iso(D, F_2^r, g, I) = 1 + \sum_{h=1}^{r} \frac{(2^B - 1)(2^{B-1} - 1)\cdots(2^{B-h+1} - 1)}{(2^h - 1)(2^{h-1} - 1)\cdots(2 - 1)},$$

where $B = B(D)$.

Proof. By Corollary 2 of [10]. □

§3. Isomorphisms of cyclic F_2^r-covers

Let D be a connected symmetric digraph and $\Gamma \leq Aut\ D$. Let $\mathcal{D}_g$ be the set of all g-cyclic F_2^r-covers of D for each $g \in F_2^r$, and let $\mathcal{D} = \bigcup_{g \in F_2^r} \mathcal{D}_g$. Then $\mathcal{D}$ is the set of all cyclic F_2^r-covers of D. Let $\mathcal{D}/\cong_\Gamma$ and $\mathcal{D}_g/\cong_\Gamma$ be the set of all Γ-isomorphism classes over $\mathcal{D}$ and $\mathcal{D}_g$, respectively. Furthermore, let $Iso(D, F_2^r, \Gamma)$ be the number of Γ-isomorphism classes of cyclic F_2^r-covers of D. The Γ-isomorphism class of $\mathcal{D}_g$ containing $D_g(\alpha)$ is denoted by $[D_g(\alpha)]$.

Theorem 4 *Let D be a connected bipartite symmetric digraph and $\Gamma \leq Aut\ D$. Then*

$$Iso(D, F_2^r, \Gamma) = Iso(D, F_2^r, g, \Gamma) \ for\ each\ g \in F_2^r.$$

Proof. Let $0 = (00\cdots 0)^t \in F_2^r$ and $\Pi = GL_r(F_2)$. For any $g \neq 0$ and any $\alpha \in C(D)$, let

$$\beta = A^{-1}(\alpha^\gamma - \delta s),\ A \in \Pi,\ \gamma \in \Gamma,\ s \in C^0(G; F_2^r),$$

where G is the underlying graph of D. By Corollary 4, we have $D_g(\alpha) \cong_\Gamma D_0(\beta)$.

For each $g \neq 0 \in F_2^r$, we define a map $\Phi_g : \mathcal{D}_0/\cong_\Gamma \longrightarrow \mathcal{D}_g/\cong_\Gamma$ by

$$\Phi_g([D_0(\rho)]) = [D_g(\beta)],$$

where $D_0(\rho) \cong_\Gamma D_g(\beta)$. Since $\cong_\Gamma$ is an equivalence relation over $\mathcal{D}$, Φ_g is injective. By Corollary 7, we have

$$\mid \mathcal{D}_0/\cong_\Gamma \mid = \mid \mathcal{D}_g/\cong_\Gamma \mid < \infty.$$

Thus Φ_g is a bijection. Therefore, it follows that $Iso(D, F_2^r, \Gamma) = Iso(D, F_2^r, g, \Gamma)$. □

Corollary 9 *Let D be a connected bipartite symmetric digraph. Then the number of I-isomorphism classes of cyclic F_2^r-covers of D is*

$$Iso(D, F_2^r, I) = 1 + \sum_{h=1}^{m} \frac{(2^B - 1)(2^{B-1} - 1)\cdots(2^{B-h+1} - 1)}{(2^h - 1)(2^{h-1} - 1)\cdots(2 - 1)},$$

Now, we state the structure of Γ-isomorphism classes of cyclic F_2^r-covers of D.

Theorem 5 *Let D be a connected bipartite symmetric digraph, G its underlying graph, $\Gamma \leq Aut\ D$ and $\Pi = GL_r(F_2)$. Then any Γ-isomorphism class of cyclic F_2^r-covers of D is of the form*

$$\bigcup_{g \in F_2^r} \{D_g(\beta) \mid \beta = A\alpha^{\gamma} + \delta s,\ A \in \Pi,\ \gamma \in \Gamma,\ s \in C^0(G; F_2^r)\},$$

where $\alpha \in C(D)$.

Proof. Let $\rho \in C(D)$, $h \neq 0 \in F_2^r$ and $[[D_h(\rho)]]$ the Γ-isomorphism class of $\mathcal{D}$ containing $D_h(\rho)$. By the first half of the proof of Theorem 4, there exists a 0-cyclic F_2^r-cover $D_0(\alpha)$ such that $D_h(\rho) \cong_{\Gamma} D_0(\beta)$. Thus it follows that $[[D_h(\rho)]] = [[D_0(\beta)]]$.

In the proof of Theorem 4, the map Φ_g is a bijection from $\mathcal{D}_0 / \cong_{\Gamma}$ into $\mathcal{D}_g / \cong_{\Gamma}$ for any $g \neq 0 \in F_2^r$. Thus there exists a g-cyclic F_2^r-cover $D_g(\beta)$ such that $D_0(\alpha) \cong_{\Gamma} D_g(\beta)$ for any $g \neq 0 \in F_2^r$. We define a map $\Psi_g : [D_0(\alpha)] \longrightarrow [D_g(\beta)]$ by

$$\Psi_g(D_0(\alpha_1)) = D_g(\beta_1),\ \beta_1 = A\alpha_1^{\gamma} + \delta s,$$

where $A \in \Pi,\ \gamma \in \Gamma,\ s \in C^0(G; F_2^r)$ are fixed. By Corollary 4, Ψ_g is well-defined. It is clear that Ψ_g is injective.

Now, let $D_g(\tau)$ be any element of $[D_g(\beta)]$. Then we have $D_g(\tau) \cong_{\Gamma} D_0(\alpha)$. By Corollary 4, there exist $B \in \Pi,\ \theta \in \Gamma$ and $t \in C^0(G; F_2^r)$ such that $\tau = B\alpha^{\theta} + \delta t$. Let

$$\eta = A^{-1}B\alpha^{\theta\gamma^{-1}} + \delta A^{-1}(t^{\gamma^{-1}} - s^{\gamma^{-1}}).$$

Then we have $\tau = A\eta^{\gamma} + \delta s$, i.e.,

$$\Psi_g(D_0(\eta)) = D_g(\tau).$$

Therefore Ψ_g is surjective, i.e., bijective.

But, by Corollary 5, we have

$$\alpha_1 = B_1\alpha^{\kappa} + \delta s_1,\ B_1 \in \Pi,\ \kappa \in \Gamma,\ s_1 \in C^0(G; F_2^r),$$

and so

$$\beta_1 = B\alpha^{\kappa\gamma} + \delta(As_1^{\kappa} + s).$$

Hence it follows that

$$[D_g(\beta)] = \{D_g(\beta_1) \mid \beta_1 = B'\alpha^{\lambda} + \delta s',\ B' \in \Pi,\ \lambda \in \Gamma,\ s' \in C^0(G; F_2^r)\}.$$

By Theorem 4, the result follows. □

In the case of $p > 2$, the results corresponding to Theorems 4 and 5 were given by [14] and [15].

§4. Isomorphisms of cyclic Z_{2^m}-covers

Let Z_n be the cyclic group of order n. Then Z_n have the IEP.

Let D be a connected symmetric digraph and G its underlying graph. Let T be a spanning tree of G and w a base vertex in G. Set $C_T(D) = C_T^1(G; Z_n) = \{\alpha_T \mid \alpha \in C(D) = C^1(G; Z_n)\}$.

Lemma 1 *Let D be a connected symmetric digraph, G the underlying graph of D, T a spanning tree of G. Furthermore, let $\sigma \in Aut\ Z_n$, α, $\beta \in C(D)$ and $g \in Z_n$. Then the following are equivalent:*

1. *$\beta = \sigma\alpha + \delta s$ and $\sigma(2g) = 2g$ for some $s \in C^0(G; Z_n)$.*
2. *$\beta_T = \sigma\alpha_T$ and $\sigma(2g) = 2g$.*

We shall consider the number of I-isomorphism classes of g-cyclic Z_{2^m}-covers of D, for any $g \in Z_{2^m}$. Set $\Pi_{2g} = \{\sigma \in Aut\ Z_{2^m} \mid \sigma(2g) = 2g\}$. By Corollary 5 and Lemma 1, the number of I-isomorphism classes of g-cyclic Z_{2^m}-covers of D is equal to that of Π_{2g}-orbits on $C^1_T(G; Z_{2^m})$.

Theorem 6 *Let D be a connected bipartite symmetric digraph and $n = 2^m$. Let $g \in Z_{2^m}$ and $ord(2g) = 2^{m-\mu}$. Set $B = B(D)$. Then the number of I-isomorphism classes of g-cyclic Z_{2^m}-covers of D is*

$$Iso(D, Z_{2^m}, g, I) = \begin{cases} 2^{mB-\mu} + 2^{(m-\mu)B-1}(2^{\mu(B-1)} - 1)/(2^{B-1} - 1) & \textit{if } \mu \neq m \textit{ and } B > 1, \\ 2^{m-\mu-1}(\mu + 2) & \textit{if } \mu \neq m \textit{ and } B = 1, \\ 2^{m(B-1)+1} - 1 + (2^{m(B-1)} - 1)/(2^{B-1} - 1) & \textit{if } \mu = m \textit{ and } B > 1, \\ m + 1 & \textit{if } \mu = m \textit{ and } B = 1, \end{cases}$$

Proof. By the above note and Burnside's Lemma, we have

$$Iso(D, Z_{2^m}, g, I) = \frac{1}{\mid \Pi_{2g} \mid} \sum_{\rho \in \Pi_{2g}} \mid C_T(D)^\rho \mid .$$

Let $F(\rho) = \{h \in Z_{2^m} \mid \rho(h) = h\}$. Then, by Corollary 3 of [8], we have $\mid C_T(D)^\rho \mid = \mid F(\rho) \mid^{B(D)}$.

But we have

$$\Pi_{2g} = \{\lambda \in Z_{2^m} \mid (\lambda, 2^m) = 1 \textit{ and } \lambda 2g = 2g\}.$$

Then

$$\lambda \in \Pi_{2g} \Leftrightarrow 2\lambda g \equiv 2g\ (mod\ 2^m) \Leftrightarrow 2g(\lambda - 1) \equiv 0\ (mod\ 2^m) \Leftrightarrow \lambda - 1 \in < ord(2g) > .$$

Thus we have $\mid \Pi_{2g} \mid = n/ord(2g)$. That is, $\mid \Pi_{2g} \mid = 2^\mu$ if $2g \in K_\mu(m)$, where $K_\mu(m) = \{k \in Z_{2^m} \mid k \in < 2^\mu >,\ k \notin < 2^{\mu+1} >\}$. Let $ord(2g) = 2^{m-\mu}$. If $ord(2g) = 1$, then $2g = 2^m$. Otherwise $\Pi_{2g} = \{2^{m-\mu}\nu + 1 \mid \nu = 0, 1, \cdots, 2^{\mu-1}\}$.

By Lemma 3 of [8], $\mid F(\rho) \mid = 2^\mu$ if $\rho - 1 \in K_\mu(m)$. Thus we have

$$\begin{aligned} \mid \{\lambda \in \Pi_{2g} \mid \mid F(\lambda) \mid = 2^{m-\mu+t}\} \mid &= 2^{\mu-t-1}\ (0 \leq t \leq \mu - 1), \\ \mid \{\lambda \in \Pi_{2g} \mid \mid F(\lambda) \mid = 2^m\} \mid &= 1. \end{aligned}$$

Furthermore, we have

$$\mid \{\lambda \in \Pi_{2g} \mid \mid F(\lambda) \mid = 1\} \mid = 0 \quad if \mu = m.$$

Therefore the result follows. Specially, the third and fourth parts of the formula are given by Theorem 8 of [8]. □

In Table 1, we give some values of $Iso(D, Z_{2^6}, g, I)$.

$\mu \setminus B$	1	2	3	4	5	6
1	48	2560	147456	8912896	553648128	34896609280
2	32	1408	75776	4489216	277348352	17456693248
3	20	736	38144	2246656	138690560	8728477696
4	12	376	19104	1123456	69345792	4364240896
5	7	190	9556	561736	34672912	2182120480
6	7	190	9556	561736	34672912	2128120480

Table 1

Acknowledgment. The present work was supported by Grant-in-Aid for Science Research (C).

References

[1] Y. Cheng and A. L. Wells, Jr., *Switching classes of directed graphs*, J. Combin. Theory Ser. B, 40 (1986), 169-186.

[2] K. Dresbach, *Über die strenge Isomorphie von Graphenüberlagerungen*, Diplomarbeit, Univ. of Cologne, 1989.

[3] J. L. Gross and T. W. Tucker, *Generating all graph coverings by permutation voltage assignments*, Discrete Math., 18 (1977), 273-283.

[4] J. L. Gross and T. W. Tucker, *Topological Graph Theory*, Wiley-Interscience, New York, 1987.

[5] M. Hofmeister, *Counting double covers of graphs*, J. Graph Theory, 12 (1988), 437-444.

[6] M. Hofmeister,*Graph covering projections arising from finite vector spaces over finite fields*, submitted.

[7] M. Hofmeister, *Isomorphisms and automorphisms of graph coverings*, Discrete Math., 98 (1991), 175-185.

[8] S. Hong, J. H. Kwak and J. Lee, *Regular graph coverings whose covering transformation groups have the isomorphism extension property*, Discrete Math., 148 (1996), 85-105.

[9] A. Kerber, *Algebraic Combinatorics via Finite Group Actions*, BI-Wiss. Verlag, Mannheim, Wien, Zürrich, 1991.

[10] J. H. Kwak, J. Chun and J. Lee, *Enumeration of regular graph coverings having finite abelian covering transformation groups*, SIAM. J. Disc. Math., 11 (1998), 273-285.

[11] J. H. Kwak and J. Lee, *Isomorphism classes of graph bundles*, Canad. J. Math., XLII (1990), 747-761.

[12] H. Mizuno, J. Lee and I. Sato, *Isomorphisms of connected cyclic abelian covers of symmetric digraphs*, submitted.

[13] H. Mizuno and I. Sato, *Characteristic polynomials of some covers of symmetric digraphs*, Ars Combinatoria, 45 (1997), 3-12.

[14] H. Mizuno and I. Sato, *Isomorphisms of cyclic abelian covers of symmetric digraphs*, Ars Combinatoria, 54 (2000), 51-64.

[15] H. Mizuno and I. Sato, *Isomorphisms of some covers of symmetric digraphs* (in Japanese), Trans. Japan SIAM., 5-1 (1995), 27-36.

[16] I. Sato, *Isomorphisms of some graph coverings*, Discrete Math., 128 (1994), 317-326.

[17] D. A. Waller, *Double covers of graphs*, Bull. Austral. Math. Soc., 14 (1976), 233-248.

Author's address:

Iwao Sato,
Oyama National College of Technology,
Oyama, Tochigi, 323-0806, JAPAN,
Email: isato@oyama-ct.ac.jp