Theorems of Hicks and LeChatelier-Samuelson type in ideal Banach spaces of infinite sequences

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Abstract

It is shown in this paper that the classical Hicks and LeChatelier-Samuelson theorem, concening the behavior of solutions x of a system of linear equations $x = Ax + c$, can be generalized to the case when A is a nonnegative matrix that generates a bounded linear operator in an arbitrary ideal sequence space.

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Key words: nonnegative matrix, input-output matrix, Orlicz space, Marcinkiewicz space, perfect ideal Banach space, Riesz property **1. Statement of the problem.**
The classical Hicks and LeChatelier-Samuelson theorems are concerned with the behavior of solutions x of a system of linear equations

$$(0.1) \qquad\qquad x = Ax + c,$$

where the matrix A is nonnegative with spectral radius $\rho(A) < 1$, one among the coordinates of the vector c is increasing, and the other coordinates of the vector c are preserved. Results of this type were obtained in case of an irreducible and positive matrix A in [6, 15, 16], see also [2, 12, 20]. In [24] these results were recently generalized to arbitrary nonnegative matrices.

In [10, 11] the classical Hicks and LeChatelier-Samuelson theorems have been extended to infinite nonnegative matrices A which generate bounded linear operators in the classical sequence spaces ℓ_p ($1 \le p \le \infty$), c, and c_0, but under the same restriction, viz. the irreducibility of A in the first theorem, and the positivity of A in the second. Generalizations of these theorems to infinite nonnegative matrices in the above mentioned sequence spaces were obtained in [19].

The aim of this article is to show that all constructions and results from [19] (and [10, 11]) can be generalized to the case when A is a nonnegative matrix that generates a bounded linear operator in an arbitrary ideal sequence space. We point out that our formulation of the analogue of the LeChatelier-Samuelson theorem is different from that of the corresponding analogue in [19], and it seems to be more convenient. Our proofs of this analogue are also new. Of course, we deal only with real sequence spaces in what follows.

Consider the infinite system

$$x_1 = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1k}x_k + \ldots + c_1,$$

$$x_2 = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2k}x_k + \ldots + c_2,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$x_j = a_{j1}x_1 + a_{j2}x_2 + \ldots + a_{jk}x_k + \ldots + c_j,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

which may be written in the compact form (0.1) with $A = (a_{jk})$ being a nonnegative matrix and $x = (x_1, \ldots, x_j, \ldots)$ and $c = (c_1, \ldots, c_j, \ldots)$ being nonnegative sequences. Equation (0.1), as its finite analogue, describes the balance of inputs and outputs in an economical model with an infinite number of branches and products (goods) under the assumption that each branch produces only one product, and each product is produced by only one branch. In this interpretation A is called a *technological matrix* or *input-output matrix*, x an *output vector*, and c an *input vector*. Without loss of generality, the set of indices $\{1, 2, \ldots\}$ can be identified with the set of products (goods).

In what follows we assume that A is a matrix representing a linear operator in an ideal sequence space X (for definitions see below). By virtue of Banach's classical theorem [3], this operator (which we denote again by A) is then automatically bounded.

2. Ideal Banach spaces. Let us recall some definitions and results about ideal sequence spaces and linear operators acting in them [4, 5, 8].

By s we denote the space of all real sequences $x = (x_1, \ldots, x_n, \ldots)$ with the usual metric. A normed linear space $X \subset s$ is called *ideal space* if the relations $|x| \le |y|$, $x \in s$, and $y \in X$ imply the relations $x \in X$ and $\|x\| \le \|y\|$. An ideal space X is complete (i.e., a Banach space) if and only if it possesses the *Riesz property* which means that, if

$$\sum_{n=1}^{\infty} \|x_n\| < \infty,$$

then the sum x of the series

$$x = \sum_{n=1}^{\infty} x_n$$

belongs to X and this series is convergent in X. An ideal Banach space X is called *perfect* if it has the *Fatou property*: if a sequence $x_n \in X$ converges in s to $x \in s$ and

$$\limsup_{n \to \infty} \|x_n\| < \infty,$$

then $x \in X$ and

$$\|x\| \le \limsup_{n \to \infty} \|x_n\|.$$

The classical spaces ℓ_p $(1 \leq p \leq \infty)$ and c_0 are ideal spaces, but c is not. Other important examples are Orlicz, Lorentz and Marcinkiewicz spaces whose definition we recall now.

Given a continuous non-decreasing and convex function $M : [0, \infty) \to [0, \infty)$ with $M(0) = 0$, the *large Orlicz space* (or simply *Orlicz space*) ℓ_M is the Banach space of all sequences $x \in s$ such that

$$(0.2) \qquad \sum_{n=1}^{\infty} M\left(\frac{|x_n|}{\lambda}\right) < \infty$$

for some $\lambda > 0$, equipped with the Luxemburg norm

$$\|x\|_L = \inf\left\{\lambda > 0 : \ \sum_{n=1}^{\infty} M\left(\frac{|x_n|}{\lambda}\right) \leq 1\right\},$$

or Orlicz norm

$$\|x\|_O = \inf_{0 < \lambda < \infty} \frac{1}{\lambda}\left(1 + \sum_{n=1}^{\infty} M\left(\lambda|x_n|\right)\right).$$

The *small Orlicz space* ℓ_M° is the subspace of ℓ_M of all sequences $x \in s$ for which (0.2) holds for all $\lambda > 0$. The space ℓ_M is perfect, the space ℓ_M° in general is only ideal; it is perfect if and only if $\ell_M^\circ = \ell_M$. On the other hand, the space ℓ_M° is always separable, but ℓ_M is separable if and only if $\ell_M = \ell_M^\circ$.

Now let $\phi : [0, \infty) \to [0, \infty)$ be a continuous non-decreasing and concave function with $\phi(0) = 0$. The *Lorentz space* λ_ϕ is the Banach space of all sequences $x \in s$ such that

$$(0.3) \qquad \sum_{h=1}^{\infty} \phi(\lambda(x, h)) < \infty,$$

equipped with the norm

$$\|x\|_L = \sum_{h=1}^{\infty} \phi(\lambda(x, h)).$$

Here $\lambda(x, h) = \operatorname{card}\{n : |x_n| > h\}$, with $\operatorname{card} D$ denoting the number of elements in D, is the distribution function of $x = (x_1, \ldots, x_j, \ldots)$. The Lorentz space is always perfect and separable. Similarly, the *large Marcinkiewicz space* (or simply *Marcinkiewicz space*) μ_ϕ is the Banach space of all sequences $x \in s$ such that

$$(0.4) \qquad \sup_{D \subseteq \{1,2,\ldots\}} \frac{\phi(\operatorname{card} D)}{\operatorname{card} D} \sum_{n \in D} |x_n| < \infty,$$

equipped with the norm

$$\|x\|_L = \sup_{D \subseteq \{1,2,\ldots\}} \frac{\phi(\operatorname{card} D)}{\operatorname{card} D} \sum_{n \in D} |x_n|.$$

The *small Marcinkiewicz space* μ_ϕ° is the subspace of μ_ϕ of all sequences $x \in s$ for which

$$\lim_{\nu(D)\to 0} \sup_{D\subseteq\{1,2,\ldots\}} \frac{\phi(\operatorname{card} D)}{\operatorname{card} D} \sum_{n\in D} |x_n| = 0,$$

where

$$\nu(D) = \sum_{n\in D} \frac{1}{2^n}.$$

The large Marcinkiewicz space is always perfect (and nonseparable), the small one is only ideal (but always separable).

The Orlicz, Lorentz, and Marcinkiewicz spaces introduced above are important examples of symmetric sequence spaces. An ideal space is called *symmetric* if the relations $\lambda(x_1, h) = \lambda(x_2, h)$, $x_1 \in s$, and $x_2 \in X$ imply that $x_1 \in X$ and $\|x_1\| = \|x_2\|$. A detailed description of the theory of symmetric spaces may be found in [5].

We omit here other examples of ideal Banach sequence spaces. In particular, one can consider analogues of Orlicz, Lorentz, and Marcinkiewicz spaces generated by functions $\phi(n, \cdot)$ defined on $\{1, 2, \ldots\} \times [0, \infty)$; these spaces, however, are not symmetric.

In what follows we additionally assume that all ideal spaces contain sequences (so called *units*) whose components are all strictly positive. The Orlicz, Lorentz, and Marcinkiewicz spaces and all symmetric spaces have this property.

For $x, y \in s$ we denote by (x, y) the real number

$$(x, y) = \sum_{n=1}^{\infty} x_n y_n$$

provided that the series in the right hand side is convergent. Given an ideal Banach space X, the *dual space* X' is the Banach space of all sequences $x \in s$ such that

$$(x, y) < \infty \qquad (y \in X),$$

equipped with the norm

$$\|x\| = \sup_{\substack{y\in X \\ \|y\|\leq 1}} (x, y).$$

The dual space is always perfect. The second dual X'' coincides with the original space X if and only if the space X is perfect. In general X is continuously (with norm 1) embedded in X'' in the category of topological linear spaces, but not in the category of normed linear spaces. It is known that the latter holds if the convergence of a sequence $x_n \in X$ to $x \in X$ in s implies that

$$\|x\| \leq \limsup_{n\to\infty} \|x_n\|.$$

The dual space X' to an ideal Banach space X is a closed (possibly proper) subspace of the usual conjugate space X^*. The equality $X' = X^*$ holds if and only if the space X is *regular*, which means that

$$\lim_{\nu(D) \to 0} \|P_D x\| = 0 \qquad (x \in X),$$

where $P_D x = \chi_D x$ is the multiplication operator by the characteristic function χ_D of a set $D \subseteq \{1, 2, \ldots\}$. In the general case the space X^* is a direct sum of X' and the subspace of so-called *singular* continuous functionals.

For concrete ideal Banach sequence spaces the dual spaces may be described explicitly. In particular, the equations

$$(\ell_M)' = (\ell_M^\circ)' = \ell_N, \quad (\lambda_\phi)' = \mu_\psi, \quad (\mu_\phi)' = (\mu_\phi^\circ)' = \lambda_\psi$$

hold, where the dual functions N and ψ are defined by the formulas

$$N(t) = \sup_{0 < s < \infty} \{ts - M(s)\}, \qquad \psi(t) = \frac{t}{\phi(t)}.$$

A nonnegative matrix $A = (a_{ij})$ defines an operator in an ideal space X if and only if it satisfies the condition

$$\sum_{i,j=1}^{\infty} a_{ij} x_i y_j < \infty \qquad (x = (x_1, \ldots, x_i, \ldots) \in X, \ y = (y_1, \ldots, y_j, \ldots) \in X').$$

This condition implies that the operator A is bounded in X and its norm is defined by

$$\|A\| = \sup_{\substack{x \in X, y \in X' \\ \|x\| \leq 1, \|y\| \leq 1}} \sum_{i,j=1}^{\infty} a_{ij} x_i y_j.$$

Of course, an effective use of this criterion for the study of concrete operators is hardly possible. In basic cases it usually suffices to use two simple sufficient conditions which can be formulated in arbitrary ideal Banach spaces. Both conditions were obtained in the case of the Lebesgue spaces ℓ_p ($1 \leq p \leq \infty$) by Hille and Tamarkin [7].

In order to formulate these conditions we must recall two definitions. Note that all definitions of ideal Banach spaces can also be formulated for spaces of double sequences.

Let X be a perfect ideal Banach space. We denote by $(X \leftarrow X')$ the space of double sequences $z = (z_{11}, z_{12}, z_{21}, z_{22}, \ldots)$ such that $\xi_i = (z_{i1}, z_{i2}, z_{i3}, \ldots) \in X'$ for $i = 1, 2, 3, \ldots$, and the sequence of norms $\xi = (\|\xi_1\|, \|\xi_2\|, \|\xi_3\|, \ldots)$ belongs to X. Similarly, by $(X \to X')$ we denote the space of double sequences $z = (z_{11}, z_{12}, z_{21}, z_{22}, \ldots)$ such that $\eta_j = (z_{1j}, z_{2j}, z_{3j}, \ldots) \in X$ for $j = 1, 2, 3, \ldots$, and the sequence of norms $\eta = (\|\eta_1\|, \|\eta_2\|, \|\eta_3\|, \ldots)$ belongs to X'. Equipped with the natural norms

$$\|z\|_{(X \leftarrow X')} = \|\xi\|, \qquad \|z\|_{(X \to X')} = \|\eta\|,$$

respectively, these spaces are perfect ideal spaces.

Note that both norms are the same in the case when $X = \ell_2$; moreover, in this case they coincide with the usual Hilbert-Schmidt norm. In the general case these

norms are different; in particular, in case $X = \ell_p$ $(1 \leq p \leq \infty)$ we have $(\ell_p)' = \ell_{p'}$, where $p' = p/(p-1)$, and the inequalities

$$\|z\|_{(L_p \leftarrow L_{p'})} \leq \|z\|_{(L_p \rightarrow L_{p'})}, \qquad 1 \leq p < 2$$

and

$$\|z\|_{(L_p \leftarrow L_{p'})} \geq \|z\|_{(L_p \rightarrow L_{p'})}, \qquad 2 \leq p < \infty$$

hold. If $A = (a_{jk})$ is an infinite matrix as above, each of the inequalities

(0.5) $$\|(a_{jk})\|_{(X \leftarrow X')} < \infty$$

and

(0.6) $$\|(a_{jk})\|_{(X \rightarrow X')} < \infty$$

guarantees (see [21]) that the corresponding operator A acts in X, and its norm satisfies the estimates

(0.7) $$\|A\| \leq \|(a_{jk})\|_{(X \leftarrow X')},$$

and

(0.8) $$\|A\| \leq \|(a_{jk})\|_{(X \rightarrow X')}.$$

In case $X = \ell_p$ $(1 \leq p \leq \infty)$ these are precisely the Hille-Tamarkin conditions (see, e.g., [19]). In case $p = 1$ the first of them is not only sufficient but also necessary. Similarly, in case $p = \infty$ the second is not only sufficient but also necessary. Finally, in both these special cases the inequalities (0.7) and (0.8) are exact equalities.

There exist other sufficient conditions under which the operator A acts in an ideal space. We do not consider them, but just remark that the operator A generated by the matrix (a_{jk}) acts in an arbitrary perfect symmetric ideal space X if both inequalities

$$\|(a_{jk})\|_{(L_\infty \leftarrow L_1)} < \infty, \qquad \|(a_{jk})\|_{(L_1 \rightarrow L_\infty)} < \infty$$

are satisfied. Some other sufficient conditions may be found in [22, 23].

3. Main results. We are now ready to prove three general theorems on equation (0.1). Recall that a system described by (0.1) is called *productive* if it can satisfy an arbitrary nonnegative input c. It is well-known [9] that the productivity of (0.1) in any sequence space with normal and reproducing cone (and, in particular, in ideal sequence spaces) is equivalent to the fact that the spectral radius $\rho(A)$ of the corresponding matrix A is less than 1.

Since $\rho(A) \leq \|A\|$, the norm estimates for A obtained in the previous section can be considered as rough estimates of the spectral radius $\rho(A)$. Consequently, they allow us to formulate some tests of the productivity of the system (0.1). Other tests of productivity may be formulated on the base of inequalities of the type $Au_0 \leq \rho u_0$ and $A'v_0 \leq \rho v_0$ with $\rho \leq 1$; for details see [9].

As before, we assume that the operator A acts in an ideal Banach space X. In the case when the matrix (a_{jk}) is productive, we have

(0.9) $$x(c) = (I - A)^{-1}c = c + Ac + A^2c + \ldots = \sum_{k=0}^{\infty} A^k c.$$

Suppose that the input of the product j is increased and the inputs of the other products are not changed. The Hicks theorem describes then the behavior of all components of the output vector x. To be specific, let $c = (c_1, \ldots, c_j, \ldots)$ and $c' = (c'_1, \ldots, c'_j, \ldots)$ denote the input before and after the change, and so $c'_j > c_j$ and $c'_i = c_i$ for $i \neq j$. A product i is called *connected* with a product j if there exists a sequence of products $k_0, k_1, \ldots, k_s$ (with $k_0 = i$ and $k_s = j$) such that $a_{k_t k_{t+1}} > 0$ for $t = 0, \ldots, s-1$. Below we denote by $L(j)$ the set containing j and all products which may be connected with the product j. Recall that the *relative growth* of the product i is the number $(x'_i - x_i)/x_i$. Of course, the relative growth of the product i is defined only if $x_i > 0$; in case $x_i = 0$ one can think of it as being infinite. The analysis of relatives growths of products is equivalent to the analysis of the numbers $\theta_i = x_i/x'_i$.

Theorem 1. *Let an economical system described by equation (0.1) be productive, and suppose that the input c_j is increased, but the inputs of all other products are unchanged. Then the output x_i increases if and only if $i \in L(j)$. Moreover, the relative growth of the product j is maximal, and if the relative growth of the product i coincides with the relative growth of the product j then $c_i = 0$.*

Proof. It is evident that

(0.10) $$x_i(c') - x_i(c) = x_i(c' - c) = (c'_i - c_i) + a_{ij}^{(1)}(c'_j - c_j) + a_{ij}^{(2)}(c'_j - c_j)+,$$

where $a_{ij}^{(l)}$ $(i, j = 1, 2, \ldots)$ are the elements of the matrix A^l $(l = 1, 2, \ldots)$. It is easy to see that $x_j(c') - x_j(c) > 0$ if and only if either $i = j$, or $i \neq j$ and $a_{ij}^{(l)} > 0$ for some l. Since

$$a_{ij}^{(l)} = \sum_{k_1=1}^{\infty} \cdots \sum_{k_{l-1}=1}^{\infty} a_{ik_1} a_{k_1 k_2} \ldots a_{k_{l-1} j},$$

the latter is equivalent to the relation $i \in L(j)$.

The numbers θ_i are defined for $i = 1, 2, \ldots$, by assumption. It is evident that $0 < \theta_i < 1$ if $i \in L(j)$, and $\theta_i = 1$ if $i \notin L(j)$. Since $x(c)$ is a solution of (0.1), we have the identity

$$x_i(c) = \sum_{k=1}^{\infty} a_{ik} x_k(c) + c_i \qquad (i = 1, 2, \ldots).$$

Putting in this equality the equations $x_k(c) = \theta_k x_k(c')$ $(k = 1, 2, \ldots)$ yields, for all products $i \neq j$,

(0.11) $$\theta_i x_i(c') = \sum_{k=1}^{\infty} a_{ik} \theta_k x_k(c') + c_i.$$

On the other hand, the vector $x(c')$ is also a solution of (0.1) (for c instead of c'); therefore we get, for $i \neq j$,

$$(0.12) \qquad \theta_j x_i(c') = \sum_{k=1}^{\infty} a_{ik}\theta_j x_k(c') + \theta_j c_i.$$

A comparison of (0.11) and (0.12) shows that, for $i = 1, 2, \ldots$ with $i \neq j$,

$$(\theta_i - \theta_j)x_i(c') = \sum_{k=1}^{\infty} a_{ik}(\theta_k - \theta_j)x_k(c') + (1 - \theta_j)c_i.$$

These equations may be considered as an infinite system with new unknowns $\xi_i = (\theta_i - \theta_j)x_i(c')$ $(i = 1, 2, \ldots, i \neq j)$. This system has the form

$$(0.13) \qquad \xi_i = \left(\sum_{k=1}^{j-1} + \sum_{k=j+1}^{\infty}\right) a_{ik}\xi_k + (1 - \theta_j)c_i \qquad (i = 1, 2, \ldots, i \neq j)$$

and thus is a system with a nonnegative matrix A' which is obtained from A by omitting the j-th row and column. Its spectral radius (see, e.g., [9]) is less than 1. Therefore it has a nonnegative solution $\xi = (\xi_1, \ldots, \xi_{j-1}, \xi_{j+1}, \ldots)$. It is evident that $\xi_i > 0$ if $c_i > 0$ (since $\theta_j < 1$). But this means that $\theta_i > \theta_j$ and $x_i(c') > 0$. $\qquad \square$

The LeChatelier-Samuelson theorem describes the behavior of solutions of (0.1) when the input of a product j is increased, but the level of production of some part U among the other products is not changed in comparison with the situation when the production of products from U is not fixed.

Let $V = \{j\}$, and $W = \{1, 2, \ldots\} \setminus (U \cup V)$. We can write $x = (u, v, w)$ and $c = (f, g, h)$, where u and f are components of vectors x and c corresponding to the set of products from U, v and g are (scalar) components of vectors x and c corresponding to the product j, and w and h are components of vectors x and c corresponding to the set of products from W. In this notation the system (0.1) can be written in the form

$$(0.14) \qquad \begin{cases} u - A_{11}u - A_{12}v - A_{13}w = f, \\ v - A_{21}u - A_{22}v - A_{23}w = g, \\ w - A_{31}u - A_{32}v - A_{33}w = h, \end{cases}$$

where

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

is the block representation of A corresponding to the decomposition $\{1, 2, \ldots\} = U \cup V \cup W$.

Now we are interested in three solutions of equation (0.1). The first solution is the original one $x(c) = (u, v, w)$ that corresponds to the input $c = (f, g, h)$ and for which (0.14) holds. The second one is the "perturbed" one $x(c') = (u', v', w')$ which

corresponds to the input $c' = (f, g', h)$ $(g' > g)$ with increasing component g and satisfies the system

(0.15)
$$\begin{cases} u' - A_{11}u' - A_{12}v' - A_{13}w' = f, \\ v' - A_{21}u' - A_{22}v' - A_{23}w' = g', \\ w' - A_{31}u' - A_{32}v' - A_{33}w' = h. \end{cases}$$

Finally, the third is the solution $x(c'') = (u, v'', w'')$ which corresponds to the input $c'' = (f'', g', h)$ with increasing component g and fixed production u of products from U (with f replaced by a new value f'') and satisfies the system

(0.16)
$$\begin{cases} u - A_{11}u - A_{12}v'' - A_{13}w'' = f'', \\ v'' - A_{21}u - A_{22}v'' - A_{23}w'' = g', \\ w'' - A_{31}u - A_{32}v'' - A_{33}w'' = h. \end{cases}$$

If the economical model described by the matrix A is productive we can represent $x(c') - x(c'')$ in the form

$$x(c') - x(c'') = x(c' - c'') = (I - A)^{-1}(c' - c'') = (I - A)^{-1}(f - f'', 0, 0),$$

and so $x(c') \geq x(c'')$ if $f \geq f''$. Let

$$\widetilde{A} = \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix}.$$

It is evident that this matrix is also productive; therefore, the matrix $\widetilde{B} = (I - \widetilde{A})^{-1}$ exists and is nonnegative. Moreover, the identities

$$(v, w) = (I - \widetilde{A})^{-1} \begin{pmatrix} g + A_{12}u \\ h + A_{13}u \end{pmatrix}, \qquad (v'', w'') = (I - \widetilde{A})^{-1} \begin{pmatrix} g' + A_{12}u \\ h + A_{13}u \end{pmatrix}$$

imply that

$$(v'' - v, w'' - v) = (I - \widetilde{A})^{-1} \begin{pmatrix} g' - g \\ 0 \end{pmatrix}.$$

From this equality it follows in turn that

(0.17)
$$f - f'' = \begin{pmatrix} A_{12} & A_{13} \end{pmatrix} (I - \widetilde{A})^{-1} \begin{pmatrix} g' - g \\ 0 \end{pmatrix}.$$

We conclude that $f \geq f''$, and hence $x(c') \geq x(c'')$.

In what follows we are interested in products i for which $x_i(c') < x_i(c'')$. To this end, we need some notation. Given $K \subseteq \{1, 2, \ldots\}$, we denote by $L_K(j)$ the set of products $i \in \{j\} \cup K$ which can be connected with the product j using only products from K; consequently, $L(j) = L_{\{1,2,\ldots\}}(j)$. Moreover, by $P_K(j)$ we denote the set of all products $i \in \{j\} \cup K$ such that $a_{ij} > 0$. To simplify the notation, we will write

$$L_K(I) = \bigcup_{i \in I} L_K(i), \qquad P_K(I) = \bigcup_{i \in I} P_K(i)$$

in the sequel.

Theorem 2. *Let an economical system described by equation* (0.1) *be productive, and suppose that the input c_j is increased. Then the output of any product i, in the case when the levels of production of products from U do not change, is less than in the case when the levels of production of products U are not fixed if, and only if, $i \in L(P_U(V \cup L_W(j)))$.*

Proof. It is evident that

$$x_i(c') - x_i(c'') = \sum_{k \in U} b_{ik}(f_k - f_k''),$$

where b_{ij} $(i, j = 1, 2, \ldots)$ are elements of the matrix $B = (I - A)^{-1}$. Thus, $x_i(c') > x_i(c'')$ if and only if there exists an index $k \in U$ such that $b_{ik} > 0$ and $f_k - f_k'' > 0$. Clearly, $b_{ik} > 0$ if and only if $i \in L(k)$.

From the equality (0.17) it follows that

$$f_k - f_k'' = \sum_{s \in V \cup W} a_{ks}\widetilde{b}_{sj}.$$

Furthermore, $\widetilde{b}_{sj} = 0$ for $s \in W \setminus L_W(j)$ implies that

$$f_k - f_k'' = \sum_{s \in V \cup L_W(j)} a_{ks}\widetilde{b}_{sj}.$$

This equality shows that $f_k - f_k'' > 0$ holds if and only if there exists $s \in V \cup L_W(j)$ such that $a_{ks} > 0$ and $\widetilde{b}_{sj} > 0$ or, in other words, if there exists $s \in V \cup L_W(j)$ such that $k \in P_{V \cup L_W(j)}(s)$ and $s \in L_W(j)$. The latter condition may be stated equivalently as $k \in P_{L_W(j)}(L_W(j))$ if either $a_{ik} > 0$ (i.e., $k \in P_W(j)$), or

$$\sum_{s \in W} a_{ks}\widetilde{b}_{s1} > 0,$$

where $\widetilde{b}_{ij}$ $(i, j \in V \cup W)$ are elements of the matrix $\widetilde{B} = (I - \widetilde{A})^{-1}$. The latter is true if there exists $l \in W$ such that $a_{kl} > 0$ $(l \in W)$ (or, in other words, if $k \in P_W(l)$) and $\widetilde{b}_{l1} > 0$ (or in other words, if $l \in L_W(j)$). But this means precisely that $k \in P_W(L_W(j))$.

Thus we have shown that $x_i(c') > x_i(c'')$ if and only if $i \in L(L(P_U(V \cup L_W(j))))$, and the proof is complete. □

Now we consider the basic case when the matrix A is irreducible. Recall that a matrix A is *irreducible* (see [1]) if there does not exist a decomposition $\{1, 2, \ldots\} = P \cup Q$ such that $P, Q \neq \varnothing$ and $a_{ij} = 0$ for $i \in P$ and $j \in Q$.

Theorem 3. *Let an economical system described by equation* (0.1) *be productive, and suppose that the matrix A is irreducible. Assume that the input c_j is increased. Then the output of any product i, in the case when the levels of production of products from U do not change, is less than in the case when the levels of production of products U are not fixed.*

Proof. Since A is an irreducible matrix it is sufficient to prove that the set $P_U(V \cup L_W(j))$ is nonempty (see Theorem 2).

First, one can see that, for all $s \in W \setminus L_W(j)$ and $s \in V \cup L_W(j)$, the equality $a_{ks} = 0$ must hold. In all other case, $a_{ks} > 0$ and, since s is connected in $\widetilde{A}$ with j, s is also connected in $\widetilde{A}$ with j. But this contradicts the fact that $k \in W \setminus L_W(j)$.

Second, if $P_U(V \cup L_W(j)) = \varnothing$, it is easy to see that $a_{ks} = 0$ for $k \in U$ and $s \in V \cup L_W(j)$. So we have $a_{ks} = 0$ if either $k \in U \cup (W \setminus L_W(j))$ and $s \in V \cup L_W(j)$, or $k \in U$ and $s \in V \cup L_W(j)$). But this means that the matrix A is irreducible, by definition. $\square$

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