Harmonic maps relative to α-connections on statistical manifolds

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Abstract. In this paper we study harmonic maps relative to α-connections, and not always relative to Levi-Civita connections, on statistical manifolds. In particular, harmonic maps on α-conformally equivalent statistical manifolds are discussed, and conditions for harmonicity are given by parameters α and dimensions n. As the application we also describe harmonic maps between level surfaces of a Hessian domain with α-conformally flat connections.

M.S.C. 2010: 53A15, 53C43.
Key words: harmonic map; statistical manifold; dual connection; conformal transformation; Hessian domain.

1 Introduction

Harmonic maps are important to research for geometry, physics, and so on. On the other hand statistical manifolds have been studied in terms of affine geometry, information geometry, statistical mechanics, and so on [1]. In relation to them Shima gave conditions for harmonicity of gradient mappings of level surfaces on a Hessian domain, which is a typical example for a dually flat statistical manifold [7] [8].

Level surfaces on a Hessian domain are known as 1- and (-1)-conformally flat statistical manifolds for the primal connection and for the dual connection, respectively [10]. Then the gradient mappings are considered harmonic maps relative to the dual connection, i.e., the (-1)-connection. However Shima investigated harmonic maps on n-dimensional level surfaces into an $(n+1)$-dimensional dual affine space, and not into the other level surfaces. In addition Nomizu and Sasaki calculated the Laplacian of centro-affine immersions into an affine space, which generate projectively flat statistical manifolds, i.e., (-1)-conformally flat statistical manifolds. However they show no harmonic maps between two centro-affine hypersurfaces in [6].

Then we treat harmonic maps relative to α-connections between α-conformally equivalent statistical manifolds including the case of $\alpha = -1, 0$ (The 0-connection means the Levi-Civita connection.). In this paper, existence of non trivial harmonic maps for α-connections is shown with conditions of α-parameters and dimensions n. Finally, we describe harmonic maps between level surfaces of a Hessian domain for α-conformally flat connections.

Applied Sciences, Vol.14, 2012, pp. 82-88.

2 Statistical manifolds and α-conformal equivalence

We recall definitions of terms on statistical manifolds.

For a torsion-free affine connection ∇ and a pseudo-Riemannian metric h on a manifold N, the triple (N, ∇, h) is called a statistical manifold if ∇h is symmetric. If the curvature tensor R of ∇ vanishes, (N, ∇, h) is said to be flat.

For a statistical manifold (N, ∇, h), let ∇' be an affine connection on N such that

$$Xh(Y, Z) = h(\nabla_X Y, Z) + h(Y, \nabla'_X Z) \quad \text{for } X, Y \text{ and } Z \in \Gamma(TN),$$

where $\Gamma(TN)$ is the set of smooth tangent vector fields on N. The affine connection ∇' is torsion free, and $\nabla' h$ symmetric. Then ∇' is called the dual connection of ∇, the triple (N, ∇', h) the dual statistical manifold of (N, ∇, h), and (∇, ∇', h) the dualistic structure on N. The curvature tensor of ∇' vanishes if and only if that of ∇ does, and then (∇, ∇', h) is called the dually flat structure [1].

For a real number α, statistical manifolds (N, ∇, h) and $(N, \bar{\nabla}, \bar{h})$ are said to be α-conformally equivalent if there exists a function ϕ on N such that

$$\bar{h}(X, Y) = e^{\phi} h(X, Y), \tag{2.1}$$

$$\begin{aligned} h(\bar{\nabla}_X Y, Z) = h(\nabla_X Y, Z) &- \frac{1+\alpha}{2} d\phi(Z) h(X, Y) \\ &+ \frac{1-\alpha}{2} \{ d\phi(X) h(Y, Z) + d\phi(Y) h(X, Z) \} \end{aligned} \tag{2.2}$$

for X, Y and $Z \in \Gamma(TN)$. Two statistical manifolds (N, ∇, h) and $(N, \bar{\nabla}, \bar{h})$ are α-conformally equivalent if and only if the dual statistical manifolds (N, ∇', h) and $(N, \bar{\nabla}', \bar{h})$ are $(-\alpha)$-conformally equivalent. A statistical manifold (N, ∇, h) is called α-conformally flat if (N, ∇, h) is locally α-conformally equivalent to a flat statistical manifold [4].

3 Harmonic maps for α-conformal equivalence

Let (N, ∇, h) and $(N, \bar{\nabla}, \bar{h})$ be α-conformally equivalent statistical manifolds of dim $n \geq 2$, and $\{x^1, \cdots x^n\}$ a local coordinate system on N. Suppose that h and $\bar{h}$ are Riemannian metrices. We set $h_{ij} = h(\partial/\partial x^i, \partial/\partial x^j)$ and $[h^{ij}] = [h_{ij}]^{-1}$. Let $\pi_{id} : N \to N$ be the identity map, i.e., $\pi_{id}(x) = x$ for $x \in N$, and π_{id*} the differential of π_{id}. If cautioning about metrics and connections, we denote by $\pi_{id} : (N, \nabla, h) \to (N, \bar{\nabla}, \bar{h})$.

We define a harmonic map relative to $(h, \nabla, \bar{\nabla})$ as follows.

Definition 3.1. If a tension field $\tau_{(h,\nabla,\bar{\nabla})}(\pi_{id})$ vanishes, i.e., $\tau_{(h,\nabla,\bar{\nabla})}(\pi_{id}) \equiv 0$ on N, the map $\pi_{id} : (N, \nabla, h) \to (N, \bar{\nabla}, \bar{h})$ is said to be a harmonic map relative to $(h, \nabla, \bar{\nabla})$, where the tension field is defined by

$$\tau_{(h,\nabla,\bar{\nabla})}(\pi_{id}) := \sum_{i,j=1}^{n} h^{ij} \{ \bar{\nabla}_{\frac{\partial}{\partial x^i}} (\pi_{id*}(\frac{\partial}{\partial x^j})) - \pi_{id*}(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}) \} \in \Gamma(\pi_{id}^{-1} TN) \tag{3.1}$$

$$=\sum_{i,j=1}^{n} h^{ij}(\bar{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}-\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j})\in\Gamma(TN). \tag{3.2}$$

Then the next theorem holds.

Theorem 3.1. *For α-conformally equivalent statistical manifolds (N,∇,h) and $(N,\bar{\nabla},\bar{h})$ of $\dim n\geq 2$ satisfying (2.1) and (2.2), if $\alpha=-(n-2)/(n+2)$ or ϕ is a constant function on N, the identity map $\pi_{id}:(N,\nabla,h)\to(N,\bar{\nabla},\bar{h})$ is a harmonic map relative to $(h,\nabla,\bar{\nabla})$.*

Proof. By (2.2) and (3.2), for $k\in\{1,\cdots,n\}$ we have

$$h(\tau_{(h,\nabla,\bar{\nabla})}(\pi_{id}),\frac{\partial}{\partial x^k})=h(\sum_{i,j=1}^{n}h^{ij}(\bar{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}-\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}),\frac{\partial}{\partial x^k})$$

$$=\sum_{i,j=1}^{n}h^{ij}\{-\frac{1+\alpha}{2}d\phi(\frac{\partial}{\partial x^k})h(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j})+\frac{1-\alpha}{2}\{d\phi(\frac{\partial}{\partial x^i})h(\frac{\partial}{\partial x^j},\frac{\partial}{\partial x^k})$$

$$+d\phi(\frac{\partial}{\partial x^j})h(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^k})\}\}$$

$$=\sum_{i,j=1}^{n}h^{ij}\{-\frac{1+\alpha}{2}\frac{\partial\phi}{\partial x^k}h_{ij}+\frac{1-\alpha}{2}(\frac{\partial\phi}{\partial x^i}h_{jk}+\frac{\partial\phi}{\partial x^j}h_{ik})\}$$

$$=\{-\frac{1+\alpha}{2}\cdot n\cdot\frac{\partial\phi}{\partial x^k}+\frac{1-\alpha}{2}(\sum_{i=1}^{n}\frac{\partial\phi}{\partial x^i}\delta_{ik}+\sum_{j=1}^{n}\frac{\partial\phi}{\partial x^j}\delta_{jk})\}$$

$$=(-\frac{1+\alpha}{2}\cdot n+\frac{1-\alpha}{2}\cdot 2)\frac{\partial\phi}{\partial x^k}=-\frac{1}{2}\{(n+2)\alpha+(n-2)\}\frac{\partial\phi}{\partial x^k},$$

where δ_{ij} is the Kronecker's delta. Therefore, if $\tau_{(h,\nabla,\bar{\nabla})}(\pi_{id})\equiv 0$, it holds that $(n+2)\alpha+(n-2)=0$ or $\partial\phi/\partial x^k=0$ for all $k\in\{1,\cdots,n\}$ at each point in N. Thus we obtain Theorem 3.1. □

4 α-connections on level surfaces of a Hessian domain

In this section we show relations with α-connections and Hessian domains.

Let N be a manifold with a dualistic structure (∇,∇',h). For $\alpha\in\mathbf{R}$, an affine connection defined by

$$\nabla^{(\alpha)}:=\frac{1+\alpha}{2}\nabla+\frac{1-\alpha}{2}\nabla' \tag{4.1}$$

is called an α-connection of (N,∇,h). The triple $(N,\nabla^{(\alpha)},h)$ is also a statistical manifold, and $\nabla^{(-\alpha)}$ the dual connection of $\nabla^{(\alpha)}$. The 1-connection, the (-1)-connection and the 0-connection coincide with ∇, ∇' and the Levi-Civita connection of (N,h), respectively. An α-connection is not always flat [1].

Let D and $\{x^1, \ldots, x^{n+1}\}$ be the canonical flat affine connection and the canonical affine coordinate system on $\mathbf{A}^{n+1}$, i.e., $Ddx^i = 0$. If the Hessian $Dd\varphi = \sum_{i,j=1}^{n+1}(\partial^2\varphi/\partial x^i\partial x^j)dx^i dx^j$ is non-degenerate for a function φ on a domain Ω in $\mathbf{A}^{n+1}$, we call $(\Omega, D, g = Dd\varphi)$ a Hessian domain. A Hessian domain is a flat statistical manifold. Conversely, a flat statistical manifold is locally a Hessian domain [1] [8].

Let $\mathbf{A}^*_{n+1}$ and $\{x^*_1, \ldots, x^*_{n+1}\}$ be the dual affine space of $\mathbf{A}^{n+1}$ and the dual affine coordinate system of $\{x^1, \ldots, x^{n+1}\}$, respectively. We define the gradient mapping ι from Ω to $\mathbf{A}^*_{n+1}$ by

$$x^*_i \circ \iota = -\frac{\partial\varphi}{\partial x^i},$$

and a flat affine connection D' on Ω by

$$\iota_*(D'_X Y) = D^*_X \iota_*(Y) \quad \text{for} \quad X, Y \in \Gamma(T\Omega),$$

where $D^*_X \iota_*(Y)$ is covariant derivative along ι induced by the canonical flat affine connection D^* on $\mathbf{A}^*_{n+1}$. Then (Ω, D', g) is the dual statistical manifold of (Ω, D, g) [7] [8].

For a simply connected level surface M of φ with $\dim n \geq 2$, we denote by D^M and g^M the connection and the Riemannian metric on M induced by D and g, respectively. Then (M, D^M, g^M) is a 1-conformally flat statistical submanifold of (Ω, D, g) by Theorem 2.1 in [10].

We consider two simply connected level surfaces of $\dim n \geq 2$ (M, D, g), $(\hat{M}, \hat{D}, \hat{g})$ 1-conformally flat statistical submanifolds of (Ω, D, g). For $p \in M$, let λ be a function on M such that $e^{\lambda(p)}\iota(p) \in \hat{\iota}(\hat{M})$, where $\hat{\iota}$ is the restriction of the gradient mapping ι to $\hat{M}$, and set $(e^\lambda)(p) = e^{\lambda(p)}$. Note that the function e^λ means the projection of M to $\hat{M}$ with respect to the dual affine coordinate system of Ω.

We define a map $\pi : M \to \hat{M}$ by

$$\hat{\iota} \circ \pi = e^\lambda \iota,$$

denoting also by ι the restriction of the gradient mapping ι to M. We denote by $\bar{D}'$ an affine connection on M defined by

$$\pi_*(\bar{D}'_X Y) = \hat{D}'_{\pi_*(X)}\pi_*(Y) \quad \text{for} \quad X, Y \in \Gamma(TM),$$

and by $\bar{g}$ a Riemannian metric on M such that

$$\bar{g}(X, Y) = e^\lambda g(X, Y) = \hat{g}(\pi_*(X), \pi_*(Y)).$$

Then the next theorem is known (cf. [4] [5]).

Theorem 4.1. *([11]) For affine connections $D', \bar{D}'$ on M, we have*
(i) D' and $\bar{D}'$ are projectively equivalent.
(ii) (M, D', g) and $(M, \bar{D}', \bar{g})$ are (-1)-conformally equivalent.

We denote by $\bar{D}$ an affine connection on M defined by

$$\pi_*(\bar{D}_X Y) = \hat{D}_{\pi_*(X)}\pi_*(Y) \quad \text{for} \quad X, Y \in \Gamma(TM).$$

From duality of $\hat{D}$ and $\hat{D}'$, $\bar{D}$ is the dual connection of $\bar{D}'$ on M. Then the next theorem holds (cf. [3] [4]).

Theorem 4.2. *([11]) For affine connections $D, \bar{D}$ on M, we have*
(i) D and $\bar{D}$ are dual-projectively equivalent.
(ii) (M, D, g) and $(M, \bar{D}, \bar{g})$ are 1-conformally equivalent.

For α-connections $D^{(\alpha)}$, $\bar{D}^{(\alpha)} = D^{(-\alpha)}$ defined similarly to (4.1), we obtain the next corollary by Theorem 4.1, Theorem 4.2 and by (2.2) with $\phi = \lambda$ [9].

Corollary 4.3. *For affine connections $D^{(\alpha)}$, $\bar{D}^{(\alpha)}$ on M, $(M, D^{(\alpha)}, g)$ and $(M, \bar{D}^{(\alpha)}, \bar{g})$ are α-conformally equivalent.*

5 Harmonic maps relative to α-connections on level surfaces

We denote $\hat{D}^{(\alpha)}_{\pi_*(X)}\pi_*(Y)$ by $\hat{D}^{(\alpha)}_X\pi_*(Y)$, considering it in the induced section $\Gamma(\pi^{-1}T\hat{M})$. Let $\{x^1, \dots, x^n\}$ be a local coordinate system on M. A harmonic map between level surfaces $(M, D^{(\alpha)}, g)$ and $(\hat{M}, \hat{D}^{(\alpha)}, \hat{g})$ is defined as follows.

Definition 5.1. If a tension field $\tau_{(g,D^{(\alpha)},\hat{D}^{(\alpha)})}(\pi)$ vanishes, i.e., $\tau_{(g,D^{(\alpha)},\hat{D}^{(\alpha)})}(\pi) \equiv 0$ on M, the map $\pi : (M, D^{(\alpha)}, g) \to (\hat{M}, \hat{D}^{(\alpha)}, \hat{g})$ is said to be a harmonic map relative to $(g, D^{(\alpha)}, \hat{D}^{(\alpha)})$, where the tension field defined by

$$\tau_{(g,D^{(\alpha)},\hat{D}^{(\alpha)})}(\pi) := \sum_{i,j=1}^{n} g^{ij}\{\hat{D}^{(\alpha)}_{\frac{\partial}{\partial x^i}}(\pi_*(\frac{\partial}{\partial x^j})) - \pi_*(D^{(\alpha)}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j})\} \in \Gamma(\pi^{-1}T\hat{M}). \tag{5.1}$$

Now we give conditions for harmonicity of a map $\pi : M \to \hat{M}$ relative to $(g, D^{(\alpha)}, \hat{D}^{(\alpha)})$.

Theorem 5.1. *Let $(M, D^{(\alpha)}, g)$ and $(\hat{M}, \hat{D}^{(\alpha)}, \hat{g})$ be simply connected n-dimensional level surfaces of an $(n+1)$-dimensional Hessian domain (Ω, D, g) with $n \geq 2$. If $\alpha = -(n-2)/(n+2)$ or λ is a constant function on M, a map $\pi : (M, D^{(\alpha)}, g) \to (\hat{M}, \hat{D}^{(\alpha)}, \hat{g})$ is a harmonic map relative to $(g, D^{(\alpha)}, \hat{D}^{(\alpha)})$, where*

$$\hat{\iota} \circ \pi = e^{\lambda}\iota, \quad (e^{\lambda})(p) = e^{\lambda(p)}, \quad e^{\lambda(p)}\iota(p) \in \hat{\iota}(\hat{M}), \quad p \in M,$$

and ι, $\hat{\iota}$ are the restrictions of the gradient mapping on Ω to M, $\hat{M}$, respectively.

Proof. The tension field of the map π relative to $(g, D^{(\alpha)}, \hat{D}^{(\alpha)})$ is described with $(M, \bar{D}^{(\alpha)}, \bar{g})$, which is the pull-back of $(\hat{M}, \hat{D}^{(\alpha)}, \hat{g})$, as follows.

$$\tau_{(g,D^{(\alpha)},\hat{D}^{(\alpha)})}(\pi) = \sum_{i,j=1}^{n} g^{ij}\{\hat{D}^{(\alpha)}_{\frac{\partial}{\partial x^i}}(\pi_*(\frac{\partial}{\partial x^j})) - \pi_*(D^{(\alpha)}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j})\}$$

$$= \sum_{i,j=1}^{n} g^{ij}\{\pi_*(\bar{D}^{(\alpha)}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}) - \pi_*(D^{(\alpha)}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j})\} = \pi_*(\sum_{i,j=1}^{n} g^{ij}(\bar{D}^{(\alpha)}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} - D^{(\alpha)}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}))$$

Identifying $T_{\pi(x)}M$ with T_xM, and considering the definition of π, we have

$$\tau_{(g,D^{(\alpha)},\hat{D}^{(\alpha)})}(\pi) = e^{\lambda}\sum_{i,j=1}^{n} g^{ij}(\bar{D}^{(\alpha)}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} - D^{(\alpha)}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}).$$

By Corollary 4.3, $(M, D^{(\alpha)}, g)$ and $(M, \bar{D}^{(\alpha)}, \bar{g})$ are α-conformally equivalent, so that we have the equation (2.2) with $\phi = \lambda$, $h = g$, $\nabla = D^{(\alpha)}$ and $\bar{\nabla} = \bar{D}^{(\alpha)}$ for X, Y and $Z \in \Gamma(TM)$. Then it holds similarly to the proof of Theorem 3.1 that for $k \in \{1, \cdots, n\}$

$$g(\tau_{(g,D^{(\alpha)},\hat{D}^{(\alpha)})}(\pi), \frac{\partial}{\partial x^k}) = g(e^\lambda \sum_{i,j=1}^n g^{ij}(\bar{D}^{(\alpha)}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - D^{(\alpha)}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}), \frac{\partial}{\partial x^k})$$

$$= e^\lambda \sum_{i,j=1}^n g^{ij}\{-\frac{1+\alpha}{2} d\lambda(\frac{\partial}{\partial x^k}) g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) + \frac{1-\alpha}{2}\{d\lambda(\frac{\partial}{\partial x^i}) g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k})$$

$$+ d\lambda(\frac{\partial}{\partial x^j}) g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k})\}\}$$

$$= (-\frac{1+\alpha}{2} \cdot n + \frac{1-\alpha}{2} \cdot 2)\, e^\lambda \frac{\partial \lambda}{\partial x^k} = -\frac{1}{2}\{(n+2)\alpha + (n-2)\}\, e^\lambda \frac{\partial \lambda}{\partial x^k}.$$

Therefore, if $\tau_{(g,D^{(\alpha)},\hat{D}^{(\alpha)})}(\pi) \equiv 0$, it holds that $(n+2)\alpha + (n-2) = 0$ or $\partial\lambda/\partial x^k = 0$ for all $k \in \{1, \cdots, n\}$ at each point in N. Thus we obtain Theorem 5.1. □

Comparing proofs of Theorem 3.1 and Theorem 5.1, we have the following about two tension fields.

Corollary 5.2. *Let* $\pi : (M, D^{(\alpha)}, g) \to (\hat{M}, \hat{D}^{(\alpha)}, \hat{g})$ *be the map defined at Theorem 5.1, and* $\pi_{id} : (M, D^{(\alpha)}, g) \to (M, \bar{D}^{(\alpha)}, \bar{g})$ *the identity map, where* $(M, \bar{D}^{(\alpha)}, \bar{g})$ *is the pull-back of* $(\hat{M}, \hat{D}^{(\alpha)}, \hat{g})$ *by* π*. Then it holds that*

$$\tau_{(g,D^{(\alpha)},\hat{D}^{(\alpha)})}(\pi) = e^\lambda \tau_{(g,D^{(\alpha)},\bar{D}^{(\alpha)})}(\pi_{id}).$$

Remark 5.2. For $n = 2$, if and only if $\alpha = 0$, there exist harmonic maps π_{id} and π with non constant functions ϕ and λ, respectively.

Remark 5.3. For $n \geq 3$, it holds that $-1 < \alpha < 0$ if a map π_{id} or π is a harmonic map with a non constant function ϕ or λ, respectively.

Remark 5.4. For $\alpha \leq -1$ and $\alpha > 0$, there exist no harmonic maps π_{id} and π with non constant functions ϕ and λ, respectively.

Acknowledgement. The author is partially supported by AOARD Grant No.10-1-4060.

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