On a class of generalized quasi-Einstein manifolds

Cihan Ö zgür

Abstract. In this study, we find the necessary conditions in order that a special class of generalized quasi-Einstein manifolds to be pseudo Ricci-symmetric and $R$-harmonic. We also consider these type manifolds with cyclic parallel Ricci tensor.

Key words: Einstein, quasi-Einstein, generalized quasi-Einstein, pseudo Ricci-symmetric, $R$-harmonic manifold.

1 Introduction

A non-flat Riemannian manifold $(M^n, g)$, $n = \dim M \geq 3$, is said to be an Einstein if the condition $S = \kappa g$ is fulfilled on $M^n$, where $S$ and $\kappa$ denote the Ricci tensor and the scalar curvature of $(M^n, g)$ respectively.

A non-flat Riemannian manifold $(M^n, g)$, $n \geq 3$, is defined to be quasi-Einstein if its Ricci tensor $S$ is not identically zero and satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y),$$

where $a$, $b$ are scalars of which $b \neq 0$ and $A$ is non-zero 1-form such that $g(X, U) = A(X)$ for every vector field $X$ and $U$ is a unit vector field.

In [2], [3], [4] and [8], the authors studied quasi-Einstein manifolds and gave some examples of quasi-Einstein manifolds. In [6] and [7], quasi-Einstein hypersurfaces in semi-Euclidean spaces and semi-Riemannian space forms were considered, respectively.

A non-flat Riemannian manifold $(M^n, g)$, $n \geq 3$, is called generalized quasi-Einstein if its Ricci tensor $S$ is non-zero and satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y),$$

where $a, b, c$ are certain non-zero scalars and $A, B$ are two non-zero 1-forms defined by

$$g(X, U) = A(X), \quad g(X, V) = B(X)$$
On a class of generalized quasi-Einstein manifolds

139

and the unit vector fields $U$ and $V$ are orthogonal, i.e., $g(U, V) = 0$. The vector fields $U$ and $V$ are called the generators of the manifold. If $c = 0$ then the manifold reduces to a quasi-Einstein manifold (see [5]).

In [5], U. C. De and G. C. Ghosh studied generalized quasi-Einstein manifolds and as an example they showed that a 2-quasi-umbilical hypersurface of the Euclidean space is generalized quasi-Einstein.

In this study, we consider a special class of generalized quasi-Einstein manifolds such that the generators $U$ and $V$ are parallel vector fields.

2 Preliminaries

A non-flat Riemannian manifold $(M^n, g)$ is called pseudo-Ricci symmetric (see [1]) and $R$-harmonic (see [9]) if the Ricci tensor $S$ of $M^n$ satisfy the following conditions

\[
(\nabla_X S)(Y, Z) = 2\alpha(X)S(Y, Z) + \alpha(Y)S(X, Z) + \alpha(Z)S(X, Y)
\]

and

\[
(\nabla_X S)(Y, Z) = (\nabla_Z S)(X, Y),
\]

respectively, where $\alpha$ is a one form, $X, Y, Z$ are vector fields on $M^n$ and $\nabla$ is the Levi-Civita connection of $M^n$.

3 Main Results

In this section, we consider generalized quasi-Einstein manifolds under the condition that $U$ and $V$ are parallel vector fields.

Suppose that $M^n$ is a generalized quasi-Einstein manifold and the vector fields $U$ and $V$ are parallel. Then $\nabla_X U = 0$ and $\nabla_X V = 0$, which implies $R(X, Y)U = 0$ and $R(X, Y)V = 0$. Hence contracting these equations with respect to $Y$ we see that $S(X, U) = 0$ and $S(X, V) = 0$. So from (1.1)

\[
S(X, U) = (a + b)A(X) = 0
\]

and

\[
S(X, V) = (a + c)B(X) = 0,
\]

which implies that $a = -b = -c$. Then the equation (1.1) turns the form

\[
S(X, Y) = a (g(X, Y) - A(X)A(Y) - B(X)B(Y)),
\]

(for more details see [5]). On the other hand, it is well-known that

\[
(\nabla_X S)(Y, Z) = \nabla_X S(Y, Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z).
\]

Since $M^n$ is a generalized quasi-Einstein manifold, by the use of (3.1) and (3.2) we can write

\[
(\nabla_X S)(Y, Z) = X[a] (g(Y, Z) - A(Y)A(Z) - B(Y)B(Z)),
\]
where \( X[a] \) denotes the derivative of \( a \) with respect to the vector field \( X \). Since \( M^n \) is pseudo Ricci-symmetric, by the use of (2.1) and (3.3), we can write

\[
X[a] (g(Y, Z) - A(Y)A(Z) - B(Y)B(Z))
= 2a\alpha(X) (g(Y, Z) - A(Y)A(Z) - B(Y)B(Z))
+ a\alpha(Y) (g(X, Z) - A(X)A(Z) - B(X)B(Z))
+ a\alpha(Z) (g(X, Y) - A(X)A(Y) - B(X)B(Y)).
\]

Taking \( X = U \) and \( X = V \) in (3.4) we find

\[
U[a] = 2a\alpha(U)
\]
and

\[
V[a] = 2a\alpha(V),
\]
respectively.

Putting \( Z = U \) and \( Z = V \) in (3.4) we have

\[
\alpha(U) = 0
\]
and

\[
\alpha(V) = 0,
\]
respectively. So in view of (3.5), (3.6), (3.7) and (3.8) we obtain

\[
U[a] = 0, \quad V[a] = 0,
\]
which implies \( a \) is constant along the vector fields \( U \) and \( V \).

Hence we can state the following theorem:

**Theorem 3.1.** Let \( M^n \) be a generalized quasi-Einstein manifold under the condition that \( U, V \) are parallel vector fields. If \( M^n \) is pseudo Ricci-symmetric then the scalar function \( a \) is constant along the vector fields \( U \) and \( V \).

Assume that \( M^n \) is a R-harmonic generalized quasi-Einstein manifold. If \( U \) and \( V \) are parallel vector fields then from (2.2) and (3.3) we have

\[
(\nabla_X S)(Y, Z) - (\nabla_Z S)(X, Y)
= X[a] (g(Y, Z) - A(Y)A(Z) - B(Y)B(Z))
- Z[a] (g(X, Y) - A(X)A(Y) - B(X)B(Y)) = 0.
\]

Then taking \( X = U \) and \( X = V \) in (3.9) we find

\[
U[a] = 0 \quad \text{and} \quad V[a] = 0,
\]
respectively, which implies that \( a \) is constant along the vector fields \( U \) and \( V \).

So we have proved the following theorem:

**Theorem 3.2.** Let \( M^n \) be a generalized quasi-Einstein manifold under the condition that \( U, V \) are parallel vector fields. If \( M^n \) is R-harmonic then the scalar function \( a \) is constant along the vector fields \( U \) and \( V \).
Now assume that $M^n$ has cyclic parallel Ricci tensor. Then

$$\nabla_X S(Y, Z) + \nabla_Y S(X, Z) + \nabla_Z S(X, Y) = 0,$$

holds on $M^n$. If $M^n$ is a generalized quasi-Einstein manifold under the condition that $U$ and $V$ are parallel vector fields then from (3.10) and (3.3) we get

$$0 = X[a](g(Y, Z) - A(Y)A(Z) - B(Y)B(Z)) + Y[a](g(X, Z) - A(X)A(Z) - B(X)B(Z)) + Z[a](g(X, Y) - A(X)A(Y) - B(X)B(Y)).$$

Taking $X = U$ in (3.11) we have $U[a] = 0$. Putting $X = V$ in (3.11) we find $V[a] = 0$. So we have the following theorem:

**Theorem 3.3.** Let $M^n$ be a generalized quasi-Einstein manifold under the condition that $U, V$ are parallel vector fields. If $M^n$ has cyclic parallel Ricci tensor then the scalar function $a$ is constant along the vector fields $U$ and $V$.

**References**


**Author’s address:**

Cihan Özgür
Balikesir University, Department of Mathematics, Faculty of Arts and Sciences, Campus of Çağış, 10145, Balikesir, Turkey.

email: cozgur@balikesir.edu.tr