Cost analysis of a redundant system using GERT

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Abstract

In this note, we consider two models: 1) a two-unit and a three-unit system with a cold standby; 2) a two-unit as well as a k-unit system working in parallel; in both the models, the failure and repair rates are exponentially distributed with parameters $\lambda_i, (i = 1, 2), \mu_i (i = 1, 2)$ and μ with $\mu > \max\{\mu_1, \mu_2\}$ respectively. The MTSF is obtained for a k-unit system for model 2 and we obtain the MTSF for a two-unit and three-unit system in model 1.Various reliability parameters and the average regeneration cost have been derived and illustrated numerically for a two-unit system of models 1 and 2 respectively using Graphical Evaluation and Review Technique (GERT). Finally a concluding remark is given.

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Key words: average regeneration cost, GERT, redundant system, reliability, two-unit system, w-function.

§1. Introduction

The statistical properties of two-unit redundant system has been discussed since last several decades by many authors using the techniques of Markov process, point process and renewal theory. GERT (Graphical Evaluation and Review Technique) is a new graphical procedure for the analysis of stochastic networks having nodes and directed branches. GERT has been successfully applied by several authors like Whitehouse, Ohta and Yin et al. for a few types of queueing and reliability problems. The basic reason for the wide use of GERT network technique is the ease with which the system can be modeled in the network form.

In this article, an attempt has been made to analyze through GERT, the dynamic characteristics as well as the average cost of a two-unit system, one model with cold standby and the other model working in parallel. Further, we have found the various reliability characteristics of interest to system designers and operation managers.

Definition of Reliability: Reliability is the probability that a system will perform its function during a specified interval of time under the condition that the system is used in a certain specified environment.

$$\begin{aligned} R(t) &= \Pr\{\text{system is up in } (0, t]\} \\ &= \Pr\{\psi(u) = 1, \forall\, u \in (0, t)\} \end{aligned}$$

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§2. Notations

s, θ	real parameter
p_{ij}	probability of realization of the activity (i, j)
$\lambda_i (i = 1, 2)$	lifetime of units
$\mu_i (i = 1, 2)$	repair time of units
$\mu > Max\{\mu_1, \mu_2\}$	replacement rate when both the units have failed since the down state of the system results in loss of revenue the system should be replaced immediately.
π_i	steady-state probabilities of different states $i(i = 1, 2, \ldots, 5)$
$E_R(c)$	average regeneration cost
$c_d(c_s)$	cost of down-time (carrying a standby unit except model 2)
$c_1(c_2)$	cost per unit for unit 1's (unit 2's) repair for both the models
X_i	various states of model 1 $(i = 1, 2, \ldots, 5)$
Y_i	various states of model 2 $(i = 1, 2, 3, 4)$.

§3. Description of the system

The system consists of two independent units in cold standby configuration in model 1 whereas in model 2, it comprises of two-units working in parallel. Initially one unit is switched online and the other is kept as a cold standby for model 1 and in model 2, it requires at least one-unit to operate successfully. In model 1 , whenever an operating unit fails, the standby unit is switched online and the failed unit goes for repair. The lifetime and repair time of the units are independent random variables and the system fails when all the units have failed. The following general assumptions are made for both model 1 and model 2 respectively.

1. A GERT network is to have exactly one source and at least one sink.
2. Each node of a GERT network must be reachable from the source and at least one sink must be reachable from each node.
3. This paper consists of two models. In model 1, we consider a two-unit and a three-unit system with cold standby which means that it will not fail in the standby state. Some results are also given for a three-unit system. In the case of two-unit system one will be operating online and the other will be in standby state whereas in three-unit system two-units will be working in online and one-unit will be in the cold standby state
4. In model 1, state 1 (state 3) represents that both the units are in working condition.
5. In state 2 the online unit is under repair and the standby unit is operating.
6. In state 4 the standby unit operating online is under repair and the online unit is operating whereas in state 5 both the units are under repair (down state).

7. In model 2, we present a two-unit system working in parallel and we give some results for a k-unit system.
8. In model 2, state 1 describes that both unit 1 and unit 2 are operating online whereas in state 2 (state 3) unit 1 (unit 2) is under repair and unit 2 (unit 1) is operating online. Finally in state 4 both the units 1 and 2 are under repair (down state).
9. The units are completely new after each repair.
10. All the switch over times are negligible and switching is perfect.
11. The lifetime and repair time of units are exponentially distributed with parameters $\lambda_I (i = 1, 2)$ and $\mu_i (i = 1, 2)$ respectively whereas the replacement rate is μ when both the units have failed where $\mu > \max\{\mu_1, \mu_2\}$.

§4. GERT analysis of the system

A GERT network representation of the system for model 1 is shown in figure 1 whereas for model 2 it is shown in figure 2. For an activity drawn between two nodes i and j, the moment generating function (m.g.f.) of time (t_{ij}) and cost (c_{ij}) are given respectively by

$$\begin{aligned} M_{ij}(s) &= E[e^{st}] \quad \text{and} \\ M_{ij}(\theta) &= E[e^{c\theta}]. \end{aligned} \tag{4.1}$$

where s and θ are real parameter. Now such multi-parameter functions may be combined in the following way to result in a W-function

$$w_{ij}(s, \theta) = p_{ij} M_{ij}(s) M_{ij}(\theta) \tag{4.2}$$

where p_{ij} indicates the probability of realization of the activity (i, j). Once the equivalent W-function is known, the probability of realization of the network is given by

$$p = [w(s, \theta)]_{s=0,\ \theta=0}. \tag{4.3}$$

The mean-time and cost of realization of the network are respectively given by

$$\begin{aligned} E(t) &= [d/ds M_t(s)]_{s=0} \quad \text{and} \\ E(c) &= [d/d\theta M_c(\theta)]_{\theta=0} \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} M_t(s) &= w(s, 0)/p \qquad \text{and} \\ M_c(\theta) &= w(0, \theta)/p \end{aligned} \tag{4.5}$$

§5. Model 1

5.1 Determination of MTSF

The mean-time to system failure (MTSF) is defined as the time until the system is completely inoperative. This is achieved by finding the W-function from the initial node to the terminal node. Thus by applying Mason's rule in figure 1, we have

$$w(s) = \frac{w_{12}(s)[w_{25}(s) + w_{23}(s)w_{34}(s)w_{45}(s)]}{1 - w_{12}(s)w_{23}(s)w_{34}(s)w_{41}(s)} \tag{5.1.1}$$

where

$$\begin{aligned}
w_{12}(s) &= (1 - s/\lambda_1)^{-1} \\
w_{23}(s) &= (\mu_1/(\lambda_2 + \mu_1))(1 - s/\lambda_2 + \mu_1)^{-1} \\
w_{25}(s) &= (\lambda_2/(\lambda_2 + \mu_1))(1 - s/\lambda_2 + \mu_1)^{-1} \\
w_{34}(s) &= (1 - s/\lambda_2)^{-1} \\
w_{45}(s) &= (\lambda_1/(\lambda_1 + \mu_2))(1 - s/\lambda_1 + \mu_2)^{-1} \\
w_{41}(s) &= (\mu_2/(\lambda_1 + \mu_2))(1 - s/\lambda_1 + \mu_2)^{-1} \\
w_{51}(s) &= (1 - s/\mu)^{-1}
\end{aligned} \tag{5.1.2}$$

with $w(0) = 1$. Hence one finds

$$\begin{aligned}
MTSF = \frac{1}{w(0)}\left[\frac{d}{ds}[w(s)]\right]_{s=0} &= \text{Busy time + Idle time of service facility} \\
&= \frac{(\lambda_1 + \lambda_2)(\lambda_1 + \mu_2)(\lambda_2 + \mu_1) + \lambda_1\mu_1\lambda_2}{\lambda_1\lambda_2((\lambda_1\lambda_2 + \lambda_2\mu_2 + \mu_1\lambda_1)}
\end{aligned} \tag{5.1.3}$$

Now for $\lambda_1 = \lambda_2 = \lambda$ and $\mu_1 = \mu_2 = \mu$

$$MTSF = \frac{2(\lambda + \mu)^2 + \lambda\mu}{\lambda^2(\lambda + 2\mu)} \tag{5.1.4}$$

If no repair facility is available ie. $\mu_1 = \mu_2 = 0$, MTSF $= \dfrac{1}{\lambda} + \dfrac{1}{\lambda_2} = 2/\lambda$ for $\lambda_1 = \lambda_2$ which is a standard result. Now the MTSF for a three-unit system is given by

$$MTSF = \frac{(4\lambda + \mu)(3\lambda^2 + 6\lambda\mu + 3\mu^2)}{2\lambda^2(4\lambda + 3\mu)(2\lambda + \mu)}$$

and for this model we cannot generalize the MTSF for a k-unit system.

5.2 Determination of steady-state probabilities

The steady-state probabilities for a given state is equal to expected time in the given state during a regeneration of the system divided by the expected total time of the regeneration. The expected time of regeneration is equal to the first moment of m.g.f. representing the time to return to any node in the system. Thus the mean-recurrence time from initial state to the same state is found from the representation in figure 1. Now the W-function is given by

$$w(s) = w_{12}(s)w_{25}(s)w_{51}(s) + w_{12}(s)w_{23}(s)w_{34}(s)[w_{41}(s) + w_{45}(s)w_{51}(s)]$$

(5.2.1)

Hence the time spent in any state can be obtained from equation (5.2.1) as follows. Thus

(5.2.2) $$\text{mean-time spent in state 1} = \frac{1}{\lambda_1}$$

(5.2.3) $$\text{mean-time spent in state 2} = \frac{1}{\lambda_2 + \mu_1}$$

(5.2.4) $$\text{mean-time spent in state 3} = \frac{\mu_1}{\lambda_2(\lambda_2 + \mu_1}$$

(5.2.5) $$\text{mean-time spent in state 4} = \frac{\mu_1}{(\lambda_1 + \mu_2)(\mu_1 + \lambda_2)}$$

(5.2.6) $$\text{and mean-time spent in state 5} = [\lambda_1\mu_1/(\mu_1 + \lambda_2)(\lambda_1 + \mu_2) + \lambda_2/(\mu_1 + \lambda_2)]/\mu.$$

The steady-state probabilities of different states are defined as follows

$$\pi_i = \frac{\text{Mean-time spent in state i}}{\text{Expected total time of regeneration}}, \qquad i = 1, 2, \ldots, 5.$$

Therefore one has

(5.2.7)
$$\begin{aligned}
\pi_i &= \frac{(\lambda_1 + \mu_2)(\lambda_2 + \mu_1)\mu\lambda_2}{A} \\
\pi_2 &= \frac{(\lambda_1 + \mu_2)\lambda_2\mu\lambda_1}{A} \\
\pi_3 &= \frac{(\lambda_1 + \mu_2)\mu_1\lambda_1\mu}{A} \\
\pi_4 &= \frac{\lambda_2\mu\mu_1\lambda_1}{A} \\
\text{and} \quad \pi_5 &= \frac{\lambda_1\lambda_2(\lambda_1\lambda_2 + \lambda_2\mu_2 + \mu_1\lambda_1)}{A}
\end{aligned}$$

where

$$\begin{aligned}
A &= \mu[\lambda_2(\lambda_1 + \mu_2)(\lambda_2 + \mu_1) + (\mu_1 + \lambda_1)\lambda_1\lambda_2) + \lambda_1\mu_1(\lambda_1 + \mu_2)(\lambda_1\lambda_2\mu_1] \\
&\quad + \mu_1\mu_2\lambda_2\lambda_1 + \lambda_1\lambda_2^2(\lambda_1 + \mu_2).
\end{aligned}$$

5.3 Determination of average regeneration cost

Cost function is the total expenditure incurred per cycle. The average regeneration cost $E_R(c)$ can be obtained by incorporating the following costs for model 1.

c_s : the cost of carrying a standby unit

c_1 : cost per unit for unit 1's repair

c_2 : cost per unit for unit 2's repair

c_d : the cost of down-time

at paths $w_{12}, w_{23}, w_{25}, w_{34}, w_{41}$ and w_{45} respectively. Now on taking

$$(5.3.1) \qquad \begin{aligned} w_{12} &= e^{c_s\theta} \\ w_{23} &= \frac{\mu_1}{\lambda_2+\mu_1}e^{c_1\theta} \\ w_{25} &= \frac{\lambda_2}{\lambda_2+\mu_1}e^{c_1\theta} \\ w_{34} &= e^{c_s\theta} \\ w_{41} &= \frac{\lambda_2}{\lambda_1+\mu_2}e^{c_1\theta} \\ w_{45} &= \frac{\lambda_1}{\lambda_1+\mu_2}e^{c_2\theta} \\ \text{and} \qquad w_{51} &= e^{c_d\theta} \end{aligned}$$

in equation (5.2.1), we get

$$\begin{aligned} E_R(c) &= [d/d\theta W(0,\theta)]_{\theta=0}/W(0,0) \\ (5.3.2) \qquad &= x(c_s+c_1+c_d)+ya(2c_s+c_1+c_2)+zy(2c_s+c_1+c_2+c_d) \end{aligned}$$

where

$$\begin{aligned} x &= (\lambda_2/(\lambda_2+\mu_1)), \quad y=(\mu_1/(\lambda_2+\mu_1)), \\ z &= (\lambda_1/(\lambda_1+\mu_2)), \quad \text{and } a=(\mu_2/(\lambda_1+\mu_2)). \end{aligned}$$

5.4 Numerical example

As an illustration by taking $\lambda_1 = 0.3, \lambda_2 = 0.4, \mu_1 = 0.6, \mu_2 = 0.8, c_s = 3, c_1 = 3.5, c_2 = 4.5$ the average cost $E_R(c)$ is tabulated for various values of down-time cost c_d in Table 1 for a two-unit system. Further for $x = 0.4, y = 0.6, z = 3/11, a = 8/11$ (defined in equation (5.3.2)) $c_1 = 3.5, c_2 = 4$ and $c_d = 6$ and for various values of c_s, c_1 and c_2 the average cost $E_R(c)$ is shown in Tables 2, 3 and 4 respectively. In this model, the average cost increases with variations in c_s, c_1, c_d, and c_2 respectively.

Table 1

c_d	$E_R(c)$
6.0	14.38182
6.250	14.52273
6.500	14.66364
6.750	14.80455
7.0	14.94545
7.250	15.08636
7.500	15.22727
7.750	15.36818
8.0	15.50909
8.250	15.65000
8.500	15.79091
8.750	15.93182
9.0	16.07273
9.250	16.21364
9.500	16.35455

Average cost versus c_d

Table 2

c_d	$E_R(c)$
3.0	13.58182
3.25	13.83182
3.5	14.08182
3.75	14.33182
4.0	14.58182
4.25	14.83182
4.5	15.08182
4.75	15.33182
5.0	15.58182
5.25	15.83182

Average cost versus c_1

Table 3

4.0	14.08182
4.25	14.23182
4.5	14.38182
4.75	14.53182
5.0	14.68182
5.25	14.83182
5.5	14.98182
5.75	15.13182
6.0	15.28182
6.25	15.43182

Average cost versus c_2

Table 4

3.0	14.08182
3.25	14.48182
3.5	14.88182
3.75	15.28182
4.0	15.68182
4.25	16.08182
4.5	16.48182
4.75	16.88182
5.0	17.28182
5.25	17.60182

Average cost versus c_s

§6. Model 2

We give below the various results obtained for model 2, since the steps involved in obtaining these results are the same as in model 1 (see figure 2).

$$MTSF = \frac{(\lambda_1+\mu_2)(\lambda_2+\mu_1)+\lambda_1(\lambda_1+\mu_2)+\lambda_2(\lambda_2+\mu_1)}{\lambda_1\lambda_2(\lambda_1+\mu_2+\lambda_2+\mu_1)} \tag{6.1}$$

Now, for $\lambda_1=\lambda_2=\lambda, \mu_1=\mu_2=\mu$

$$MTSF = (3\lambda+\mu)/2\lambda^2. \tag{6.2}$$

In this model, we can obtain the MTSF for a k-unit system given by $MTSF = [(k-1)\lambda + \mu]/[k(k-1)\lambda^2] + 1/(k-1)\lambda$. Hence for $k = 3$ we have the MTSF for a three-unit system as $MTSF = (5\lambda + \mu)/6\lambda^2$. If no repair facility is available, then $MTSF = 3/2\lambda$ (for a two-unit system) which is again a standard result. Further, the time spent in any state is given by

$$\begin{aligned} T_1 &= \frac{1}{(\lambda_1+\lambda_2)} \\ T_2 &= \frac{\lambda_1}{(\lambda_1+\lambda_2)(\lambda_2+\mu_1)} \\ T_3 &= \frac{\lambda_2}{(\lambda_1+\mu_2)(\lambda_2+\lambda_1)} \\ T_4 &= \frac{1}{\mu}\left[\frac{\lambda_1\lambda_2}{(\lambda_1+\lambda_2)(\lambda_2+\mu_1)} + \frac{\lambda_1\lambda_2}{(\lambda_1+\lambda_2)(\lambda_1+\mu)}\right] \end{aligned} \tag{6.3}$$

where $T = T_1 + T_2 + T_3 + T_4$

Hence the steady-state probabilities are given by

$$\pi_1 = T_1/T, \pi_2 = T_2/T, \pi_3 = T_3/T, \pi_4 = T_4/T. \tag{6.4}$$

Similarly, the average cost is given by

$$E_R(c) = [ac_1 + cc_2 + c_d(ab + cd)] \tag{6.5}$$

where

$$\begin{aligned} a &= \lambda_1/\lambda_1+\lambda_2, b = \lambda_2/\lambda_2+\mu_1, c = \lambda_2/\lambda_1+\lambda_2, d = \lambda_1/\lambda_1+\mu_2 \\ e &= \mu_1/\mu_1+\lambda_2, f = \mu_2/\mu_2+\lambda_1. \end{aligned}$$

By taking $\lambda_1 = 0.6, \lambda_2 = 0.7, \mu_1 = 0.3, \mu_2 = 0.35, c_2 = 0.75, c_d = 0.8$ and for various values of c_1 , the average cost is tabulated in Table 5 where

c_d : cost of down-time

c_1 : cost per unit for unit 1's repair

c_2 : cost per unit for unit 2's repair

Again for various values of c_d, c_2, and equal failure rate $\lambda_1 = \lambda_2 = \lambda$, the average cost is shown in Tables 6, 7 and 8 respectively. In this model also the average cost increases with variations in c_d, c_1, c_2 and $\lambda_1 = \lambda_2 = \lambda$ respectively.

Table 5

c_1	$E_R(c)$
0.5	1.165141
0.6	1.211295
0.7	1.257449
0.8	1.303603
0.9	1.349757
1.0	1.395910
1.1	1.442064
1.2	1.488218
1.3	1.534372

Average cost versus c_1

Table 6

c_d	$E_R(c)$
0.5	0.966194
0.6	1.03251
0.7	1.098826
0.8	1.165142
0.9	1.231452
1.0	1.297773
1.1	1.364089
1.2	1.430405
1.3	1.496721
1.4	1.563036

Average cost versus c_d

Table 7

c_2	$E_R(c)$
0.75	1.165142
1.25	1.434372
1.75	1.703603
2.25	1.972834
2.75	2.242065
3.25	2.511296
3.75	2.780526
4.25	3.049757
4.75	3.318988
5.25	3.588219

Average cost versus c_2

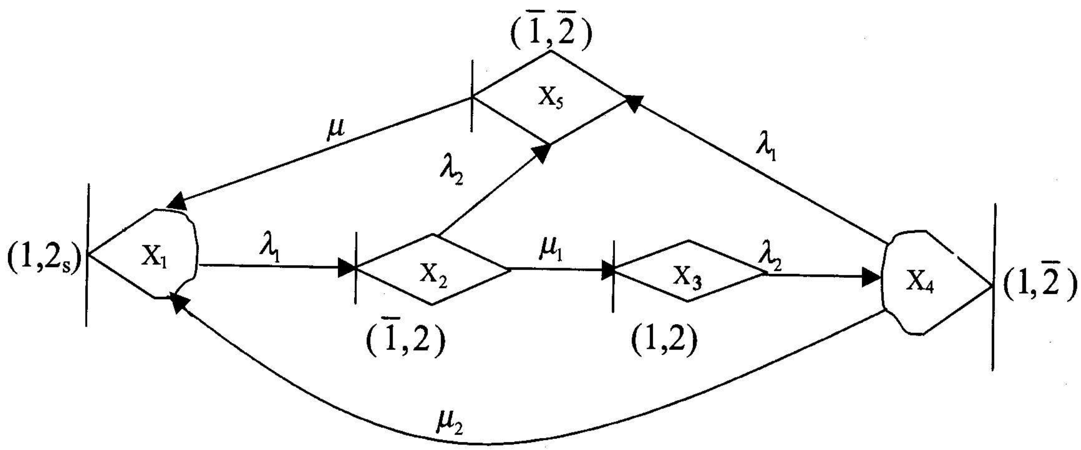
Table 8

$\lambda_1 = \lambda_2$	$E_R(c)$
0.5	1.0
0.6	1.033333
0.7	1.06
0.8	1.081828
0.9	1.1
1.0	1.115385
1.1	1.128574
1.2	1.14
1.3	1.15
1.4	1.158824

Average cost versus $\lambda_1 = \lambda_2$

§7. Concluding remark for model 1 and model 2

In this paper, we have considered two models namely model 1 and model 2. In model 1, we have a two-unit and a three –unit system with cold standby whereas in model 2 we consider a k-unit system working in parallel and for numerical we have taken k = 2. In model 1 the average cost E_R (c) increases with the cost of carrying a standby unit, c_s, cost per unit for unit 1's repair c_1, cost per unit for unit 2's repair c_2 and the cost of down-time c_d respectively. Similarly it is the case with model 2. We have also found the MTSF for a k-unit system for model 2 and MTSF for a two-unit as well as a three-unit system for model 1



State X_1: Systems good; unit 1 online and unit 2 cold standby

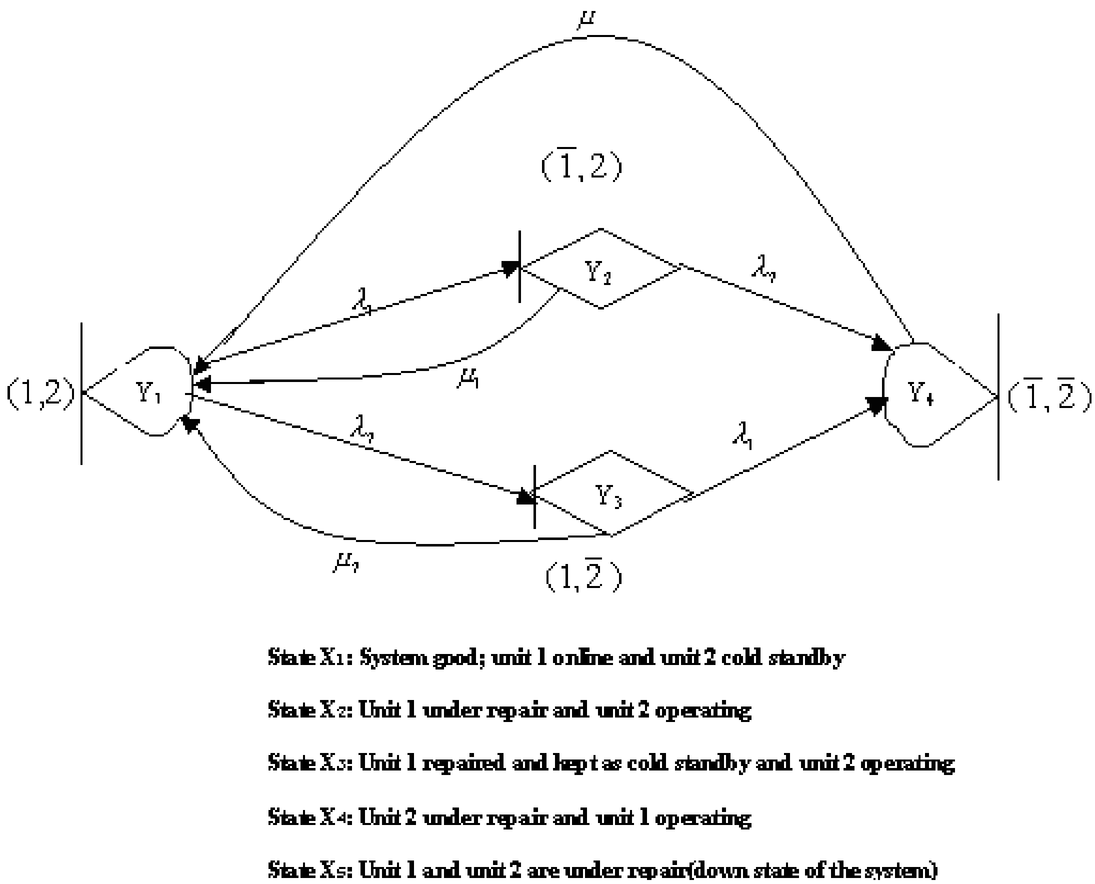
State X_2: Unit 1 under repair and unit 2 operating

State X_3: Unit 1 repaired and kept as cold standby and unit 2 operating

State X_4: Unit 2 under repair and unit 1 opertaing

State X_5: Unit 1 and unit 2 are under repair (down state of the system)

Fig. 1. GERT network for model 1



GERT network for model 2

References

[1] S.J. Mason, *Some properties of signal flowgraphs*, Proc. IRE, 41, 9 (1953), 1144–1156.

[2] H. Ohta, *GERT analysis of a single conveyor system*, Intl. J. Prod. Res., 17 (1979), 405–410.

[3] G.E. Whitehouse, *GERT a useful technique for analysing reliability problems*, Technometrics, 12 (1970), no. 1, 33–48.

[4] D. Yin, H. Ohta, and F. Hashimoto, *GERT analysis of reliability problems*, Bull. Univ. Osaka Prefecture A, 33, 2 (1984), 171–179.

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