# A relativistic approach on 1-jet spaces of the rheonomic Berwald-Moór metric

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**Abstract.** The aim of this paper is to develop on the 1-jet space $J^1(\mathbb{R}, M^4)$ the Finsler-like geometry (the d-framework), for the rheonomic Berwald-Moór metric $\mathring{F}(t,y) = \sqrt{h^{11}(t)}\sqrt[4]{y_1^1 y_1^2 y_1^3 y_1^4}$. The d-connections, d-torsions and d-curvatures are determined, and a natural geometrical gravitational field-like theory produced by the above rheonomic Berwald-Moór metric is constructed.

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## 1 Introduction

It is obvious that our natural physical intuition distinguishes four dimensions in a natural correspondence with the material reality. Consequently, the four dimensionality plays a special role in almost all modern physical theories.

On the other hand, it is an well known fact that, in order to create the Relativity Theory, Einstein used the Riemannian geometry instead of the classical Euclidean geometry, the first one representing the natural mathematical model for the local *isotropic* space-time. Although the use of Riemannian geometry was indeed a genial idea, there are recent studies of physicists which suggest a *non-isotropic* perspective of the space-time (for example, in Pavlov's opinion [16], the concept of inertial body mass emphasizes the necessity of study of local non-isotropic spaces). Obviously, for the study of non-isotropic physical phenomena, the Finsler geometry is very useful as mathematical framework.

The studies of Russian scholars (Asanov [1], Mikhailov [10], Garas'ko and Pavlov [8]) emphasize the importance of the Finsler geometry which is characterized by the total equality in rights of all non-isotropic directions. For such a reason, Asanov, Pavlov and their co-workers underline the important role played by the Berwald-Moór metric

$$F : TM \to \mathbb{R}, \qquad F(y) = \left(y^1 y^2 ... y^n\right)^{\frac{1}{n}}, \tag{1.1}$$

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in the theory of space-time structure and gravitation, as well as in unified gauge field theories. Because any of such directions can be related to the proper time of an inertial reference frame, Pavlov considers that it is appropriate as such spaces to be generically called *"multi-dimensional times"* [16]. In the framework of the 4-dimensional linear space with Berwald-Moór metric (i.e. the four-dimensional time), Pavlov and his co-workers [8], [16] offer some new physical approaches and geometrical interpretations such as:

1. physical events = points in the 4-dimensional space;
2. straight lines = shortest curves;
3. intervals = distances between the points along of a straight line;
4. simultaneous surfaces = the surfaces of simultaneous physical events.

It is known that, on the tangent bundle $TM$, the Finsler geometry of the Berwald-Moór metric (1.1) is deeply studied by the Japanese geometers Matsumoto and Shimada [9]. Moreover, the Legendre transformation produced by the Berwald-Moór metric (1.1) is explicitly described and the Hamiltonian Berwald-Moór dual geometrical structure on the cotangent bundle $T^*M$ is also investigated by Balan and Lebedev [5]. Based on the works of Matsumoto and Shimada, in the paper [6], Balan and Nicola present the equation of motion in the $x$-conformally deformed 4-dimensional Berwald-Moór framework, together with some KCC Finslerian geometrical elements (intimately related by the Jacobi stability).

In such a geometrical and physical context, this paper is devoted to the development on the 1-jet space $J^1(\mathbb{R}, M^4)$ of the Finsler-like geometry (together with a theoretical-geometric gravitational field-like theory) for the *rheonomic Berwald-Moór metric*

$$\mathring{F} : J^1(\mathbb{R}, M^4) \to \mathbb{R}, \qquad \mathring{F}(t, y) = \sqrt{h^{11}(t)}\sqrt[4]{y_1^1 y_1^2 y_1^3 y_1^4},$$

where $h_{11}(t)$ is a Riemannian metric on $\mathbb{R}$ and $(t, x^1, x^2, x^3, x^4, y_1^1, y_1^2, y_1^3, y_1^4)$ are the coordinates of the 1-jet space $J^1(\mathbb{R}, M^4)$.

**Remark 1.1.** (a) For the particular Riemannian metric $h_{11}(t) = e^{-2\sigma(t)} > 0$, the *rheonomic Berwald-Moór metric* $\mathring{F}$ becomes a $t$-conformal deformation of the *jet relativistic Berwald-Moór metric*

$$BM(y) = \sqrt[4]{y_1^1 y_1^2 y_1^3 y_1^4}.$$

(b) In the paper [15], Neagu and Udrişte construct the differential equations of the stream lines that characterize plasma regarded as a non-isotropic medium geometrized by the jet rheonomic Berwald-Moor metric $\mathring{F}$.

The differential geometry (in the sense of distinguished (d-) connections, d-torsions, d-curvatures, gravitational and electromagnetic geometrical theories) produced by a jet rheonomic Lagrangian function $L : J^1(\mathbb{R}, M^n) \to \mathbb{R}$ is now completely done in the author's paper [14]. We point out that the geometrical ideas from [14] are similar, but however distinct ones, with those exposed by Miron and Anastasiei in the classical Lagrangian geometry on tangent bundles (see [11]). In fact, the geometrical ideas from [14] (that we called the jet geometrical theory of the *relativistic rheonomic Lagrange spaces*) were initially stated by Asanov in [2] and developed further by Udrişte (see [18], [19], [20] and [21]) and Neagu (see [13]).

In the sequel, we apply the general geometrical results from [14] to the rheonomic Berwald-Moór metric $\mathring{F}$.

## 2 Preliminary notations and formulas

Let $(\mathbb{R}, h_{11}(t))$ be a Riemannian manifold, where $\mathbb{R}$ is the set of real numbers. The Christoffel symbol of the Riemannian metric $h_{11}(t)$ is

$$\varkappa_{11}^1 = \frac{h^{11}}{2}\frac{dh_{11}}{dt}, \qquad h^{11} = \frac{1}{h_{11}} > 0.$$

Let also $M^4$ be a manifold of dimension four, whose local coordinates are $(x^1, x^2, x^3, x^4)$. Let us consider the 1-jet space $J^1(\mathbb{R}, M^4)$, whose local coordinates are

$$(t, x^1, x^2, x^3, x^4, y_1^1, y_1^2, y_1^3, y_1^4).$$

These transform by the rules (the Einstein convention of summation is used throughout this work):

$$\text{(2.1)} \qquad \widetilde{t} = \widetilde{t}(t), \quad \widetilde{x}^p = \widetilde{x}^p(x^q), \quad \widetilde{y}_1^p = \frac{\partial \widetilde{x}^p}{\partial x^q}\frac{dt}{d\widetilde{t}} \cdot y_1^q, \qquad p, q = \overline{1, 4},$$

where $d\widetilde{t}/dt \neq 0$ and rank $(\partial \widetilde{x}^p/\partial x^q) = 4$. We consider that the manifold $M^4$ is endowed with a tensor of kind $(0, 4)$, given by the local components $G_{pqrs}(x)$, which is totally symmetric in the indices $p$, $q$, $r$ and $s$. Suppose that the d-tensor

$$G_{ij11} = 12 G_{ijpq} y_1^p y_1^q,$$

is non-degenerate, that is there exists the d-tensor $G^{jk11}$ on $J^1(\mathbb{R}, M^4)$ such that $G_{ij11}G^{jk11} = \delta_i^k$.

In this geometrical context, if we use the notation $G_{1111} = G_{pqrs} y_1^p y_1^q y_1^r y_1^s$, we can consider the rheonomic Finsler-like function (it is 1-positive homogenous in the variable $y$):

$$\text{(2.2)} \qquad F(t, x, y) = \sqrt[4]{G_{pqrs}(x) y_1^p y_1^q y_1^r y_1^s} \cdot \sqrt{h^{11}(t)} = \sqrt[4]{G_{1111}(x, y)} \cdot \sqrt{h^{11}(t)},$$

where the Finsler function $F$ has as domain of definition all values $(t, x, y)$ which verify the condition $G_{1111}(x, y) > 0$ (i.e. the domain where we can $y$-differentiate the function $F(t, x, y)$). If we denote $G_{i111} = 4 G_{ipqr}(x) y_1^p y_1^q y_1^r$, then the 4-positive homogeneity of the "$y$-function" $G_{1111}$ (this is in fact a d-tensor on $J^1(\mathbb{R}, M^4)$) leads to the equalities:

$$G_{i111} = \frac{\partial G_{1111}}{\partial y_1^i}, \quad G_{i111} y_1^i = 4 G_{1111}, \quad G_{ij11} y_1^j = 3 G_{i111},$$

$$G_{ij11} = \frac{\partial G_{i111}}{\partial y_1^j} = \frac{\partial^2 G_{1111}}{\partial y_1^i \partial y_1^j}, \quad G_{ij11} y_1^i y_1^j = 12 G_{1111}.$$

The *fundamental metrical d-tensor* produced by $F$ is given by the formula

$$g_{ij}(t, x, y) = \frac{h_{11}(t)}{2}\frac{\partial^2 F^2}{\partial y_1^i \partial y_1^j}.$$

By direct computations, the fundamental metrical d-tensor takes the form

$$g_{ij}(x,y) = \frac{1}{4\sqrt{G_{1111}}}\left[G_{ij11} - \frac{1}{2G_{1111}}G_{i111}G_{j111}\right]. \tag{2.3}$$

Moreover, taking into account that the d-tensor $G_{ij11}$ is non-degenerate, we deduce that the matrix $g = (g_{ij})$ admits the inverse $g^{-1} = (g^{jk})$. The entries of the inverse matrix $g^{-1}$ are

$$g^{jk} = 4\sqrt{G_{1111}}\left[G^{jk11} + \frac{G_1^j G_1^k}{2\left(G_{1111} - \mathcal{G}_{1111}\right)}\right], \tag{2.4}$$

where $G_1^j = G^{jp11}G_{p111}$ and $2\mathcal{G}_{1111} = G^{pq11}G_{p111}G_{q111}$.

# 3 The rheonomic Berwald-Moór metric

Beginning with this Section we will focus only on the *rheonomic Berwald-Moór metric*, which is the Finsler-like metric (2.2) for the particular case

$$G_{pqrs} = \begin{cases} \dfrac{1}{4!}, & \{p,q,r,s\} \text{ - distinct indices} \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, the rheonomic Berwald-Moór metric is given by

$$\mathring{F}(t,y) = \sqrt{h^{11}(t)} \cdot \sqrt[4]{y_1^1 y_1^2 y_1^3 y_1^4}. \tag{3.1}$$

Moreover, using preceding notations and formulas, we obtain the following relations:

$$G_{1111} = y_1^1 y_1^2 y_1^3 y_1^4, \quad G_{i111} = \frac{G_{1111}}{y_1^i},$$

$$G_{ij11} = (1 - \delta_{ij})\,\frac{G_{1111}}{y_1^i y_1^j} \text{ (no sum by } i \text{ or } j),$$

where $\delta_{ij}$ is the Kronecker symbol. Because we have

$$\det\left(G_{ij11}\right)_{i,j=\overline{1,4}} = -3\left(G_{1111}\right)^2 \neq 0,$$

we find

$$G^{jk11} = \frac{(1 - 3\delta^{jk})}{3G_{1111}} y_1^j y_1^k \text{ (no sum by } j \text{ or } k).$$

It follows that we have $\mathcal{G}_{1111} = (2/3)G_{1111}$ and $G_1^j = (1/3)y_1^j$.

Replacing now the preceding computed entities into the formulas (2.3) and (2.4), we get

$$g_{ij} = \frac{(1 - 2\delta_{ij})\sqrt{G_{1111}}}{8}\,\frac{1}{y_1^i y_1^j} \text{ (no sum by } i \text{ or } j) \tag{3.2}$$

and

$$g^{jk}=\frac{2(1-2\delta^{jk})}{\sqrt{G_{1111}}}y_1^jy_1^k \text{ (no sum by } j \text{ or } k). \tag{3.3}$$

Using a general formula from the paper [14], we find the following geometrical result:

**Proposition 1.** *For the rheonomic Berwald-Moór metric (3.1), the* energy action functional

$$\overset{\circ}{\mathbb{E}}(t,x(t))=\int_a^b \overset{\circ}{F}{}^2(t,y)\sqrt{h_{11}}dt=\int_a^b\sqrt{y_1^1y_1^2y_1^3y_1^4}\cdot h^{11}\sqrt{h_{11}}dt$$

*produces on the* 1*-jet space* $J^1(\mathbb{R},M^4)$ *(via the attached Euler-Lagrange equations, see [14]) the canonical nonlinear connection*

$$\Gamma=\left(M^{(i)}_{(1)1}=-\varkappa^1_{11}y^i_1,\ N^{(i)}_{(1)j}=0\right). \tag{3.4}$$

Because the canonical nonlinear connection (3.4) has the spatial components equal to zero, it follows that our subsequent geometrical theory becomes trivial, in a way. For such a reason, in order to avoid the triviality of our theory, we will use on the 1-jet space $J^1(\mathbb{R},M^4)$, by an *"a priori"* definition, the following nonlinear connection:

$$\overset{\circ}{\Gamma}=\left(M^{(i)}_{(1)1}=-\varkappa^1_{11}y^i_1,\ N^{(i)}_{(1)j}=-\frac{\varkappa^1_{11}}{3}\delta^i_j\right). \tag{3.5}$$

Note that the non-trivial nonlinear connection (3.5) has the quality that it does not curve the space, in the sense that its *autoparallel curves* [11] (or *harmonic curves* [12]) are pieces of straight lines. This fact is also true for the trivial canonical nonlinear connection (3.4).

**Remark 3.1.** The spatial components of the nonlinear connection (3.5), which are given in the local chart $\mathcal{U}$ by the functions

$$\overset{\circ}{N}=\left(N^{(i)}_{(1)j}=-\frac{\varkappa^1_{11}}{3}\delta^i_j\right),$$

have not a global character on the 1-jet space $J^1(\mathbb{R},M^4)$, but have only a local character. Consequently, taking into account the transformation rules of a spatial nonlinear connection (see the paper [12]), it follows that $\overset{\circ}{N}$ has in the local chart $\widetilde{\mathcal{U}}$ the following components:

$$\widetilde{N}^{(k)}_{(1)l}=-\frac{\widetilde{\varkappa}^1_{11}}{3}\delta^k_l+\frac{1}{3}\frac{d\widetilde{t}}{dt}\frac{d^2t}{d\widetilde{t}^2}\delta^k_l+\frac{\partial\widetilde{x}^k}{\partial x^m}\frac{\partial^2x^m}{\partial\widetilde{x}^l\partial\widetilde{x}^r}\widetilde{y}^r_1.$$

# 4 Cartan canonical connection. d-Torsions and d-curvatures

The importance of the nonlinear connection (3.5) is coming from the possibility of construction of the dual *adapted bases* of distinguished (d-) vector fields

$$\left\{\frac{\delta}{\delta t}=\frac{\partial}{\partial t}+\varkappa^1_{11}y^p_1\frac{\partial}{\partial y^p_1}\ ;\ \frac{\delta}{\delta x^i}=\frac{\partial}{\partial x^i}+\frac{\varkappa^1_{11}}{3}\frac{\partial}{\partial y^i_1}\ ;\ \frac{\partial}{\partial y^i_1}\right\}\subset\mathcal{X}(E) \tag{4.1}$$

and distinguished covector fields

$$\left\{dt\ ;\ dx^i\ ;\ \delta y_1^i = dy_1^i - \varkappa_{11}^1 y_1^i dt - \frac{\varkappa_{11}^1}{3} dx^i\right\} \subset \mathcal{X}^*(E), \tag{4.2}$$

where $E = J^1(\mathbb{R}, M^4)$. Note that, under a change of coordinates (2.1), the elements of the adapted bases (4.1) and (4.2) transform as classical tensors. Consequently, all subsequent geometrical objects on the 1-jet space $J^1(\mathbb{R}, M^4)$ (as Cartan canonical linear connection, torsion, curvature etc.) will be described in local adapted components.

Using a general result from [14], by direct computations, we can give the following important geometrical result:

**Proposition 2.** *The Cartan canonical $\mathring{\Gamma}$-linear connection, produced by the rheonomic Berwald-Moór metric (3.1), has the following adapted local components:*

$$C\mathring{\Gamma} = \left(\varkappa_{11}^1,\ G_{j1}^k = 0,\ L_{jk}^i = \frac{\varkappa_{11}^1}{3} C_{j(k)}^{i(1)},\ C_{j(k)}^{i(1)}\right),$$

*where, if we use the notation*

$$A_{jk}^i = \frac{2\delta_j^i + 2\delta_k^i + 2\delta_{jk} - 8\delta_j^i \delta_{jk} - 1}{8} \text{ (no sum by } i,\ j \text{ or } k)$$

*we have*

$$C_{j(k)}^{i(1)} = A_{jk}^i \cdot \frac{y_1^i}{y_1^j y_1^k} \text{ (no sum by } i,\ j \text{ or } k).$$

*Proof.* Via the Berwald-Moór derivative operators (4.1), we use the general formulas which give the adapted components of the Cartan canonical connection, namely [14]

$$G_{j1}^k = \frac{g^{km}}{2}\frac{\delta g_{mj}}{\delta t}, \quad L_{jk}^i = \frac{g^{im}}{2}\left(\frac{\delta g_{jm}}{\delta x^k} + \frac{\delta g_{km}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^m}\right),$$

$$C_{j(k)}^{i(1)} = \frac{g^{im}}{2}\left(\frac{\partial g_{jm}}{\partial y_1^k} + \frac{\partial g_{km}}{\partial y_1^j} - \frac{\partial g_{jk}}{\partial y_1^m}\right) = \frac{g^{im}}{2}\frac{\partial g_{jm}}{\partial y_1^k}.$$

□

**Remark 4.1.** The below properties of the vertical d-tensor $C_{j(k)}^{i(1)}$ are true (see also the papers [3] and [9]):

$$C_{j(k)}^{i(1)} = C_{k(j)}^{i(1)}, \quad C_{j(m)}^{i(1)} y_1^m = 0, \quad C_{j(m)}^{m(1)} = 0 \text{ (sum by } m). \tag{4.3}$$

**Remark 4.2.** The coefficients $A_{ij}^l$ have the following values:

$$A_{ij}^l = \begin{cases} -\dfrac{1}{8}, & i \neq j \neq l \neq i \\ \dfrac{1}{8}, & i = j \neq l \text{ or } i = l \neq j \text{ or } j = l \neq i \\ -\dfrac{3}{8}, & i = j = l. \end{cases} \tag{4.4}$$

**Proposition 3.** *The Cartan canonical connection $C\overset{\circ}{\Gamma}$ of the rheonomic Berwald-Moór metric (3.1) has* three *effective local torsion d-tensors:*

$$P^{(k)\ (1)}_{(1)i(j)} = -\frac{1}{3}\varkappa^1_{11} C^{k(1)}_{i(j)}, \qquad P^{k(1)}_{i(j)} = C^{k(1)}_{i(j)},$$

$$R^{(k)}_{(1)1j} = \frac{1}{3}\left[\frac{d\varkappa^1_{11}}{dt} - \varkappa^1_{11}\varkappa^1_{11}\right]\delta^k_j.$$

*Proof.* A general $h$-normal $\Gamma$-linear connection on the 1-jet space $J^1(\mathbb{R}, M^4)$ is characterized by *eight* effective d-tensors of torsion (for more details, please see [14]). For our Cartan canonical connection $C\overset{\circ}{\Gamma}$ these reduce to the following *three* (the other five cancel):

$$P^{(k)\ (1)}_{(1)i(j)} = \frac{\partial N^{(k)}_{(1)i}}{\partial y^j_1} - L^k_{ji}, \quad R^{(k)}_{(1)1j} = \frac{\delta M^{(k)}_{(1)1}}{\delta x^j} - \frac{\delta N^{(k)}_{(1)j}}{\delta t}, \quad P^{k(1)}_{i(j)} = C^{k(1)}_{i(j)}.$$

□

**Proposition 4.** *The Cartan canonical connection $C\overset{\circ}{\Gamma}$ of the rheonomic Berwald-Moór metric (3.1) has* three *effective local curvature d-tensors:*

$$R^l_{ijk} = \frac{1}{9}\varkappa^1_{11}\varkappa^1_{11} S^{l(1)(1)}_{i(j)(k)}, \qquad P^{l\ (1)}_{ij(k)} = \frac{1}{3}\varkappa^1_{11} S^{l(1)(1)}_{i(j)(k)},$$

$$S^{l(1)(1)}_{i(j)(k)} = \frac{\partial C^{l(1)}_{i(j)}}{\partial y^k_1} - \frac{\partial C^{l(1)}_{i(k)}}{\partial y^j_1} + C^{m(1)}_{i(j)} C^{l(1)}_{m(k)} - C^{m(1)}_{i(k)} C^{l(1)}_{m(j)}.$$

*Proof.* A general $h$-normal $\Gamma$-linear connection on the 1-jet space $J^1(\mathbb{R}, M^4)$ is characterized by *five* effective d-tensors of curvature (for more details, please see [14]). For our Cartan canonical connection $C\overset{\circ}{\Gamma}$ these reduce to the following *three* (the other two cancel):

$$R^l_{ijk} = \frac{\delta L^l_{ij}}{\delta x^k} - \frac{\delta L^l_{ik}}{\delta x^j} + L^m_{ij} L^l_{mk} - L^m_{ik} L^l_{mj},$$

$$P^{l\ (1)}_{ij(k)} = \frac{\partial L^l_{ij}}{\partial y^k_1} - C^{l(1)}_{i(k)|j} + C^{l(1)}_{i(m)} P^{(m)\ (1)}_{(1)j(k)},$$

$$S^{l(1)(1)}_{i(j)(k)} = \frac{\partial C^{l(1)}_{i(j)}}{\partial y^k_1} - \frac{\partial C^{l(1)}_{i(k)}}{\partial y^j_1} + C^{m(1)}_{i(j)} C^{l(1)}_{m(k)} - C^{m(1)}_{i(k)} C^{l(1)}_{m(j)},$$

where

$$C^{l(1)}_{i(k)|j} = \frac{\delta C^{l(1)}_{i(k)}}{\delta x^j} + C^{m(1)}_{i(k)} L^l_{mj} - C^{l(1)}_{m(k)} L^m_{ij} - C^{l(1)}_{i(m)} L^m_{kj}.$$

□

**Remark 4.3.** The vertical curvature d-tensor $S^{l(1)(1)}_{i(j)(k)}$ has the properties

$$S^{l(1)(1)}_{i(j)(k)} + S^{l(1)(1)}_{i(k)(j)} = 0, \quad S^{l(1)(1)}_{i(j)(j)} = 0 \text{ (no sum by } j).$$

**Proposition 5.** *The vertical curvature d-tensor* $S^{l(1)(1)}_{i(j)(k)}$ *has the values:*

1. $S^{l(1)(1)}_{i(j)(k)} = 0$ *for* $\{i,\ j,\ k,\ l\}$ *distinct indices;*

2. $S^{l(1)(1)}_{i(i)(k)} = -\dfrac{1}{16}\dfrac{y^l_1}{\left(y^i_1\right)^2 y^k_1}$ *$(i \neq k \neq l \neq i$ and no sum by $i)$;*

3. $S^{l(1)(1)}_{i(j)(i)} = \dfrac{1}{16}\dfrac{y^l_1}{\left(y^i_1\right)^2 y^j_1}$ *$(i \neq j \neq l \neq i$ and no sum by $i)$;*

4. $S^{i(1)(1)}_{i(j)(k)} = 0$ *$(i \neq j \neq k \neq i$ and no sum by $i)$;*

5. $S^{l(1)(1)}_{i(l)(k)} = \dfrac{1}{16 y^i_1 y^k_1}$ *$(i \neq k \neq l \neq i$ and no sum by $l)$;*

6. $S^{l(1)(1)}_{i(j)(l)} = -\dfrac{1}{16 y^i_1 y^j_1}$ *$(i \neq j \neq l \neq i$ and no sum by $l)$;*

7. $S^{l(1)(1)}_{i(i)(l)} = \dfrac{1}{8\left(y^i_1\right)^2}$ *$(i \neq l$ and no sum by $i$ or $l)$;*

8. $S^{l(1)(1)}_{i(l)(i)} = -\dfrac{1}{8\left(y^i_1\right)^2}$ *$(i \neq l$ and no sum by $i$ or $l)$;*

9. $S^{l(1)(1)}_{l(l)(k)} = 0$ *$(k \neq l$ and no sum by $l)$;*

10. $S^{l(1)(1)}_{l(j)(l)} = 0$ *$(j \neq l$ and no sum by $l)$.*

*Proof.* For $j \neq k$, the expression of the curvature tensor $S^{l(1)(1)}_{i(j)(k)}$ takes the form (no sum by $i$, $j$, $k$ or $l$, but with sum by $m$)

$$\begin{aligned} S^{l(1)(1)}_{i(j)(k)} \quad = \quad & \left[\frac{A^l_{ij}\delta^l_k}{y^i_1 y^j_1} - \frac{A^l_{ik}\delta^l_j}{y^i_1 y^k_1}\right] + \left[\frac{A^l_{ik}\delta_{ij} y^l_1}{\left(y^i_1\right)^2 y^k_1} - \frac{A^l_{ij}\delta_{ik} y^l_1}{\left(y^i_1\right)^2 y^j_1}\right] + \\ & + \left[A^m_{ij}A^l_{mk} - A^m_{ik}A^l_{mj}\right]\frac{y^l_1}{y^i_1 y^j_1 y^k_1}, \end{aligned}$$

where the coefficients $A^l_{ij}$ are given by the relations (4.4). □

# 5 Geometric gravitational-like model produced by the rheonomic Berwald-Moór metric

From a physical point of view, on the 1-jet space $J^1(\mathbb{R}, M^4)$, the rheonomic Berwald-Moór metric (3.1) produces the adapted metrical d-tensor

$$\mathbb{G} = h_{11} dt \otimes dt + g_{ij} dx^i \otimes dx^j + h^{11} g_{ij} \delta y^i_1 \otimes \delta y^j_1, \tag{5.1}$$

where $g_{ij}$ is given by (3.2). This may be regarded as a *"non-isotropic gravitational potential"*. In such a "physical" context, the nonlinear connection $\mathring{\Gamma}$ (used in the construction of the distinguished 1-forms $\delta y_1^i$) prescribes, probably, a kind of *"interaction"* between $(t)$-, $(x)$- and $(y)$-fields.

We postulate that the non-isotropic gravitational potential $\mathbb{G}$ is governed by the *geometrical Einstein equations*

$$\text{Ric}\left(C\mathring{\Gamma}\right) - \frac{\text{Sc}\left(C\mathring{\Gamma}\right)}{2}\mathbb{G} = \mathcal{K}\mathcal{T}, \tag{5.2}$$

where $\text{Ric}\left(C\mathring{\Gamma}\right)$ is the *Ricci d-tensor* associated to the Cartan canonical connection $C\mathring{\Gamma}$ (in Riemannian sense and using adapted bases), $\text{Sc}\left(C\mathring{\Gamma}\right)$ is the *scalar curvature*, $\mathcal{K}$ is the *Einstein constant* and $\mathcal{T}$ is the intrinsic *stress-energy* d-tensor of matter.

In this way, working with the adapted basis of vector fields (4.1), we can find the local geometrical Einstein equations for the rheonomic Berwald-Moór metric (3.1). Firstly, by direct computations, we find:

**Lemma 6.** *The Ricci d-tensor of the Cartan canonical connection $C\mathring{\Gamma}$ of the rheonomic Berwald-Moór metric (3.1) has the following effective local Ricci d-tensors:*

$$\begin{aligned} &R_{ij} = R^m_{ijm} = \frac{\varkappa^1_{11}\varkappa^1_{11}}{9}S^{(1)(1)}_{(i)(j)}, \quad P_{i(j)}^{\ (1)} = P^{(1)}_{(i)j} = P^{m\ (1)}_{ij(m)} = \frac{\varkappa^1_{11}}{3}S^{(1)(1)}_{(i)(j)},\\ &S^{(1)(1)}_{(i)(j)} = S^{m(1)(1)}_{i(j)(m)} = \frac{7\delta_{ij}-1}{8}\frac{1}{y_1^i y_1^j} \ \textit{(no sum by } i \textit{ or } j\textit{).} \end{aligned} \tag{5.3}$$

**Remark 5.1.** The vertical Ricci d-tensor $S^{(1)(1)}_{(i)(j)}$ has the following value:

$$S^{(1)(1)}_{(i)(j)} = \begin{cases} -\dfrac{1}{8}\dfrac{1}{y_1^i y_1^j}, & i \neq j \\[2mm] \dfrac{3}{4}\dfrac{1}{\left(y_1^i\right)^2}, & i = j. \end{cases}$$

**Remark 5.2.** Using the third equality of (5.3) and the equality (3.3), we deduce that the following equality is true (sum by $r$):

$$S_i^{m11} \stackrel{def}{=} g^{mr}S^{(1)(1)}_{(r)(i)} = \frac{5-14\delta_i^m}{4}\cdot\frac{1}{\sqrt{G_{1111}}}\cdot\frac{y_1^m}{y_1^i} \text{ (no sum by } i \text{ or } m\text{).} \tag{5.4}$$

Moreover, by a direct calculation, we obtain the equalities

$$\sum_{m,r=1}^{4} S_r^{m11}C^{r(1)}_{i(m)} = 0, \quad \sum_{m=1}^{4}\frac{\partial S_i^{m11}}{\partial y_1^m} = \frac{3}{\sqrt{G_{1111}}}\frac{1}{y_1^i}. \tag{5.5}$$

**Lemma 7.** *The scalar curvature of the Cartan canonical connection $C\mathring{\Gamma}$ of the rheonomic Berwald-Moór metric (3.1) is given by*

$$Sc\left(C\mathring{\Gamma}\right) = -\frac{9h_{11}+\varkappa^1_{11}\varkappa^1_{11}}{\sqrt{G_{1111}}}.$$

*Proof.* The general formula for the scalar curvature of a Cartan connection is (for more details, please see [14])

$$\mathrm{Sc}\ \left(C\overset{\circ}{\Gamma}\right) = g^{pq}R_{pq} + h_{11}g^{pq}S^{(1)(1)}_{(p)(q)}.$$

□

Describing the global geometrical Einstein equations (5.2) in the adapted basis of vector fields (4.1), we find the following important geometrical and physical result (for more details, please see [14]):

**Proposition 8.** *The local* geometrical Einstein equations *that govern the non-isotropic gravitational potential* $\mathbb{G}$ *(produced by the rheonomic Berwald-Moór metric (3.1) and nonlinear connection (3.5)) are given by*

$$(5.6)\qquad \begin{cases} \dfrac{\xi_{11}h_{11}}{\sqrt{G_{1111}}} = \mathcal{T}_{11} \\ \dfrac{\varkappa^1_{11}\varkappa^1_{11}}{9\mathcal{K}}S^{(1)(1)}_{(i)(j)} + \dfrac{\xi_{11}}{\sqrt{G_{1111}}}g_{ij} = \mathcal{T}_{ij} \\ \dfrac{1}{\mathcal{K}}S^{(1)(1)}_{(i)(j)} + \dfrac{\xi_{11}}{\sqrt{G_{1111}}}h^{11}g_{ij} = \mathcal{T}^{(1)(1)}_{(i)(j)}, \end{cases}$$

$$(5.7)\qquad \begin{cases} 0 = \mathcal{T}_{1i}, \quad 0 = \mathcal{T}_{i1}, \quad 0 = \mathcal{T}^{(1)}_{(i)1}, \\ 0 = \mathcal{T}_{1\binom{(1)}{(i)}}, \quad \dfrac{\varkappa^1_{11}}{3\mathcal{K}}S^{(1)(1)}_{(i)(j)} = \mathcal{T}_{i\binom{(1)}{(j)}}, \quad \dfrac{\varkappa^1_{11}}{3\mathcal{K}}S^{(1)(1)}_{(i)(j)} = \mathcal{T}^{(1)}_{(i)j}, \end{cases}$$

*where*

$$(5.8)\qquad \xi_{11} = \frac{9h_{11} + \varkappa^1_{11}\varkappa^1_{11}}{2\mathcal{K}}.$$

**Remark 5.3.** The local geometrical Einstein equations (5.6) and (5.7) impose the stress-energy d-tensor of matter $\mathcal{T}$ to be symmetric. In other words, the stress-energy d-tensor of matter $\mathcal{T}$ must verify the local symmetry conditions

$$\mathcal{T}_{AB} = \mathcal{T}_{BA}, \quad \forall\ A, B \in \left\{1,\ i,\ {}^{(1)}_{(i)}\right\}.$$

By direct computations, the local geometrical Einstein equations (5.6) and (5.7) imply the following identities of the stress-energy d-tensor (sum by $r$):

$$\mathcal{T}^1_1 \overset{def}{=} h^{11}\mathcal{T}_{11} = \frac{\xi_{11}}{\sqrt{G_{1111}}}, \quad \mathcal{T}^m_1 \overset{def}{=} g^{mr}\mathcal{T}_{r1} = 0,$$

$$\mathcal{T}^{(m)}_{(1)1} \overset{def}{=} h_{11}g^{mr}\mathcal{T}^{(1)}_{(r)1} = 0, \quad \mathcal{T}^1_i \overset{def}{=} h^{11}\mathcal{T}_{1i} = 0,$$

$$\mathcal{T}^m_i \overset{def}{=} g^{mr}\mathcal{T}_{ri} = \frac{\varkappa^1_{11}\varkappa^1_{11}}{9\mathcal{K}}S^{m11}_i + \frac{\xi_{11}}{\sqrt{G_{1111}}}\delta^m_i,$$

$$\mathcal{T}^{(m)}_{(1)i} \stackrel{def}{=} h_{11}g^{mr}\mathcal{T}^{\ (1)}_{(r)i} = \frac{h_{11}\varkappa^1_{11}}{3\mathcal{K}}S^{m11}_i, \quad \mathcal{T}^{1(1)}_{\ \ (i)} \stackrel{def}{=} h^{11}\mathcal{T}_{1(i)}^{\ (1)} = 0,$$

$$\mathcal{T}^{m(1)}_{\ \ (i)} \stackrel{def}{=} g^{mr}\mathcal{T}_{r(i)}^{\ (1)} = \frac{\varkappa^1_{11}}{3\mathcal{K}}S^{m11}_i,$$

$\mathcal{T}^{(m)(1)}_{(1)(i)} \stackrel{def}{=} h_{11}g^{mr}\mathcal{T}^{(1)(1)}_{(r)(i)} = \frac{h_{11}}{\mathcal{K}}S^{m11}_i + \frac{\xi_{11}}{\sqrt{G_{1111}}}\delta^m_i$, where the d-tensor $S^{m11}_i$ is given by (5.4) and $\xi_{11}$ is given by (5.8).

**Corollary 9.** *The stress-energy d-tensor of matter $\mathcal{T}$ must verify the following* geometrical conservation laws *(summation by $m$):*

$$\begin{cases} \mathcal{T}^1_{1/1} + \mathcal{T}^m_{1|m} + \mathcal{T}^{(m)}_{(1)1}|^{(1)}_{(m)} = \dfrac{\left(h^{11}\right)^2}{8\mathcal{K}}\dfrac{dh_{11}}{dt}\left[2\dfrac{d^2h_{11}}{dt^2} - \dfrac{3}{h_{11}}\left(\dfrac{dh_{11}}{dt}\right)^2\right]\cdot\dfrac{1}{\sqrt{G_{1111}}} \\ \mathcal{T}^1_{i/1} + \mathcal{T}^m_{i|m} + \mathcal{T}^{(m)}_{(1)i}|^{(1)}_{(m)} = \dfrac{\varkappa^1_{11}\xi_{11}}{18}\cdot\dfrac{1}{\sqrt{G_{1111}}}\cdot\dfrac{1}{y^i_1} \\ \mathcal{T}^{1(1)}_{\ \ (i)/1} + \mathcal{T}^{m(1)}_{\ \ (i)|m} + \mathcal{T}^{(m)(1)}_{(1)(i)}|^{(1)}_{(m)} = \dfrac{\xi_{11}}{6}\cdot\dfrac{1}{\sqrt{G_{1111}}}\cdot\dfrac{1}{y^i_1}, \end{cases}$$

*where (summation by $m$ and $r$)*

$$\mathcal{T}^1_{1/1} \stackrel{def}{=} \frac{\delta\mathcal{T}^1_1}{\delta t} + \mathcal{T}^1_1\varkappa^1_{11} - \mathcal{T}^1_1\varkappa^1_{11} = \frac{\delta\mathcal{T}^1_1}{\delta t},$$

$$\mathcal{T}^m_{1|m} \stackrel{def}{=} \frac{\delta\mathcal{T}^m_1}{\delta x^m} + \mathcal{T}^r_1 L^m_{rm} = \frac{\delta\mathcal{T}^m_1}{\delta x^m},$$

$$\mathcal{T}^{(m)}_{(1)1}|^{(1)}_{(m)} \stackrel{def}{=} \frac{\partial\mathcal{T}^{(m)}_{(1)1}}{\partial y^m_1} + \mathcal{T}^{(r)}_{(1)1}C^{m(1)}_{r(m)} = \frac{\partial\mathcal{T}^{(m)}_{(1)1}}{\partial y^m_1},$$

$$\mathcal{T}^1_{i/1} \stackrel{def}{=} \frac{\delta\mathcal{T}^1_i}{\delta t} + \mathcal{T}^1_i\varkappa^1_{11} - \mathcal{T}^1_r G^r_{i1} = \frac{\delta\mathcal{T}^1_i}{\delta t} + \mathcal{T}^1_i\varkappa^1_{11},$$

$$\mathcal{T}^m_{i|m} \stackrel{def}{=} \frac{\delta\mathcal{T}^m_i}{\delta x^m} + \mathcal{T}^r_i L^m_{rm} - \mathcal{T}^m_r L^r_{im} = \frac{\varkappa^1_{11}}{3}\frac{\partial\mathcal{T}^m_i}{\partial y^m_1},$$

$$\mathcal{T}^{(m)}_{(1)i}|^{(1)}_{(m)} \stackrel{def}{=} \frac{\partial\mathcal{T}^{(m)}_{(1)i}}{\partial y^m_1} + \mathcal{T}^{(r)}_{(1)i}C^{m(1)}_{r(m)} - \mathcal{T}^{(m)}_{(1)r}C^{r(1)}_{i(m)} = \frac{\partial\mathcal{T}^{(m)}_{(1)i}}{\partial y^m_1},$$

$$\mathcal{T}^{1(1)}_{\ \ (i)/1} \stackrel{def}{=} \frac{\delta\mathcal{T}^{1(1)}_{\ \ (i)}}{\delta t} + 2\mathcal{T}^{1(1)}_{\ \ (i)}\varkappa^1_{11},$$

$$\mathcal{T}^{m(1)}_{\ \ (i)|m} \stackrel{def}{=} \frac{\delta\mathcal{T}^{m(1)}_{\ \ (i)}}{\delta x^m} + \mathcal{T}^{r(1)}_{\ \ (i)}L^m_{rm} - \mathcal{T}^{m(1)}_{\ \ (r)}L^r_{im} = \frac{\varkappa^1_{11}}{3}\frac{\partial\mathcal{T}^{m(1)}_{\ \ (i)}}{\partial y^m_1},$$

$$\mathcal{T}^{(m)(1)}_{(1)(i)}|^{(1)}_{(m)} \stackrel{def}{=} \frac{\partial \mathcal{T}^{(m)(1)}_{(1)(i)}}{\partial y^m_1} + \mathcal{T}^{(r)(1)}_{(1)(i)} C^{m(1)}_{r(m)} - \mathcal{T}^{(m)(1)}_{(1)(r)} C^{r(1)}_{i(m)} = \frac{\partial \mathcal{T}^{(m)(1)}_{(1)(i)}}{\partial y^m_1}.$$

*Proof.* The conservation laws are provided by direct computations, using the relations (4.3) and (5.5). □

# 6 Some physical remarks and comments

## 6.1 On gravitational theory

It is known that in the classical Relativity theory of Einstein (which characterizes the gravity in an isotropic space-time) the tensor of matter must verify the conservation laws

$$\mathcal{T}^m_{i;m} = 0, \qquad \forall\ i = \overline{1,4},$$

where ";" means the covariant derivative produced by the Levi-Civita connection associated to pseudo-Riemannian metric $g_{ij}(x)$ (i.e. the isotropic gravitational potentials).

Comparatively, in our non-isotropic gravitational theory (with respect to the rheonomic Berwald-Moór metric (3.1) and nonlinear connection (3.5)) the conservation laws are replaced with ($i = \overline{1,4}$)

$$\mathcal{T}_1 = \frac{\left(h^{11}\right)^2}{8\mathcal{K}} \frac{dh_{11}}{dt} \left[2\frac{d^2 h_{11}}{dt^2} - \frac{3}{h_{11}} \left(\frac{dh_{11}}{dt}\right)^2\right] \cdot \frac{1}{\sqrt{G_{1111}}}$$

$$\mathcal{T}_i = \frac{\varkappa^1_{11}\xi_{11}}{18} \cdot \frac{1}{\sqrt{G_{1111}}} \cdot \frac{1}{y^i_1}, \qquad \mathcal{T}^{(1)}_{(i)} = \frac{\xi_{11}}{6} \cdot \frac{1}{\sqrt{G_{1111}}} \cdot \frac{1}{y^i_1},$$

where

$$\mathcal{T}_1 \stackrel{def}{=} \mathcal{T}^1_{1/1} + \mathcal{T}^m_{1|m} + \mathcal{T}^{(m)}_{(1)1}|^{(1)}_{(m)}, \qquad \mathcal{T}_i \stackrel{def}{=} \mathcal{T}^1_{i/1} + \mathcal{T}^m_{i|m} + \mathcal{T}^{(m)}_{(1)i}|^{(1)}_{(m)},$$

$$\mathcal{T}^{(1)}_{(i)} \stackrel{def}{=} \mathcal{T}^{1(1)}{}_{(i)/1} + \mathcal{T}^{m(1)}{}_{(i)|m} + \mathcal{T}^{(m)(1)}_{(1)(i)}|^{(1)}_{(m)}.$$

By analogy with Einstein's theory, if we simultaneously impose the conditions (for all $i = \overline{1,4}$)

$$\mathcal{T}_1 = 0, \quad \mathcal{T}_i = 0, \quad \mathcal{T}^{(1)}_{(i)} = 0,$$

then we reach to the system of differential equations

$$\text{(6.1)} \qquad \begin{cases} \dfrac{dh_{11}}{dt} \left[2\dfrac{d^2 h_{11}}{dt^2} - \dfrac{3}{h_{11}} \left(\dfrac{dh_{11}}{dt}\right)^2\right] = 0 \\ 9h_{11} + \varkappa^1_{11}\varkappa^1_{11} = 0. \end{cases}$$

Obviously, because we have $h_{11} > 0$, we deduce that the ODEs system (6.1) has not any solution. Consequently, we always have

$$[\mathcal{T}_1]^2 + [\mathcal{T}_i]^2 + \left[\mathcal{T}^{(1)}_{(i)}\right]^2 \neq 0, \quad \forall\ i = \overline{1,4}.$$

In our opinion, this fact suggests that our geometrical gravitational theory (produced by the rheonomic Berwald-Moór gravitational potential (5.1)) is not suitable for media whose stress-energy d-components are

$$\mathcal{T}_{AB} = 0, \quad \forall\ A, B \in \left\{1,\ i,\ {}^{(1)}_{(i)}\right\}.$$

However, it is important to note that at "infinity"

$$\text{(this means that } y_1^i \to \infty, \quad \forall\ i = \overline{1,4}),$$

our Berwald-Moór geometrical gravitational theory seems to be appropriate even for media characterized by a null stress-energy d-tensor of matter. This is because at "infinity" the stress-energy local d-tensors tend to become zero.

At the end of this Subsection, we point out that, from a physical point of view, some interesting modifications of the gravitational theory by using anisotropic metrics (i.e., the study of classical GRT tests in the "anisotropic geometrodynamics" approach) find in Siparov's paper [17]

## 6.2 On electromagnetic theory

In the paper [14], a geometrical theory for electromagnetism was also created, using only a given Lagrangian function $L$ on the 1-jet space $J^1(\mathbb{R}, M^4)$. In the background of the jet relativistic rheonomic Lagrange geometry from [14], we work with the *electromagnetic distinguished* 2*-form*

$$\mathbb{F} = F^{(1)}_{(i)j} \delta y_1^i \wedge dx^j,$$

where

$$F^{(1)}_{(i)j} = \frac{h^{11}}{2}\left[g_{jm}N^{(m)}_{(1)i} - g_{im}N^{(m)}_{(1)j} + \left(g_{ir}L^r_{jm} - g_{jr}L^r_{im}\right) y_1^m\right].$$

This is characterized by some natural *geometrical Maxwell equations* (for more details, please see Miron [11] and Neagu [14]).

In our particular case of rheonomic Berwald-Moór metric (3.1) and nonlinear connection (3.5), we find the electromagnetic 2-form

$$\mathbb{F} := \mathring{\mathbb{F}} = 0.$$

Consequently, our jet Berwald-Moór geometrical electromagnetic theory is trivial. In our opinion, this fact suggests that the rheonomic Berwald-Moór metric (3.1) has rather strong gravitational connotations than electromagnetic ones. This is because, in our geometrical approach, the jet Berwald-Moór electromagnetism does not bring new geometrical and physical informations. In such a perspective, it seems that we need to consider a similar geometric-physical study for $x$-dependent conformal deformations of the jet Berwald-Moór structure, agreeing thus the recent geometric-physical ideas proposed by Garas'ko in the book [7].

Finally, we point out that a new approach of the electromagnetism in spaces with anisotropic metrics (which is different by the electromagnetic theory exposed above) finds in Voicu and Siparov paper [22].

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## References


[1] G.S. Asanov, *Finslerian Extension of General Relativity*, Reidel, Dordrecht, 1984.

[2] G.S. Asanov, *Jet extension of Finslerian gauge approach*, Fortschritte der Physik 38, 8 (1990), 571-610.

[3] Gh. Atanasiu, V. Balan, M. Neagu, *The Pavlov's 4-polyform of momenta* $K(p) = \sqrt[4]{p_1 p_2 p_3 p_4}$ *and its applications in Hamilton geometry*, Hypercomplex Numbers in Geometry and Physics 2 (4), 2 (2005), 134-139.

[4] V. Balan, *Notable submanifolds in Berwald-Moór spaces*, BSG Proceedings 17, Geometry Balkan Press, Bucharest (2010), 21-30.

[5] V. Balan, S. Lebedev, *On the Legendre transform and Hamiltonian formalism in Berwald-Moór geometry*, Diff. Geom. Dyn. Syst., 12 (2010), 4-11.

[6] V. Balan, I.R. Nicola, *Berwald-Moór metrics and structural stability of conformally-deformed geodesic SODE*, Appl. Sci., 11 (2009), 19-34.

[7] G.I. Garas'ko, *Foundations of Finsler Geometry for Physicists* (in Russian), Tetru Eds, Moscow, 2009.

[8] G.I. Garas'ko, D.G. Pavlov, *The notions of distance and velocity modulus in the linear Finsler spaces*, In (Eds. D.G. Pavlov, Gh. Atanasiu, V. Balan): "Space-Time Structure. Algebra and Geometry", Russian Hypercomplex Society, Lilia Print, Moscow, 2007, 104-117.

[9] M. Matsumoto, H. Shimada, *On Finsler spaces with 1-form metric. II. Berwald-Moór's metric* $L = \left(y^1 y^2 ... y^n\right)^{1/n}$, Tensor N. S. 32 (1978), 275-278.

[10] R.V. Mikhailov, *On some questions of four dimensional topology: a survey of modern research*, Hypercomplex Numbers in Geometry and Physics 1 (1), 1 (2004), 100-103.

[11] R. Miron, M. Anastasiei, *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Academic Publishers, 1994.

[12] M. Neagu, *Jet geometrical objects depending on a relativistic time*, An. Şt. Univ. "Al.I. Cuza" Iaşi (S.N.). Mat., Tomul LVI, f. 2 (2010), 407-428.

[13] M. Neagu, *Riemann-Lagrange Geometry on 1-Jet Spaces*, Matrix Rom, Bucharest, 2005.

[14] M. Neagu, *The geometry of relativistic rheonomic Lagrange spaces*, BSG Proceedings 5, Geometry Balkan Press, Bucharest 2001, 142-168.

[15] M. Neagu, C. Udrişte, *Geometric dynamics of plasma in jet spaces with Berwald-Moór metric*, Proceedings of the 9-th WSEAS International Conference on System Science and Simulation in Engineering (ICOSSSE '10), Iwate, Japan, October 4-6, 2010; Selected Topics in System Science and Simulation in Engineering (2010), 42-49.

[16] D.G. Pavlov, *Four-dimensional time*, Hypercomplex Numbers in Geometry and Physics 1 (1), 1 (2004), 31-39.

[17] S. Siparov, *Anisotropic metric for the gravitation theory: new ways to interpret the classical GRT tests*, BSG Proceedings 17, Geometry Balkan Press, Bucharest (2010), 205-217.

[18] C. Udrişte, *Geometric Dynamics*, Kluwer Academic Publishers, 2000.

[19] C. Udrişte, *Solutions of ODEs and PDEs as potential maps using first order Lagrangians*, Balkan J. Geom. Appl. 6, 1 (2001), 93-108.

[20] C. Udrişte, L. Matei, *Lagrange-Hamilton Theories* (in Romanian), Monographs and Textbooks 8, Geometry Balkan Press, Bucharest, 2008.

[21] C. Udrişte, I. Ţevy, *Multi-time Euler-Lagrange-Hamilton theory*, WSEAS Trans. Math. 6 (2007), 6, 701-709.

[22] N. Voicu, S. Siparov, *A new approach to electromagnetism in anisotropic spaces*, BSG Proceedings 17, Geometry Balkan Press, Bucharest (2010), 250-260.

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