A note on proper conformal vector fields in Bianchi type I space-times

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Abstract. Direct integration technique is used to study the proper conformal vector fields in non conformally flat Bianchi type-1 space-times. Using the above mentioned technique we have shown that a very special class of the above space-time admits proper conformal vector fields.

Key words: Direct integration techniques; proper conformal vector fields.

1 Introduction
This paper investigates the existence of proper conformal vector fields in Bianchi type-1 space-times by using the direct integration technique. The conformal vector field which preserves the metric structure up to a conformal factor carries significant interest in Einstein’s theory of general relativity. It is therefore important to study these symmetries.

Through out $M$ is represents a four dimensional, connected, hausdorff space-time manifold with Lorentz metric $g$ of signature $(-, +, +, +)$. The curvature tensor associated with $g_{ab}$ through Levi-Civita connection, is denoted in component form by $R^{c}_{abcd}$ and the Ricci tensor components are $R_{ab} = R^{c}_{acb}$. The usual covariant, partial and Lie derivatives are denoted by a semicolon, a comma and the symbol $L$, respectively. Round and square brackets denote the usual symmetrization and skew-symmetrization, respectively.

Any vector field $X$ on $M$ can be decomposed as

\begin{equation}
X_{a;b} = \frac{1}{2}h_{ab} + F_{ab},
\end{equation}

where $h_{ab}(= h_{ba}) = LXg_{ab}$ and $F_{ab} = -F_{ba}$ are symmetric and skew symmetric tensors on $M$, respectively. Such a vector field $X$ is called conformal vector field if the local diffeomorphisms $\psi_t$ (for appropriate $t$) associated with $X$ preserve the metric structure up to a conformal factor i.e. $\psi^*_t g = \phi_t g$, where $\phi$ is a nowhere zero positive function on $M$ and $\psi^*_t$ is a pullback map on $M$ [3]. This is equivalent to the condition that

$$h_{ab} = 2\phi g_{ab},$$
equivalently
\begin{equation}
(1.2) \quad g_{ab,c}X^c + g_{ac}X^c_{,b} + g_{bc}X^c_{,a} = 2\phi g_{ab},
\end{equation}

where $\phi : M \to \mathbb{R}$ is the smooth conformal function on $M$, then $X$ is a called conformal vector field. If $\phi$ is constant on $M$, $X$ is homothetic (proper homothetic if $\phi \neq 0$) while $\phi = 0$ it is Killing [3]. If the vector field $X$ is not homothetic then it is called proper conformal. It follows from [3] that for a conformal vector field $X$, the bivector $F$ and the function $\phi$ satisfy (putting $\phi_a = \phi_{,a}$)
\begin{equation}
(1.3) \quad F_{abc} = R_{abcd}X^d - 2\phi[a]g_{bc},
\end{equation}
\begin{equation}
(1.4) \quad \phi_{a;b} = -\frac{1}{2}L_{abc}X^c - \phi L + R_{c(a}F_{b)}^c,
\end{equation}

where $L_{ab} = R_{ab} - \frac{1}{6}Rg_{ab}$.

2 Main results

Consider a Bianchi type-1 space-time in the usual coordinate system $(t, x, y, z)$ with line element [1]
\begin{equation}
(2.1) \quad ds^2 = -dt^2 + h(t)dx^2 + k(t)dy^2 + f(t)dz^2,
\end{equation}
where $f$, $k$ and $h$ are some nowhere zero functions of $t$ only. The possible Segre type of the above space-time is $\{1,111\}$ or one of its degeneracies. It follows from [2, 4] the above space-time admits three linearly independent Killing vector fields which are
\begin{equation}
(2.2) \quad \frac{\partial}{\partial x} , \frac{\partial}{\partial y} , \frac{\partial}{\partial z}.
\end{equation}

A vector field $X$ is said to be a conformal vector field if it satisfy equation (1.2). One can write (1.2) explicitly using (2.1) we have
\begin{equation}
(2.3) \quad X^0_{,0} = \phi,
\end{equation}
\begin{equation}
(2.4) \quad -X^0_{,1} + hX^1_{,0} = 0,
\end{equation}
\begin{equation}
(2.5) \quad -X^0_{,2} + kX^2_{,0} = 0,
\end{equation}
\begin{equation}
(2.6) \quad -X^0_{,3} + fX^3_{,0} = 0,
\end{equation}
\begin{equation}
(2.7) \quad \dot{h}X^0 + 2hX^1_{,0} = 2h\phi,
\end{equation}
\begin{equation}
(2.8) \quad hX^1_{,2} + kX^3_{,1} = 0,
\end{equation}
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(2.9) \[ hX_3^1 + fX_3^1 = 0, \]
(2.10) \[ \dot{k}X^0 + 2kX_2^2 = 2k\phi, \]
(2.11) \[ kX_3^3 + fX_3^3 = 0, \]
(2.12) \[ \dot{j}X^0 + 2fX_2^2 = 2f\phi. \]

Equations (2.3), (2.4), (2.5) and (2.6) give

\[
\begin{align*}
X^0 & = \int \phi(t)dt + A^1(x, y, z), \\
X^1 & = A^1_2(x, y, z) \int \frac{dt}{k} + A^2(x, y, z), \\
X^2 & = A^2_3(x, y, z) \int \frac{dt}{k} + A^3(x, y, z), \\
X^3 & = A^3_1(x, y, z) \int \frac{dt}{f} + A^4(x, y, z),
\end{align*}
\]

where \( A^1(x, y, z), A^2(x, y, z), A^3(x, y, z) \) and \( A^4(x, y, z) \) are functions of integration. In order to determine \( A^1(x, y, z), A^2(x, y, z), A^3(x, y, z) \) and \( A^4(x, y, z) \) we need to integrate the remaining six equations. To avoid details, here we will present only the result when the above space-time (2.1) admits proper conformal vector field. It follows from the above calculations; there exist only one possibility when the above space-time (2.1) admits proper conformal vector field which is:

**Case 1:** Four conformal vector fields:

In this case the space-time (2.1) becomes

\[
(2.14) \quad ds^2 = -dt^2 + V^2(t)(e^{-2d_1 N(t)} dx^2 + e^{-2d_2 N(t)} dy^2 + e^{-2d_3 N(t)} dz^2)
\]

and conformal vector field is

\[
(2.15) \quad X^0 = V(t), X^1 = d_1 x + d_2, X^2 = d_1 y + d_1 z + d_1,
\]

where \( V(t) = \int \phi(t)dt + d_8, N(t) = \int \frac{dt}{V(t)}, d_1, d_2, d_3, d_11, d_12, d_13, d_14 \in R \) (\(d_1 \neq d_11, d_1 \neq d_13, d_13 \neq d_11, d_1 \neq 0, d_11 \neq 0, d_13 \neq 0\)) and \( \phi \) is no where zero function of \( t \) only. The above space-time (2.14) admits four independent conformal vector fields in which three are Killing vector fields which are given in (2.2) and one is proper conformal vector field which is

\[
(2.16) \quad Z = (V(t), d_1 x, d_11 y, d_13 z).
\]

One can easily check that the above vector field (2.16) is not a homothetic vector field by substituting it into the homothetic equations.

Now consider the case when \( d_11 = d_13, d_11 \neq d_1 \) and the above space-time (2.14) becomes

\[
(2.17) \quad ds^2 = -dt^2 + V^2(t)(e^{-2d_1 N(t)} dx^2 + e^{-2d_11 N(t)} (dy^2 + dz^2)).
\]

The above space-time admits five independent conformal vector fields in which four independent Killing vector fields which are: \( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \) and \( z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z} \) and one proper conformal vector field which is given in (2.16). The cases when \( d_11 = d_1, d_11 \neq d_13 \) and \( d_1 = d_13, d_11 \neq d_1 \) are exactly the same.
References


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