Some sufficient conditions for univalence of two integral operators

Laura Stanciu and Daniel Breaz

Abstract. New conditions for the univalence of two complex integral operators in the open unit disk are obtained. These original conditions derived by means of specializing certain parameters involved in the definitions of the integral operators involved.

M.S.C. 2010: 30C45, 30C75.
Key words: Analytic functions; open unit disk; integral operators; univalence conditions; general Schwarz Lemma.

1 Two families of integral operators

Let $\mathbb{C}$ be the set of complex numbers. Let $\mathcal{A}$ be the class of functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the *open* unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C},\ |z| < 1\}$$

and satisfy the following usual normalization condition

$$f(0) = f'(0) - 1 = 0.$$

We denote by $\mathcal{P}$ the class of the functions $p(z)$ which are analytic in $\mathbb{U}$ and satisfy the conditions

$$p(0) = 1,\ \mathrm{Re}\{p(z)\} > 0,\ z \in \mathbb{U}.$$

Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which are univalent in $\mathbb{U}$. In this paper, we define two families of integral operators. The first family of integral operators is defined by

$$K_n(z) = \left(\sum_{i=1}^{n} \frac{\alpha_i}{\gamma_i} \int_0^z t^{-1} \prod_{i=1}^{n} \left(f_i(t)\right)^{\frac{\alpha_i}{\gamma_i}} g_i(t)dt \right)^{\frac{1}{\sum_{i=1}^{n} \frac{\alpha_i}{\gamma_i}}} \tag{1.1}$$

Applied Sciences, Vol.15, 2013, pp. 104-111.

$$\alpha_i, \gamma_i \in \mathbb{C},\ \gamma_i \neq 0,\ f_i \in \mathcal{A} \text{ and } g_i \in \mathcal{P} \text{ for all } i \in \{1,2,...,n\}.$$

Remark 1.1. For $\alpha_1 = \alpha_2 = ... = \alpha_n = 1$ and $g_1 = g_2 = ... = g_n = 1$, we obtain the integral operator

$$J_{\gamma_1,\gamma_2,...,\gamma_n}(z) = \left(\sum_{i=1}^{n} \frac{1}{\gamma_i} \int_0^z t^{-1} \prod_{i=1}^{n} (f_i(t))^{\frac{1}{\gamma_i}}\, dt \right)^{\frac{1}{\sum_{i=1}^{n} \frac{1}{\gamma_i}}}$$

studied in [6].

Remark 1.2. For $n = 1$, $\alpha_1 = 1$ and $g_1 = 1$, from (1.1) we obtain the integral operator

$$J_\gamma(z) = \left(\frac{1}{\gamma} \int_0^z t^{-1} (f(t))^{\frac{1}{\gamma}}\, dt \right)^{\gamma}$$

studied in [6].

The second family of integral operators is defined by

$$L_n(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} \right)^{\frac{\alpha_i}{\gamma_i}} g_i(t)dt \tag{1.2}$$

$$\alpha_i, \gamma_i \in \mathbb{C},\ \gamma_i \neq 0,\ f_i \in \mathcal{A} \text{ and } g_i \in \mathcal{P} \text{ for all } i \in \{1,2,...,n\}.$$

Remark 1.3. From (1.2), for $n = 1$, $g_1 = 1$, $\alpha_1 = 1$ and $\frac{1}{\gamma_1} = \alpha$, we obtain the integral operator Kim-Merkes

$$H_\alpha(z) = \int_0^z \left(\frac{f(t)}{t} \right)^{\alpha} dt$$

studied in [3].

In the present paper, we obtain some sufficient conditions for the integral operators $K_n(z)$ and $L_n(z)$ to be in the class $\mathcal{S}$.

2 Preliminary results

In the proof of our main results (Theorem 3.1 and Theorem 3.3 below), we need the

Definition 2.1. (see [2]) A function $f(z) \in \mathcal{A}$ is said to be a member of the class $\mathcal{B}(\mu)$ if and only if

$$\left| \frac{z^2 f'(z)}{[f(z)]^2} - 1 \right| \leq 1 - \mu, \quad z \in \mathbb{U}; 0 \leq \mu < 1. \tag{2.1}$$

Theorem 2.1. (see [1]) *If the function $f(z) = z + a_2 z^2 + ...$ is regular in unit disk $\mathbb{U}$ and*

$$\left(1 - |z|^2\right) \left| \frac{z f''(z)}{f'(z)} \right| \leq 1, \qquad z \in \mathbb{U}$$

then the function f is univalent in $\mathbb{U}$.

Theorem 2.2. (see [5]) *Let the parameters* β, $c \in \mathbb{C}$ *be constrained by*

$$\mathrm{Re}\beta > 0 \quad \text{and} \quad |c| \leq 1, \quad c \neq -1.$$

If $f \in \mathcal{A}$ *satisfies the inequality*

$$\left| c\,|z|^{2\beta} + \left(1 - |z|^{2\beta}\right) \frac{zf''(z)}{\beta f'(z)} \right| \leq 1, \quad z \in \mathbb{U}$$

then the function $F_\beta(z)$ *given by*

$$F_\beta(z) = \left(\beta \int_0^z t^{\beta-1} f'(t)dt\right)^{\frac{1}{\beta}} = z + ...$$

is analytic and univalent in $\mathbb{U}$.

Finally, in our present investigation, we shall also need the familiar Schwarz Lemma (see, for details, [4]).

Lemma 2.3. (**General Schwarz Lemma**) [4] *Let the function* $f(z)$ *be regular in the disk*

$$\mathbb{U}_R = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < R \quad (R > 0)\},$$

with $|f(z)| < M$, *for a fixed number* $M > 0$. *If the function* $f(z)$ *has one zero with multiplicity order bigger than a positive integer* m *for* $z = 0$, *then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m, \quad z \in \mathbb{U}_R. \tag{2.2}$$

The equality (2.2) can hold true only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ *is a real constant.*

3 Univalence conditions

Our main univalence conditions for the integral operators $K_n(z)$, $L_n(z)$ defined by (1.1), (1.2) are asserted by Theorem 3.1 and Theorem 3.3 below.

Theorem 3.1. *Let the functions* $f_i(z) \in \mathcal{A}$ *be in the class* $\mathcal{B}(\mu_i)$, $0 \leq \mu_i < 1$ *satisfy the hypothesis (2.1) of Definition 2.1 and* $\alpha_i, \gamma_i \in \mathbb{C}$, $\gamma_i \neq 0$ *with* $\beta = \sum_{i=1}^n \mathrm{Re}\frac{\alpha_i}{\gamma_i}$ *for all* $i \in \{1, 2, ..., n\}$. *Suppose that for all* $i \in \{1, 2, ..., n\}$

$$M_i \geq 1, \quad N_i \geq 1, \quad g_i(z) \in \mathcal{P}, \quad z \in \mathbb{U}$$

and

$$\mathrm{Re}\beta \geq \sum_{i=1}^n \left(\frac{|\alpha_i|}{|\gamma_i|} \left((2 - \mu_i) M_i + 1\right) + N_i \right).$$

If

$$|f_i(z)| \leq M_i \quad z \in \mathbb{U}, \qquad \left|\frac{zg_i'(z)}{g_i(z)}\right| \leq N_i \quad z \in \mathbb{U} \tag{3.1}$$

and

$$|c| \leq 1 - \frac{1}{\mathrm{Re}\beta} \sum_{i=1}^{n} \left(\frac{|\alpha_i|}{|\gamma_i|} \left(\left(2 - \mu_i\right) M_i + 1\right) + N_i \right) \qquad (c \in \mathbb{C}; c \neq -1)\,, \tag{3.2}$$

for all $i \in \{1, 2, ..., n\}$, then the integral operator $K_n(z)$ defined by (1.1) is in the class $\mathcal{S}$.

Proof. We begin by observing that the integral operator $K_n(z)$ in (1.1) can be rewritten in the form

$$K_n(z) = \left(\sum_{i=1}^{n} \frac{\alpha_i}{\gamma_i} \int_0^z t^{\sum_{i=1}^{n} \frac{\alpha_i}{\gamma_i} - 1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} \right)^{\frac{\alpha_i}{\gamma_i}} g_i(t) dt \right)^{\frac{1}{\sum_{i=1}^{n} \frac{\alpha_i}{\gamma_i}}}.$$

Let us define the function $h(z)$ by

$$h(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} \right)^{\frac{\alpha_i}{\gamma_i}} g(t) dt \qquad (f_i \in \mathcal{A}; g_i \in \mathcal{P})\,.$$

The functions $f_i(z)$ are indeed regular in $\mathbb{U}$ and satisfy the following usual normalization condition

$$f(0) = f'(0) - 1 = 0.$$

Now, calculating the derivatives of $h(z)$ of the first and second orders, we readily obtain

$$h'(z) = \prod_{i=1}^{n} \left(\frac{f_i(z)}{z} \right)^{\frac{\alpha_i}{\gamma_i}} g_i(z) \tag{3.3}$$

and

$$h''(z) = \sum_{i=1}^{n} \left(\frac{\alpha_i}{\gamma_i} \left(\frac{f_i(z)}{z} \right)^{\frac{\alpha_i}{\gamma_i} - 1} \left(\frac{zf_i'(z) - f_i(z)}{z^2} \right) g_i(z) + \left(\frac{f_i(z)}{z} \right)^{\frac{\alpha_i}{\gamma_i}} g_i'(z) \right) \times$$

$$\times \prod_{\substack{k=1 \\ k \neq i}}^{n} \left(\frac{f_k(z)}{z} \right)^{\frac{\alpha_k}{\gamma_k}} g_k(z). \tag{3.4}$$

We easily find, from (3.3) and (3.4), that

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n} \left(\frac{\alpha_i}{\gamma_i} \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) + \frac{zg_i'(z)}{g_i(z)} \right), \tag{3.5}$$

which readily shows that

$$\left|c\,|z|^{2\beta}+\left(1-|z|^{2\beta}\right)\frac{zh''(z)}{\beta h'(z)}\right|$$
$$=\left|c\,|z|^{2\beta}+\left(1-|z|^{2\beta}\right)\frac{1}{\beta}\sum_{i=1}^{n}\left(\frac{\alpha_i}{\gamma_i}\left(\frac{zf_i'(z)}{f_i(z)}-1\right)+\frac{zg_i'(z)}{g_i(z)}\right)\right|$$
$$\le |c|+\frac{1}{|\beta|}\sum_{i=1}^{n}\left(\frac{|\alpha_i|}{|\gamma_i|}\left(\left|\frac{z^2f_i'(z)}{[f_i(z)]^2}\right|\left|\frac{f_i(z)}{z}\right|+1\right)+\left|\frac{zg_i'(z)}{g_i(z)}\right|\right). \tag{3.6}$$

Furthermore, from the hypothesis (3.1) of Theorem 3.1, we have

$$|f_i(z)|\le M_i,\qquad z\in\mathbb{U}\qquad\text{and}\qquad\left|\frac{zg_i'(z)}{g_i(z)}\right|\le N_i,\qquad z\in\mathbb{U}.$$

By applying the General Schwarz Lemma, we thus obtain

$$|f_i(z)|\le M_i\,|z|\,,\qquad z\in\mathbb{U}.$$

Next, by making use of (3.6), we find

$$\left|c\,|z|^{2\beta}+\left(1-|z|^{2\beta}\right)\frac{zh''(z)}{\beta h'(z)}\right|$$
$$\le |c|+\frac{1}{|\beta|}\sum_{i=1}^{n}\left(\frac{|\alpha_i|}{|\gamma_i|}\left(\left(\left|\frac{z^2f_i'(z)}{[f_i(z)]^2}-1\right|+1\right)M_i+1\right)+N_i\right)$$
$$\le |c|+\frac{1}{|\beta|}\sum_{i=1}^{n}\left(\frac{|\alpha_i|}{|\gamma_i|}\left((2-\mu_i)\,M_i+1\right)+N_i\right)$$
$$\le |c|+\frac{1}{\mathrm{Re}\beta}\sum_{i=1}^{n}\left(\frac{|\alpha_i|}{|\gamma_i|}\left((2-\mu_i)\,M_i+1\right)+N_i\right)$$
$$\le 1$$

where we have also used the hypothesis (3.2) of Theorem 3.1.
Finally, by applying Theorem 2.2, we conclude that the integral operator $K_n(z)$ defined by (1.1) is in the class $\mathcal{S}$. This evidently completes the proof of Theorem 3.1. □

Setting $n=1$ in Theorem 3.1, we immediately arrive at the following application of Theorem 3.1.

Corollary 3.2. *Let the function $f(z)\in\mathcal{A}$ be in the class $\mathcal{B}(\mu)$, $0\le\mu<1$ and satisfy the hypothesis (2.1) of Definition 2.1 and $\alpha,\gamma\in\mathbb{C}$, $\gamma\ne 0$ with $\beta=\mathrm{Re}\frac{\alpha}{\gamma}$. Suppose that*

$$M\ge 1,\qquad N\ge 1,\qquad g(z)\in\mathcal{P},\quad z\in\mathbb{U}$$

and

$$\mathrm{Re}\beta\ge\frac{|\alpha|}{|\gamma|}[(2-\mu)\,M+1]+N.$$

If

$$|f(z)| \leq M \qquad z \in \mathbb{U}, \qquad \left|\frac{zg'(z)}{g(z)}\right| \leq N \qquad z \in \mathbb{U}$$

and

$$|c| \leq 1 - \frac{1}{\mathrm{Re}\beta}\frac{|\alpha|}{|\gamma|}[(2-\mu)M+1] - \frac{N}{\mathrm{Re}\beta} \qquad (c \in \mathbb{C}; c \neq -1),$$

then the integral operator

$$K(z) = \left(\frac{\alpha}{\gamma}\int_0^z t^{-1}(f(t))^{\frac{\alpha}{\gamma}} g(t)dt\right)^{\frac{\gamma}{\alpha}}$$

is in the class $\mathcal{S}$.

Theorem 3.3. *Let* α_i, γ_i *be complex numbers,* $\gamma_i \neq 0$ *and* $f_i(z) \in \mathcal{A}$ *for all* $i \in \{1,2,...,n\}$. *Suppose that for all* $i \in \{1,2,...,n\}$

$$M_i \geq 1, \qquad N_i \geq 1, \qquad g_i(z) \in \mathcal{P}, \qquad z \in \mathbb{U}.$$

If for all $i \in \{1,2,...,n\}$

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \leq M_i \qquad z \in \mathbb{U}, \qquad \left|\frac{g_i'(z)}{g_i(z)}\right| \leq N_i \qquad z \in \mathbb{U} \tag{3.7}$$

and

$$\sum_{i=1}^n |\alpha_i| \leq \left(\frac{9}{2\sqrt{3}} - \sum_{i=1}^n N_i\right)\sum_{i=1}^n \frac{|\gamma_i|}{M_i} \tag{3.8}$$

then the integral operator $L_n(z)$ *defined by (1.2) is in the class* $\mathcal{S}$.

Proof. Defining the function

$$h(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\frac{\alpha_i}{\gamma_i}} g_i(t)dt$$

we take the same steps as in the proof of Theorem 3.1. Then, from (3.5), we obtain

$$\left|\frac{zh''(z)}{h'(z)}\right| \leq \sum_{i=1}^n \left(\frac{|\alpha_i|}{|\gamma_i|}\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| + \left|\frac{zg_i'(z)}{g_i(z)}\right|\right). \tag{3.9}$$

From (3.7) and (3.9), we have

$$\left|\frac{zh''(z)}{h'(z)}\right| \leq \sum_{i=1}^n \left(\frac{|\alpha_i|}{|\gamma_i|}M_i + N_i|z|\right).$$

Applying General Schwarz Lemma, we find

$$\left|\frac{zh''(z)}{h'(z)}\right| \leq \sum_{i=1}^n \left(\frac{|\alpha_i|}{|\gamma_i|}M_i|z| + N_i|z|\right)$$

and hence, we get

$$\left(1-|z|^2\right)\left|\frac{zh''(z)}{h'(z)}\right| \leq \left(1-|z|^2\right)|z|\sum_{i=1}^{n}\left(\frac{|\alpha_i|}{|\gamma_i|}M_i+N_i\right). \tag{3.10}$$

Since

$$\max_{|z|\leq 1}\{(1-|z|^2)\,|z|\} = \frac{2\sqrt{3}}{9},$$

from (3.8) and (3.10), we deduce

$$\left(1-|z|^2\right)\left|\frac{zh''(z)}{h'(z)}\right| \leq 1. \tag{3.11}$$

From (3.11) and by Theorem 2.1, we obtain that the integral operator $L_n(z)$ defined by (1.2) is in the class $\mathcal{S}$. □

Setting $n=1$ in Theorem 3.3, we obtain the following consequence of Theorem 3.3.

Corollary 3.4. *Let α, γ be complex numbers, $\gamma \neq 0$ and $f(z) \in \mathcal{A}$. Suppose that*

$$M \geq 1, \quad N \geq 1, \quad g(z) \in \mathcal{P}, \quad z \in \mathbb{U}.$$

If

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq M \quad z \in \mathbb{U}, \qquad \left|\frac{g'(z)}{g(z)}\right| \leq N \quad z \in \mathbb{U}$$

and

$$|\alpha| \leq \frac{|\gamma|}{M}\left(\frac{9}{2\sqrt{3}}-N\right)$$

then the integral operator

$$L_n(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\frac{\alpha}{\gamma}} g(t)dt$$

is in the class $\mathcal{S}$.

Acknowledgements. This work was partially supported by the strategic project POSDRU 107/1.5/S/77265, inside POSDRU Romania 2007-2013 co-financed by the European Social Fund-Investing in People.

References

[1] J. Becker, *Lëownersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen,* J. Reine Angew. Math. 255 (1972), 23-43.

[2] B. A. Frasin, M. Darus, *On certain analytic univalent functions,* IJMMS 25:5 (2001), 305-310.

[3] Y. J. Kim, E. P. Merkes, *On an Integral of Powers of a Spirallike Function,* Kyungpook Math. J., 12 (1972), 249-253.

[4] Z. Nehari, *Conformal Mapping*, McGraw-Hill, 1952.

[5] V. Pescar, *A new generalization of Ahlfors's and Becker's criterion of univalence*, Bull. Malaysian Math. Soc. (Second Ser.) 19 (1996), 53-54.

[6] V. Pescar, D. Breaz, N. Breaz, *Certain sufficient conditions for univalence,* General Mathematics vol. 17, 4 (2009), 97-109.

Author's address:

Laura Stanciu
University of Piteşti, Department of Mathematics
Târgul din Vale Str., No.1, 110040, Piteşti, Argeş, România.
E-mail: laura_stanciu_30@yahoo.com

Daniel Breaz
"1 Decembrie 1918" University of Alba Iulia, Department of Mathematics,
Alba Iulia, 11-13 N. Iorga Str., RO-510000, Romania.
E-mail: dbreaz@uab.ro