On $\alpha-$distance in three dimensional space

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Abstract. In this study we extend the concept $\alpha-$distance, which is a generalization of both of taxicab distance and chinese checker distance, to three dimensional space.

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1 Introduction

During the recent years, Taxicab geometry and Chinese Checker geometry have been studied and developed in many directions (see [1], [2], [3], [4], [5], [6]). Tian [5] gave a generalization of both of Taxicab and Chinese Checker distances in the plane, and named it as $\alpha-$distance. In this work we extend $\alpha-$distance to three dimensional case.

Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be two points in $\mathbb{R}^3$. Denote

$$\Delta_{P_1P_2} = \max\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\} \qquad \text{and}$$

$$\delta_{P_1P_2} = \min\{|x_1 - x_2| + |y_1 - y_2|, |x_1 - x_2| + |z_1 - z_2|, |y_1 - y_2| + |z_1 - z_2|\}.$$

The Taxicab distance and Chinese Checker distance between P_1 and P_2 are

$$d_T(P_1, P_2) = \Delta_{P_1P_2} + \delta_{P_1P_2} \quad \text{and} \quad d_c(P_1, P_2) = \Delta_{P_1P_2} + (\sqrt{2} - 1)\delta_{P_1P_2}$$

respectively.For each $\alpha \in [0, \pi/4]$,the α-distance between P_1 and P_2 is defined by

$$d_\alpha(P_1, P_2) = \Delta_{P_1P_2} + (\sec\alpha - \tan\alpha)\delta_{P_1P_2}\ .$$

Notice that $d_0(P_1, P_2)=d_T(P_1, P_2)$ and $d_{\frac{\pi}{4}}(P_1, P_2)=d_c(P_1, P_2)$. Also if $\delta_{P_1P_2} > 0$, then

$$d_E(P_1, P_2) < d_c(P_1, P_2) < d_\alpha(P_1, P_2) < d_T(P_1, P_2) \quad \text{for all } \alpha \in (0, \pi/4)\ .$$

If $\delta_{P_1P_2} = 0$, then P_1 and P_2 lie on a line which is parallel to one of coordinate axes, and

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$$d_c(P_1,P_2) = d_\alpha(P_1,P_2) = d_T(P_1,P_2) = d_E(P_1,P_2) \quad \text{for all } \alpha \in [0,\pi/4]$$

where d_E stands for the Euclidean distance.

Obviously $d_\alpha(P_1,P_2) = 0$ if and only if $P_1 = P_2$ and $d_\alpha(P_1,P_2) = d_\alpha(P_2,P_1)$ for all $P_1,P_2 \in \mathbb{R}^3$. Now, we try to prove that

$$d_\alpha(P_1,P_2) \leq d_\alpha(P_1,P_3) + d_\alpha(P_3,P_2)$$

for all $P_1,P_2,P_3 \in \mathbb{R}^3$ and $\alpha \in [0,\pi/4]$. $S_{P_1P_2}$ denote the region bounded by rectangular prism with diagonal P_1P_2 for two points $P_1 = (x_1,y_1,z_1)$, $P_2 = (x_2,y_2,z_2) \in \mathbb{R}^3$. The next two propositions follow directly from the definition of the $\alpha-$distance:

Proposition 1. *The $\alpha-$distance is invariant under all translation in R^3. That is, $T : R^3 \to R^3 \ni T(x,y,z) = (x+a, y+b, z+c)$, $a,b,c \in R$ does not change the distance between two any points in R^3.*

Let P_1,P_2,P_3 and P_4 be four points in $\mathbb{R}^3$. As a consequence of Proposition1, if $S_{P_1P_2}$ and $S_{P_3P_4}$ are congruent, then $d_\alpha(P_1,P_2)=d_\alpha(P_3,P_4)$ for all $\alpha \in [0,\pi/4]$.

Proposition 2. *Let P_1 and P_2 be two points in R^3. Then $d_\alpha(P_1,P_2) \geq d_\alpha(P_3,P_4)$ for all $P_3,P_4 \in S_{P_1P_2}$ and $\alpha \in [0,\pi/4]$.*

Notice that, according to positions of P_1 and P_2 in $\mathbb{R}^3$, the three cases of d_α are possible:

$$d_\alpha(P_1,P_2) = \begin{cases} |x_1-x_2| + (\sec\alpha\text{-}\tan\alpha)\,(|y_1-y_2|+|z_1-z_2|) & \text{, if } |x_1-x_2| \text{ is max} \\ |y_1-y_2| + (\sec\alpha\text{-}\tan\alpha)\,(|x_1-x_2|+|z_1-z_2|) & \text{, if } |y_1-y_2| \text{ is max} \\ |z_1-z_2| + (\sec\alpha\text{-}\tan\alpha)\,(|x_1-x_2|+|y_1-y_2|) & \text{, if } |z_1-z_2| \text{ is max} \end{cases} .$$

Proposition 3. *Let P_1 and P_2 be any two points in R^3. a,b,c denote values of $d_\alpha(P_1,P_2)$ for $|x_1-x_2|$, $|y_1-y_2|$, $|z_1-z_2|$, respectively. Then*

$$\begin{array}{ll} a \geq b \text{ and } a \geq c & \text{if } \Delta_{P_1P_2} = |x_1-x_2| \ , \\ b \geq a \text{ and } b \geq c & \text{if } \Delta_{P_1P_2} = |y_1-y_2| \ , \\ c \geq a \text{ and } c \geq b & \text{if } \Delta_{P_1P_2} = |z_1-z_2| \ . \end{array}$$

Proof. Let $P_1=(x_1,y_1,z_1)$ and $P_2=(x_2,y_2,z_2)$. Denote $q=\sec\alpha - \tan\alpha$. If $\Delta_{P_1P_2} = |x_1-x_2|$, then

$$\begin{aligned} a &= |x_1-x_2| + q\,(|y_1-y_2| + |z_1-z_2|) \\ &= |y_1-y_2| + q\,(|x_1-x_2| + |z_1-z_2|) - (1-q)\,|y_1-y_2| + (1-q)\,|x_1-x_2| \\ &= b + (1-q)(|x_1-x_2| - |y_1-y_2|) \ . \end{aligned}$$

Notice that $(1-q) \geq 0$ for all $\alpha \in [0,\pi/4]$ and $(|x_1-x_2| - |y_1-y_2|) \geq 0$. Thus

$$|y_1-y_2| + q\,(|x_1-x_2| + |z_1-z_2|) \leq |x_1-x_2| + q\,(|y_1-y_2| + |z_1-z_2|) \ .$$

That is, $a \geq b$. Similarly $a \geq c$. Similar proofs can easily given for the remaining cases.□

Theorem 4. *Let P_1 and P_2 be any two points in R^3 and $\alpha \in [0,\pi/4]$. Then,*

$$d_\alpha(P_1,P_2) \leq d_\alpha(P_1,P_3) + d_\alpha(P_3,P_2)$$

for all $P_3 \in R^3$.

Proof. Clearly, the result holds when $\delta_{P_1P_2}$=0. Suppose that $\delta_{P_1P_2}$>0. By Proposition 1, without loss of generality, assume that P_1 lies on the origin, and $P_2=(x_2, y_2, z_2)$ with $x_2>y_2>z_2>0$. Let $A=(x_2-y_2, 0, 0)$, $B=(x_2-\frac{z_2}{\sqrt{2}}, y_2-\frac{z_2}{\sqrt{2}}, 0)$, $C=(x_2-y_2\tan\alpha, 0, 0)$, $D=(x_2-\frac{z_2}{\sqrt{1+\tan^2\alpha}}, y_2-\frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}}z_2, 0)$,
$E=(x_2, y_2, 0)$, $F=(x_2, 0, 0)$ and $G=(x_2, 0, z_2)$.

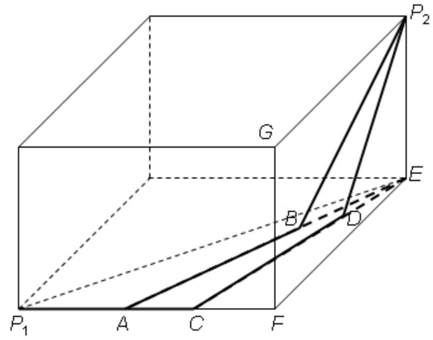

Figure 1

Now consider the triangular regions $R_1=P_1\overset{\triangle}{E}A$, $R_2=A\overset{\triangle}{E}C$, $R_3=C\overset{\triangle}{E}F$ and the region $R_4=\{(x,y,0) : x \geq 0 \text{ and } y<0, \text{ or } x>x_2 \text{ and } y \geq 0\}$ in $xy-$plane. Let $\mathcal{P}$ denote the plane which pass through the points P_1, P_2 and G. Let K_1, K_2 and K_3 be subset of $S_{P_1P_2}$ such that they lie in between the $\mathcal{P}$ and $xy-$plane; and the orthogonal projections of them to $xy-$plane are R_1, R_2 and R_3, respectively. Let K_4 denote the subset of points of $\mathbb{R}^3$ which are on the same side of the plane $\mathcal{P}$ such that orthogonal projection of K_4 is in R_4 (see Figure 1). Now, it suffices to prove the result for $P_3 \in K_1 \cup K_2 \cup K_3 \cup K_4$.

Case I. Assume that $P_3=(x_3,y_3,z_3) \in K_1$. In this case, it is easily seen that $x_3 \geq y_3$, $x_3 \geq z_3$, $x_2-x_3 \geq y_2-y_3$ and $x_2-x_3 \geq z_2-z_3$ or $x_2-x_3<z_2-z_3$. Thus,

$$d_\alpha(P_1,P_3)=x_3+q(y_3+z_3)$$

and

$$d_\alpha(P_3,P_2)=(x_2-x_3)+q(y_2-y_3+z_2-z_3)$$

or

$$d_\alpha(P_3,P_2)=(z_2-z_3)+q(x_2-x_3+y_2-y_3)\ .$$

Thus,

$$\begin{array}{rrcl}
\textbf{i)} & d_\alpha(P_1,P_3)+d_\alpha(P_3,P_2) & = & x_2+q(y_2+z_2)=d_\alpha(P_1,P_2) \\
\textbf{ii)} & d_\alpha(P_1,P_3)+d_\alpha(P_3,P_2) & = & x_3+q(y_3+z_3)+(z_2-z_3)+q(x_2-x_3+y_2-y_3) \\
 & & = & x_2+q(y_2+z_2)+(q-1)((x_2-x_3)-(z_2-z_3)) \\
 & & = & d_\alpha(P_1,P_2)+(q-1)((x_2-x_3)-(z_2-z_3)) \\
 & & \geq & d_\alpha(P_1,P_2)
\end{array}$$

where (1-q)((z_2-z_3)-(x_2-x_3)) ≥ 0. That is, $d_\alpha(P_1,P_3)+d_\alpha(P_3,P_2)\geq d_\alpha(P_1,P_2)$ by Proposition 3.

Case II. Assume that $P_3=(x_3,y_3,z_3)\in K_2$. Let P_3' be orthogonal projection of P_3 onto $xy-$plane. Consider the line segments through P_3' and parallel to the line $\overleftrightarrow{CD}$ and $x-$axis, which intersect the lines $\overleftrightarrow{P_1C}$ and $\overleftrightarrow{CD}$ at the points A_1, A_2, respectively. Draw a line segment parallel to the DP_2 through P_3 which intersects the line segment A_1P_3' at A_3. Now, consider the rectangle $P_3P_3'A_2A_4$. Draw line segment parallel to the CD through A_4 which intersects line segment DP_2 at A_5 (see Figure 2). It is easily seen that $|A_1C|=\left|P_3'A_2\right|=|P_3A_4|$ and $|A_2D|+\left|A_3P_3'\right|=|A_4A_5|$. Also $d_\alpha(P_1,P_3)=|P_1A_1|+|A_1A_3|+|A_3P_3|$ and

$$
\begin{aligned}
d_\alpha(P_1,P_2) &= |P_1C|+|CD|+|DP_2|\\
&= |P_1A_1|+|A_1C|+|A_1A_3|+|A_4A_5|+|A_3P_3|+|A_5P_2|\\
&= |P_1A_1|+|A_1A_3|+|A_3P_3|+|A_1C|+|A_4A_5|+|A_5P_2|\\
&= d_\alpha(P_1,P_3)+|P_3A_4|+|A_4A_5|+|A_5P_2|\ .
\end{aligned}
$$

Thus, $|P_3A_4|+|A_4A_5|+|A_5P_2|\leq d_\alpha(P_3,P_2)$ by proposition 3. Therefore $d_\alpha(P_1,P_2)\leq d_\alpha(P_1,P_3)+d_\alpha(P_3,P_2)$.

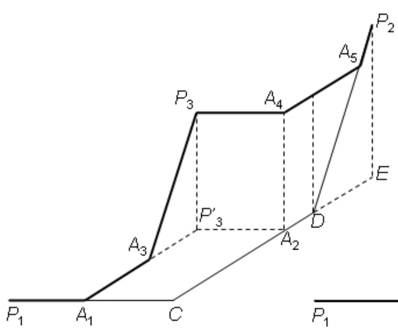

Figure 2

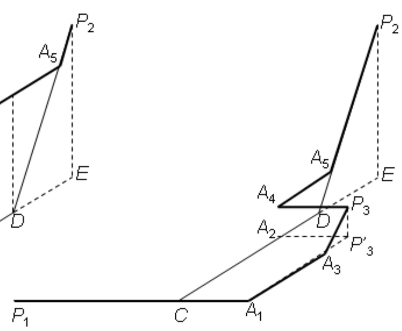

Figure 3

Case III. Assume that $P_3=(x_3,y_3,z_3)\in K_3$. Similarly A_i $(i=1,2,\ldots,5)$ points can be obtained as in Case II (see Figure 3). Similarly, it follows that

$$
\begin{aligned}
d_\alpha(P_1,P_2) &= |P_1C|+|CD|+|DP_2|\\
&= |P_1C|+\left|A_1P_3'\right|+|A_4A_5|+|A_3P_3|+|A_5P_2|\\
&\leq |P_1C|+|CA_1|+|A_1A_3|+|A_3P_3|+|P_3A_4|+|A_4A_5|+|A_5P_2|\\
&\leq d_\alpha(P_1,P_3)+d_\alpha(P_3,P_2)\ .
\end{aligned}
$$

Case IV. Assume that $P_3=(x_3,y_3,z_3)\in K_4$. Let $P_4=(\min\{x,x_2\},\min\{\max\{0,y\},y_2\},0)$. P_4 lies on the line segment P_1F and FE. By proposition 2,

$$d_\alpha(P_1,P_4)\leq d_\alpha(P_1,P_3)\qquad\text{and}\qquad d_\alpha(P_4,P_2)\leq d_\alpha(P_3,P_2)\ .$$

Based on the result from Case II and Case III,

$$d_\alpha(P_1,P_2)\leq d_\alpha(P_1,P_4)+d_\alpha(P_4,P_2)\leq d_\alpha(P_1,P_3)+d_\alpha(P_3,P_2)\ .$$

□

References

[1] Akca, Z., Kaya, R., *On The Distance Formulae in Three Dimensional Taxicab Space*, Hadronic Journal, Vol. 27**,** No. 5 (2004), 521-532 .

[2] Akca, Z., Kaya, R., *On the Norm in Higher Dimensional Taxicab Spaces*, Hadronic Journal Supplement, Vol. 19 (2004), 491-501 .

[3] Gelişgen, Ö., Kaya, R., Özcan, M., *Distance Formulae in the Chinese Checker Space,* Int. Jour. of Pure and Appl. Math. (IJPAM),To appear.

[4] Kaya, R., Gelişgen, Ö., Ekmekci, S.,Bayar, A., *Group of Isometries of CC-Plane*, Missouri Journal of Mathematical Sciences,To appear.

[5] Tian, S., *Alpha Distance-A Generalization of Chinese Checker Distance and Taxicab Distance*, Missouri Journal of Mathematical Sciences, Vol. 17 , No. 1 (2005), 35-40 .

[6] Turan, M., Özcan, M., *Two-foci CC-Ellipses*, Int. Jour. of Pure and Appl. Math. (IJPAM), Vol.16, No.1 (2004), 119-127 .

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