Stochastic analysis of a three unit standby system

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Abstract

This paper presents a stochastic analysis of a non-identical three-unit system with one-unit as standby and the other two units online by a Graphical Evaluation and Review Technique (GERT). In this model, it is assumed that failure and repair rates are constant and further we assume that standby unit functions with less efficiency. Various statistical characteristics like steady-state availability, meantime to system failure (MTSF), busy and idle time of service facility and reliability of the system have been obtained. Finally a cost analysis is also derived.

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Key words: busy-time, GERT, idle-time, MTSF, w-function.

1 Introduction

Redundancy is used to increase the reliability and availability of a system. Further standby systems play an important role in increasing the availability and reliability of the system given by Dhillon etal. and Gaver. This paper presents a GERT to a non-identical three unit systems with one unit being kept as standby. Initially two units A and B start functioning and unit C is kept as standby. When unit A and B fails, unit C starts functioning automatically, if available, and the failed unit is sent for repair. A single repair facility is available. Once unit A or B is repaired, it starts functioning replacing unit C as unit C functions with less efficiency. For the functioning of the system two units should be operating. When two units fail, the system is down and is replaced with a new system at rate μ.

2 Assumptions and notations

1. The failure, repair and replacement times are constant.

2. When two units are inoperative, the system is down.

3. After repair, the units are as good as new.

$\lambda_a, \lambda_b\,(\mu_a, \mu_b)$	constant failure repair rates for units A and B.
μ	constant replacement rate from states S_4, S_5 and S_6 to S_1.

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3 Description of the system

The GERT network of the above system is given in figure 1.

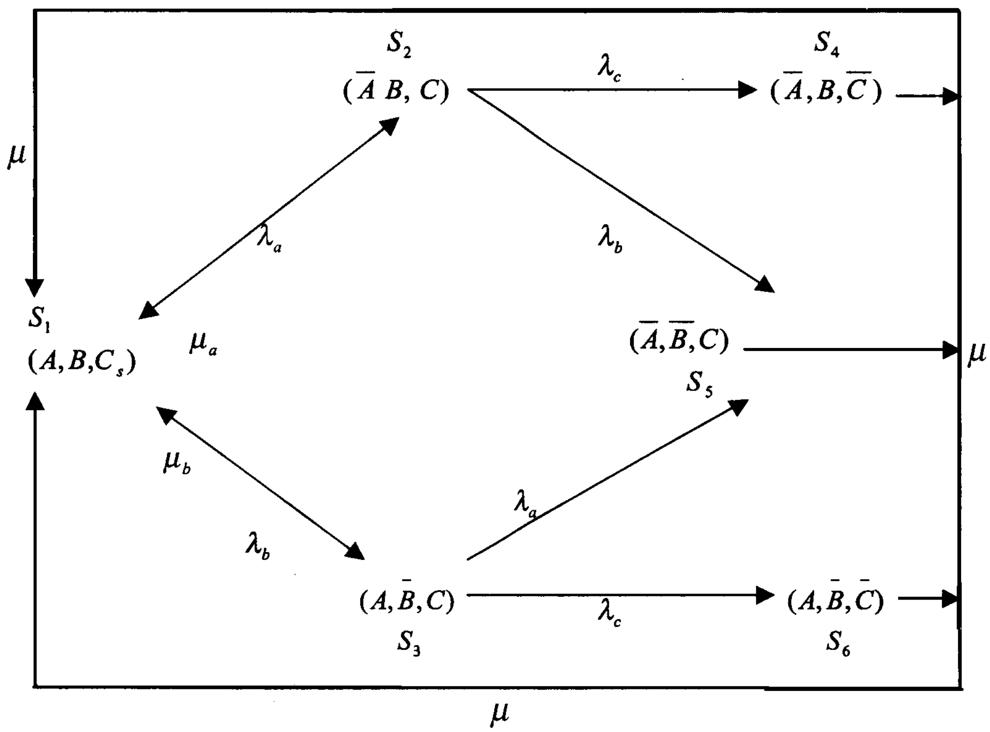

Figure 1: Possible states of the system

The possible states of the system are as follows.

S_1 : Both the units A and B are operating online with unit C as standby
$S_3(S_2)$: Unit B (Unit A)has failed and units A(Unit B) and C are operating.
S_4 : Units A and C have failed.
S_5 : Units A and B have failed
S_6 : Units B and C have failed

The state dependent of λ's and μ's are given by

Node i	1	2	3	4	5	6
λ_i	$\lambda_a + \lambda_b$	$\lambda_b + \lambda_c$	$\lambda_a + \lambda_c$	-	-	-
μ_i	$-$	μ_0	μ_b	μ	μ	μ

Now, by using these state dependent λ_i and μ_i it can be observed (see for example Whitehouse) that the time taken to move from node i to node j is distributed exponentially with mean $\frac{1}{\lambda_i+\mu_i}$. Hence the probability of moving from node i to node j is given by

$$\rho_{ij} = \begin{cases} \dfrac{\lambda_{ij}}{\lambda_i + \mu_i} & for\ j > i \\ \dfrac{\mu_{ij}}{\lambda_i + \mu_i} & for\ j < i \end{cases}$$

where $\lambda_{ij} + (\mu_{ij})$ is the failure(repair) rate from node i to node j. Therefore the W-function from node i to node j is given by

$$W_{ij}(s) = \rho_{ij}\left(1 - \frac{s}{\lambda_i + \mu_i}\right)^{-1}.$$

4 Determination of MTSF

The mean time to system failure is defined as the time for the system to be completely inoperative. This is obtained by forming the W-function from the initial node to the terminal node 4, 5 or 6. By applying Mason's rule to figure 1 we obtain

$$W(s) = \frac{W_{12}(s)\left[W_{24}(s) + W_{25}(s)\right] + W_{13}(s)\left[W_{25}(s) + W_{36}(s)\right]}{1 - W_{12}(s)\,W_{21}(s) - W_{13}(s)\,W_{31}(s)} \tag{4.1}$$

where the W- functions are given by

$$\begin{aligned}
W_{12}(s) &= \frac{\lambda_a}{\lambda_a + \lambda_b}\left(1 - \frac{s}{\lambda_a + \lambda_b}\right)^{-1} \\
W_{13}(s) &= \frac{\lambda_b}{\lambda_a + \lambda_b}\left(1 - \frac{s}{\lambda_a + \lambda_b}\right)^{-1} \\
W_{21}(s) &= \frac{\mu_a}{\mu_a + \lambda_b + \lambda_c}\left(1 - \frac{s}{\mu_a + \lambda_b + \lambda_c}\right)^{-1} \\
W_{24}(s) &= \frac{\lambda_c}{\mu_a + \lambda_b + \lambda_c}\left(1 - \frac{s}{\mu_a + \lambda_b + \lambda_c}\right)^{-1} \\
W_{25}(s) &= \frac{\lambda_b}{\mu_a + \lambda_b + \lambda_c}\left(1 - \frac{s}{\mu_a + \lambda_b + \lambda_c}\right)^{-1} \\
W_{31}(s) &= \frac{\mu_b}{\lambda_a + \mu_b + \lambda_c}\left(1 - \frac{s}{\lambda_a + \mu_b + \lambda_c}\right)^{-1} \\
W_{35}(s) &= \frac{\lambda_a}{\lambda_a + \mu_b + \lambda_c}\left(1 - \frac{s}{\lambda_a + \mu_b + \lambda_c}\right)^{-1} \\
W_{36}(s) &= \frac{\lambda_c}{\lambda_a + \mu_b + \lambda_c}\left(1 - \frac{s}{\lambda_a + \mu_b + \lambda_c}\right)^{-1} \quad and \\
W_{41}(s) &= W_{51}(S) = W_{61}(s) = \left(1 - \frac{s}{\mu}\right)^{-1}.
\end{aligned}$$

Now the mean time to system failure is given by

$$T_1 = \left[\frac{W'(s)}{W(s)}\right]_{s=0} = \frac{(\mu_a + \lambda_b + \lambda_c)(\lambda_a + \mu_b + \lambda_c) + \lambda_a(\lambda_a + \mu_b + \lambda_c) + \lambda_b(\mu_a + \lambda_b + \lambda_c)}{\lambda_a(\lambda_b + \lambda_c)\lambda_a(\lambda_a + \mu_b + \lambda_c) + \lambda_b(\lambda_a + \lambda_c)(\mu_a + \lambda_b + \lambda_c)}. \tag{4.2}$$

5 Idle-time and busy-time of service facility

Idle-time: When the system is in state S_1 the service facility is idle. This is obtained by taking, the moment generating function (m.g.f) of all paths not emanating from state S_1 to be unity in (4.1). Thus the necessary W- function is given by

$$W(s) = \frac{\frac{\lambda_a}{\lambda_a+\lambda_b}\left(\frac{\lambda_b+\lambda_c}{\mu_a+\lambda_b+\lambda_c}\right) + \frac{\lambda_b}{\lambda_a+\lambda_b}\left(\frac{\lambda_a+\lambda_c}{\lambda_a+\mu_b+\lambda_c}\right)\left(1-\frac{s}{\lambda_a+\lambda_b}\right)^{-1}}{1-\left(\frac{\lambda_b}{\lambda_a+\lambda_b}\frac{\mu_a}{\mu_a+\lambda_b+\lambda_c} + \frac{\lambda_b}{\lambda_c+\lambda_b}\frac{\mu_b}{\lambda_a+\mu_b+\lambda_c}\right)\left(1-\frac{s}{\lambda_a+\lambda_b}\right)^{-1}}.$$

Hence the idle-time T_2 of the service facility is given by

$$T_2 = \left[\frac{W'(s)}{W(s)}\right]_{s=0} = \frac{1}{(\lambda_a+\lambda_b)X} \tag{5.1}$$

where X is given by

$$X = \frac{\lambda_a(\lambda_b+\lambda_c)(\lambda_a+\mu_b+\lambda_c) + \lambda_b(\lambda_a+\lambda_c)(\mu_a+\lambda_b+\lambda_c)}{(\lambda_a+\lambda_b)(\mu_a+\lambda_b+\lambda_c)(\lambda_a+\mu_b+\lambda_c)}. \tag{5.2}$$

Further the busy-time of the service facility is given by

$$\frac{1}{X}\left[\frac{1}{\lambda_a+\lambda_b}\left(\frac{\lambda_a}{\mu_a+\lambda_a+\lambda_c} + \frac{\lambda_b}{\lambda_a+\mu_b+\lambda_c}\right)\right]. \tag{5.3}$$

Hence the operative utilization of the service facility is given by

$$OU = \frac{\lambda_a(\lambda_a+\mu_b+\lambda_c) + \lambda_b(\mu_a+\lambda_b+\lambda_c)}{\lambda_a(\lambda_a+\mu_b+\lambda_c) + \lambda_b(\mu_a+\lambda_b+\lambda_c) + (\mu_a+\lambda_b+\lambda_c)(\lambda_a+\mu_b+\lambda_c)}. \tag{5.4}$$

6 Total time of regeneration

This is the first moment of the moment generating function (m.g.f.) representing the time from start to return to any node of the system. The mean recurrence time from state S_1 to state S_1 is obtained from the $W-$function given by

$$\begin{aligned} W(s) = W_{11}(s) = {} & W_{12}(s)W_{21}(s) + W_{13}(s)W_{31}(s) + W_{12}(s) \\ & [W_{24}(s)W_{41}(s) + W_{25}(s)W_{51}(s)] \\ & + W_{13}(s)[W_{35}(s)W_{51}(s) + W_{36}(s)W_{61}(s)] \end{aligned}$$

The time spent in state i is obtained by equating to 1, the m.g.f. of all states not emanating from that state i. Therefore the total time of regeneration is $T = \left[\frac{W_{11}'(s)}{W_{11}(s)}\right]_{s=0}$ and hence

$$\begin{aligned} T &= \frac{1}{\lambda_a+\lambda_b} + \frac{\lambda_a}{(\lambda_a+\lambda_b)(\mu_a+\lambda_b+\lambda_c)} + \frac{\lambda_b}{(\lambda_a+\lambda_b)(\lambda_a+\mu_b+\lambda_c)} \\ & + \frac{1}{\mu}\left[\frac{\lambda_a(\lambda_b+\lambda_c)}{(\lambda_a+\lambda_b)(\mu_a+\lambda_b+\lambda_c)} + \frac{\lambda_b(\lambda_a+\lambda_c)}{(\lambda_a+\lambda_b)(\lambda_a+\mu_b+\lambda_c)}\right]. \end{aligned} \tag{6.1}$$

Now the time spent in state S_1 is given by the W-function as $W(s) = \left(1 - \frac{s}{\lambda_a + \lambda_b}\right)^{-1}$. Hence T_1, the time spent in state S_1 is given by $T_1 = \frac{1}{\lambda_a + \lambda_b}$.

Similarly time spent in states S_2, S_3, S_4, S_5 and S_6 are given by

$$\begin{aligned}
T_2 &= \frac{\lambda_a}{(\lambda_a + \lambda_b)(\mu_a + \lambda_b + \lambda_c)} \\
T_3 &= \frac{\lambda_b}{(\lambda_a + \lambda_b)(\lambda_a + \mu_b + \lambda_c)} \\
T_4 &= \frac{\lambda_a \lambda_c}{(\lambda_a + \lambda_b)(\mu_a + \lambda_b + \lambda_c)} \quad \frac{1}{\mu} \\
T_5 &= \frac{\lambda_a \lambda_b}{(\lambda_a + \lambda_b)} \left[\frac{1}{\mu_a + \lambda_b + \lambda_c} + \frac{1}{\lambda_a + \mu_b + \lambda_c}\right] \frac{1}{\mu} \\
T_6 &= \frac{\lambda_b \lambda_c}{(\lambda_a + \lambda_b)} . \frac{1}{(\lambda_a + \mu_b + \lambda_c)} \frac{1}{\mu}.
\end{aligned}$$

Hence we obtain the steady-state availability as $\frac{T_1+T_2+T_3}{T}$

(6.2)
$$= \frac{\mu\left((\lambda_a + \mu_b + \lambda_c)(\mu_a + \lambda_b + \lambda_c) + \lambda_a(\lambda_a + \mu_b + \lambda_c) + \lambda_b(\mu_a + \lambda_b + \lambda_c)\right)}{\begin{array}{l}(\mu\left((\lambda_a + \mu_b + \lambda_c)(\mu_a + \lambda_b + \lambda_c) + \lambda_a(\lambda_a + \mu_b + \lambda_c) + \lambda_b(\mu_a + \lambda_b + \lambda_c)\right) \\ +\lambda_a(\lambda_b + \lambda_c)(\lambda_a + \mu_b + \lambda_c) + \lambda_b(\lambda_a + \lambda_c)(\mu_a + \lambda_b + \lambda_c))\end{array}}.$$

Now taking $\lambda_a = \lambda_b = \lambda, \quad \lambda_c = \lambda', \quad \mu_a = \mu_b = \mu'$

(6.3) we obtain $$MTSF = \frac{3\lambda + \lambda' + \mu'}{2\lambda(\lambda + \lambda')}.$$

(6.4) Busy-time of the service facility $$= \frac{1}{\lambda + \lambda'}.$$

(6.5) Idle-time of the service facility $$= \frac{\mu' + \lambda + \lambda'}{2\lambda(\lambda + \lambda')}$$

with total time of regeneration as

(6.6) $$\left[\frac{1}{2\lambda} + \frac{1}{\lambda + \lambda' + \mu'} + \frac{1}{\mu}\left(\frac{\lambda + \lambda'}{\lambda + \lambda' + \mu'}\right)\right].$$

(6.7) Here the availability is given by $$\frac{\mu(3\lambda + \lambda' + \mu')}{\mu(3\lambda + \lambda' + \mu') + 2\lambda(\lambda + \lambda')}$$

and the reliability of the system is obtained by taking $\mu_a = \mu_b = \mu' = 0$.

(6.8) Hence reliability of the system $$= \frac{\mu(3\lambda + \lambda')}{\mu(3\lambda + \lambda') + 2\lambda(\lambda + \lambda')}$$

7 Special case

In the absence of standby we have

$$W_{12}(s) = \frac{\lambda_a}{\lambda_a + \lambda_b}\left(1 - \frac{s}{\lambda_a + \lambda_b}\right)^{-1}$$

$$W_{13}(s) = \frac{\lambda_b}{\lambda_a + \lambda_b}\left(1 - \frac{s}{\lambda_a + \lambda_b}\right)^{-1}$$

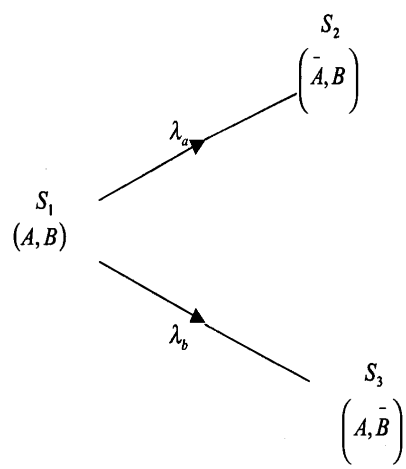

Figure 2: Possible states in the absence of standby

In this case the MTSF is given by $W(s) = W_{12}(s) + W_{13}(s)$ with $MTSF = \dfrac{1}{\lambda_a + \lambda_b}$ and obviously $MTSF = \dfrac{1}{2\lambda}$ for $\lambda_a = \lambda_a = \lambda$ and $\dfrac{3\lambda + \mu + \lambda'}{2\lambda(\lambda + \lambda')} > \dfrac{1}{2\lambda}$.

8 Cost analysis and total revenue

Let c, c_1, and c' be the cost of carrying the standby, the cost of repairing of units A and B, and the cost of repair of unit C respectively. Now the W-function of the total cost of regeneration is given by

$$\begin{aligned} W(\theta) &= W_{12}(\theta) W_{21}(\theta) + W_{13}(\theta) W_{31}(\theta) + W_{12}(\theta) W_{24}(\theta) W_{41}(\theta) \\ &+ W_{12}(\theta) W_{25}(\theta) W_{51}(\theta) + W_{13}(\theta) W_{35}(\theta) \\ & W_{51}(\theta) + W_{13}(\theta) W_{36}(\theta) W_{61}(\theta) \end{aligned} \tag{8.1}$$

where

$$
\begin{aligned}
W_{12}(\theta) &= W_{13}(\theta) = \frac{1}{2}e^{c\theta} \\
W_{21}(\theta) &= W_{31}(\theta) = \frac{\mu'}{\mu'+\lambda+\lambda'}\ e^{c_1\theta} \\
W_{24}(\theta) &= W_{25} = (\theta) = W_{35}(\theta) = W_{36}(\theta)\,\frac{\lambda'}{\lambda'+\lambda+\mu}e^{c_1\theta} \\
\text{and}\quad W_{24}(\theta) &= W_{51} = (\theta) = W_{61}(\theta)\,e^{c'\theta}
\end{aligned}
$$

Substituting for the W-function in (8.1) we obtain

$$
\mathrm{W}(\theta) = \frac{\mu'}{\lambda+\lambda'+\mu'}\ e^{(c_1+c)\theta} + \frac{(\lambda+\lambda')}{\lambda+\lambda'+\mu'}\ e^{\left(c+c_1+c'\right)\theta} \quad \text{with} \quad W(0)=1.
$$

Hence the total cost is given by

$$
\begin{aligned}
\left[\frac{W'(\theta)}{W(\theta)}\right]_{\theta=0} &= \frac{\mu'}{\mu'+\lambda+\lambda'}(c_1+c) + \frac{\lambda+\lambda'}{\lambda+\lambda'+\mu}(c+c'+c_1) \\
&= c+c_1+\frac{\lambda+\lambda'}{\lambda+\lambda'+\mu'}\ c'.
\end{aligned}
$$

Further let c, c' and c_1 denote respectively the revenue when units A and B are operating, when units A or B and C are operating and repair cost for A and B. Hence the W-functions for the revenue are given by

$$
\begin{aligned}
&W_{12}(\theta) = W_{13}(\theta) = \frac{1}{2}e^{c\theta} && W_{35}(\theta) = \frac{\lambda}{\lambda+\lambda'+\mu'}e^{c_1\theta} \\
&W_{24}(\theta) = \frac{\lambda}{\lambda+\lambda'+\mu'}e^{c_1\theta} && W_{36}(\theta) = \frac{\lambda'}{\lambda+\lambda'+\mu'}e^{c_1\theta} \\
&W_{25}(\theta) = \frac{\lambda}{\lambda+\lambda'+\mu'}e^{c_1\theta} && W_{31}(\theta) = \frac{\mu}{\lambda+\lambda'+\mu'}e^{c_1\theta} \\
&W_{21}(\theta) = \frac{\mu'}{\lambda+\lambda'+\mu'}e^{c_1\theta} && W_{41}(\theta) = W_{51}(\theta) = W_{61}(\theta) = e^{c'\theta}.
\end{aligned}
$$

Therefore the total revenue during the regeneration period is given by

$$
\begin{aligned}
W(\theta) = W_{12}(\theta)\,W_{21}(\theta) &+ W_{13}(\theta)\,W_{31}(\theta) \\
&+ W_{12}(\theta)\left(W_{24}(\theta)\,W_{41}(\theta) + W_{25}(\theta)\,W_{51}(\theta)\right) \\
&+ W_{13}(\theta)\left(W_{35}(\theta)\,W_{51}(\theta) + W_{36}(\theta)\,W_{61}(\theta)\right).
\end{aligned}
$$

Substituting for the W-functions, we have the revenue during total regeneration as

$$
R = c + c_1 - \frac{c'(\lambda+\lambda')}{\lambda+\lambda'+\mu'}.
$$

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