

# Logistic population change and the Mankiw-Romer-Weil model

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**Abstract.** This paper evaluates the dynamic effects of assuming a logistic law hypothesis for population growth into the Mankiw, Romer, Weil (hereafter MRW) neoclassical growth model [9]. Under this assumption, the model is described by a three dimensional dynamical system, which has a unique non-trivial steady state equilibrium that is a saddle point with a two dimensional stable manifold. As a result, the speed of convergence is determined by two stable roots, rather than only one as in the basic MRW model, so that its transitional adjustment paths are enriched.

**M.S.C. 2000:** 91B62.

**Key words:** MRW model; logistic population.

## 1 Introduction

Mankiw, Romer and Weil [9] have extended the Solow growth model [12] by incorporating an explicit process of human capital accumulation. In this framework, they have derived a convergence equation relating the increments of output to investment rates for both physical and human capital. This specification have allowed them to find that Solow model's predictions are consistent with empirical evidence, but the effects of saving and population growth rates are biased upward since human capital is excluded as an explanatory variable from the model.

Recently, Ferrara and Guerrini [6] have explored the implications of studying the neoclassical Solow growth model within a framework where the change over time of the labor force is given by the logistic population model. It is known that usually standard economic growth theory assumes that population grows exponentially. However, this hypothesis is realistic only for an initial period, but it cannot be valid indefinitely because population growing exponentially can be arbitrarily large. What is often observed, instead, is that, as the population grows, some members interfere with each other in competition for some critical resource. That competition diminishes the growth rate, until the population ceases to grow. Therefore, it seems reasonable for a good population model to reproduce this behavior. The logistic growth model, as proposed by Verhulst [13], is just such a model. In economic growth modelling,

this approach have been recently analyzed in different directions (see, for example, [1],[6],[4],[5],[7],[8],[3] and [10]).

The objective of this paper is to combine within the same framework the above two different research lines, that have been analyzed separately, i.e. the one studying the effects of including human capital in the Solow model, and that analyzing the role of a logistic-type population growth within the Solow model. Within this framework, we show that the economy is described by a three dimensional dynamical system, whose unique non-trivial steady-state equilibrium is saddle-point stable. The saddle-path stable system now has two negative eigenvalues, so that the stable manifold is a two dimensional locus, thereby introducing important flexibility to the convergence and transition characteristics. The crucial determinant of the asymptotic speed of convergence is the larger of the two negative eigenvalues.

## 2 The model

We consider a closed economy where the aggregate production function is represented by a Cobb-Douglas technology with constant returns to scale. More precisely, in each period  $t$ , the production is given by

$$(2.1) \quad Y = K^\alpha H^\beta L^{1-\alpha-\beta}, \quad \alpha, \beta, \alpha + \beta \in (0, 1),$$

where  $Y$  is aggregate output,  $K$  is the stock of physical capital,  $H$  is the stock of human capital, and  $L$  is labor (population). Time argument is suppressed to ease the burden of notations. All markets, both input and output markets, are assumed to be perfectly competitive, all firms are assumed to be identical. In this way, the economy can be described by a representative agent. Physical capital and human capital are assumed to be accumulating factors, i.e. the representative agent saves output to have more capital, either physical or human. Their equations of motion are given by  $\dot{K} = s_k Y - \delta K$ ,  $\dot{H} = s_h Y - \delta H$ , where  $s_k$  and  $s_h$  are the saving rates for physical capital and human capital, respectively. They are given exogenously. Note that both capital and human capital are assumed to depreciate at the same rate  $\delta$ . Following Solow [12], and Mankiw et al. [9], we rewrite income, physical and human capital in (2.1) in terms of quantities per unit of effective labor, i.e.  $y = Y/L$ ,  $k = K/L$ ,  $h = H/L$ . In particular, this gives  $y = k^\alpha h^\beta$ . If we divide both sides of the previous equations by  $L$ , we get  $\dot{K}/L = s_k y - \delta k$ ,  $\dot{H}/L = s_h y - \delta h$ . We can write  $\dot{K}/L$  and  $\dot{H}/L$  as a function of  $k$  and  $h$  by using the relations

$$\dot{k} = \frac{d(K/L)}{dt} = \frac{\dot{K}}{L} - \frac{\dot{L}}{L} k, \quad \dot{h} = \frac{d(H/L)}{dt} = \frac{\dot{H}}{L} - \frac{\dot{L}}{L} h.$$

Consequently, the two equations above yield

$$\dot{k} = s_k y - \left( \delta + \frac{\dot{L}}{L} \right) k, \quad \dot{h} = s_h y - \left( \delta + \frac{\dot{L}}{L} \right) h.$$

In standard growth models, the population growth rate  $\dot{L}/L$  is assumed to be constant, say  $n$ . The main problem of this assumption is that population grows exponentially,

$L = L_0 e^{nt}$ , and so tends to infinity as time goes to infinity, which is clearly unrealistic. Following Ferrara and Guerrini [6], we assume that  $\dot{L}/L$  is not constant, but given by the logistic law of population growth, i.e.

$$(2.2) \quad \frac{\dot{L}}{L} = a - bL \equiv n(L),$$

with  $a > b > 0$ . In addition, today's population is assumed to be normalized to one,  $L_0 = 1$ . Recall that (2.2) is a Bernoulli differential equation with solution given by  $L = ae^{at}/(a - b + be^{at})$ . In particular, this yields that  $L$  converges to  $a/b$  in the long run. Putting all these assumptions together, we can conclude that the evolution of the economy is determined by the following system of non-linear differential equations

$$(2.3) \quad \begin{cases} \dot{k} &= s_k k^\alpha h^\beta - [\delta + n(L)]k, \\ \dot{h} &= s_h k^\alpha h^\beta - [\delta + n(L)]h, \\ \dot{L} &= Ln(L). \end{cases}$$

Given  $k_0 > 0$ ,  $h_0 > 0$ , this Cauchy problem has a unique solution, say  $(k, h, L)$ , defined on  $[0, \infty)$  (see [2]).

### 3 Transitional dynamics

We are interested in characterizing a possible state of the economy in which the growth rate of per capita variables can be maintained constant forever. Such a situation, it is defined more precisely as steady state. Let us denote the steady state values of  $k, c, L$  by  $k_*, c_*, L_*$ , respectively. In our analysis, we will exclude the economically meaningless solutions such as  $k_* = 0$ ,  $c_* = 0$ , or  $L_* = 0$ .

**Proposition 3.1.** *There exists a unique steady state equilibrium  $(k_*, c_*, L_*)$ , where*

$$(3.1) \quad k_* = \left( \frac{s_k^{1-\beta} s_h^\beta}{\delta} \right)^{1/(1-\alpha-\beta)}, \quad h_* = \left( \frac{s_k^\alpha s_h^{1-\alpha}}{\delta} \right)^{1/(1-\alpha-\beta)}, \quad L_* = \frac{a}{b}.$$

*Proof.* In steady state, the left hand side of (2.3) is zero. Hence, we get the following system of equations

$$(3.2) \quad s_k k^{\alpha-1} h^\beta = \delta, \quad s_h k^\alpha h^{\beta-1} = \delta, \quad n(L) = 0.$$

It is now straightforward to show that (3.2) has a unique solution, which is exactly the one provided by (3.1).  $\square$

**Lemma 3.1.** *At steady state, the condition  $k_0/s_k = h_0/s_h$  must hold.*

*Proof.* Eliminating  $k^\alpha h^\beta$  between the equations of (2.3) shows that physical capital and human capital satisfy the equation

$$\frac{d}{dt} \left( \frac{k}{s_k} - \frac{h}{s_h} \right) = -\delta - \frac{\dot{L}}{L},$$

the solution of which is

$$(3.3) \quad \frac{k}{s_k} - \frac{h}{s_h} = \left( \frac{k_0}{s_k} - \frac{h_0}{s_h} \right) e^{-\delta t L}.$$

The statement now follows from (3.3) noting that, from (3.1),  $k_*/s_k = h_*/s_h$ .  $\square$

**Corollary 3.1.**  $\lim_{t \rightarrow \infty} \frac{k}{s_k} = \lim_{t \rightarrow \infty} \frac{h}{s_h}$ .

*Proof.* Immediate from (3.3) as  $t$  goes to infinity.  $\square$

Outside the steady state the growth rate of the economy is not constant but, rather, it behaves according to (2.3). In order to determine what the equilibrium path of the economy looks like, we need to study the transitional dynamics of the dynamical system.

**Theorem 3.1.** *The steady state equilibrium is a saddle point with a two dimensional stable manifold.*

*Proof.* The local dynamic around a steady state equilibrium  $(k_*, h_*, L_*)$  is determined by the signs of the eigenvalues of the Jacobian matrix corresponding to its linearized system, which writes as follows

$$(3.4) \quad \begin{bmatrix} \dot{k} \\ \dot{h} \\ \dot{L} \end{bmatrix} = J^* \begin{bmatrix} k - k_* \\ h - h_* \\ L - L_* \end{bmatrix}, \text{ where } J^* = \begin{bmatrix} J_{11}^* & J_{12}^* & J_{13}^* \\ J_{21}^* & J_{22}^* & J_{23}^* \\ J_{31}^* & J_{32}^* & J_{33}^* \end{bmatrix}.$$

The matrix  $J^*$ , called the Jacobian matrix, denotes the matrix of first partial derivatives evaluated at the equilibrium point, namely  $J_{11}^* = (\partial \dot{k} / \partial k) |_{(k_*, h_*, L_*)}$ ,  $J_{12}^* = (\partial \dot{k} / \partial h) |_{(k_*, h_*, L_*)}$ ,  $J_{13}^* = (\partial \dot{k} / \partial L) |_{(k_*, h_*, L_*)}$ , and so on for all the other matrix entries. Computing these elements yields

$$(3.5) \quad J^* = \begin{bmatrix} -(1-\alpha)\delta & \frac{s_k \beta \delta}{s_h} & 0 \\ \frac{s_h \alpha \delta}{s_k} & -(1-\beta)\delta & 0 \\ 0 & 0 & -a \end{bmatrix}.$$

It is immediately seen from (3.5) that one of its three eigenvalues, say  $\lambda_1$ , is  $-a$ . The remaining eigenvalues  $\lambda_2, \lambda_3$  are found solving the equation

$$\lambda^2 + [(1-\alpha) + (1-\beta)]\delta\lambda - \alpha\beta\delta^2 = 0.$$

Since its discriminant is  $[(1-\alpha) + (1-\beta)]^2\delta^2 + 4\alpha\beta\delta^2 > 0$ ,  $\lambda_2, \lambda_3$  must be real numbers. Their sign can be derived looking at the determinant and trace of  $J^*$ . After simplification, we get

$$\text{Det}(J^*) = -(1-\alpha-\beta)\delta^2 a < 0,$$

$$\text{Trace}(J^*) = J_{11}^* + J_{22}^* + J_{33}^* = -[(1-\alpha) + (1-\beta)]\delta - a < 0.$$

Recalling that the determinant of a matrix is also equal to the product of its eigenvalues, as well as the trace of a matrix is also equal to the sum of its eigenvalues, we

can conclude that one eigenvalue is negative and one is positive. In conclusion, the matrix  $J^*$  has one real positive (unstable) and two real negative (stable) roots. This proves that the steady state is (locally) a saddle point. Since there are two negative eigenvalues, the stable manifold is a plane going through the steady state. This plane is generated by the two negative eigenvalues (see [11]).  $\square$

**Remark 3.1.** *The point  $(k_*, h_*, L_*)$  is locally asymptotically stable, namely all solutions which start near it remain near the steady state for all time, and, furthermore, they tend towards  $(k_*, h_*, L_*)$  as  $t$  grows to infinity.*

The linearized system (3.4) has the following closed-form solution

$$\begin{cases} k_t - k_* = c_1 v_{11} e^{\lambda_1 t} + c_2 v_{12} e^{\lambda_2 t} + c_3 v_{13} e^{\lambda_3 t}, \\ h_t - h_* = c_1 v_{21} e^{\lambda_1 t} + c_2 d_{22} e^{\lambda_2 t} + c_3 v_{23} e^{\lambda_3 t}, \\ L_t - L_* = c_1 v_{31} e^{\lambda_1 t} + c_2 v_{32} e^{\lambda_2 t} + c_3 v_{33} e^{\lambda_3 t}, \end{cases}$$

where  $c_1, c_2, c_3$  are arbitrary constants, to be determined using the initial conditions, and  $[v_{11} \ v_{21} \ v_{31}]^T, [v_{12} \ v_{22} \ v_{32}]^T, [v_{13} \ v_{23} \ v_{33}]^T$  are the eigenvectors associated with the eigenvalues  $\lambda_1, \lambda_2, \lambda_3$ , respectively. Without loss of generality, assume  $\lambda_2 < 0$  and  $\lambda_3 > 0$ . Since  $e^{\lambda_3 t}$  diverges to infinity, it is clear that the solutions will be stable if  $c_3 = 0$ . Thus, the solutions along the stable arm of the saddle-path are given by

$$\begin{cases} k_t - k_* = c_1 v_{11} e^{\lambda_1 t} + c_2 v_{12} e^{\lambda_2 t}, \\ h_t - h_* = c_1 v_{21} e^{\lambda_1 t} + c_2 d_{22} e^{\lambda_2 t}, \\ L_t - L_* = c_1 v_{31} e^{\lambda_1 t} + c_2 v_{32} e^{\lambda_2 t}. \end{cases}$$

A particularly interesting aspect of the above results pertains to the eigenvalues, which are crucial in determining the economy's speed of convergence, namely how long it takes for the economy to adjust to the steady state. Many models of growth, including MWR model, have the property that the transitional dynamics are determined by a one dimensional stable manifold. As a consequence, all the variables converge to their respective steady states at the same constant speed, which is equal to the magnitude of the unique stable eigenvalue. By contrast, if the stable manifold is two dimensional, as for our model, then the speed of convergence of any variable at any point of time is a weighted average of the two stable eigenvalues. Clearly, over time, the weight of the smaller (more negative) eigenvalue declines, so that the larger of the two stable eigenvalues describes the asymptotic speed of convergence. It is clear that the flexibility provided by the additional eigenvalue allows the system to match some features of the data related with the timing of the key variables and growth rates along the transitional path.

## 4 Conclusions

In this paper we have considered a modified version of the MRW model, obtained by assuming a logistic law formulation for the evolution of population. As it is well known, the main problem behind the standard assumption of a constant population growth is that it yields an exponential behavior of population size over time, which

is clearly unrealistic and unsustainable in the very long run. Using the logistic population growth hypothesis has the advantage that population size tends to a finite saturation level in the very long run. Within this setup, the model is shown to be described by a three dimensional dynamical system with a unique non-trivial steady state equilibrium (a saddle). Two stable roots, rather than only one as in basic MRW model, now determine the speed of convergence, thereby introducing important flexibility to the convergence and transition characteristics. For future research it would be interesting to include in our discussion tables and real experiments.

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