Oscillatory solutions to a certain third order difference equation

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Abstract. Many authors have considered the superlinearity conditions to study the oscillatory behavior of certain second order difference equations [10] as well as dynamic equations [3, 5, 6]. In this paper, superlinearity conditions are used to study the sufficient condition for the oscillatory solutions of certain third order difference equation. An example is inserted to illustrate the result.

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1 Introduction

We consider the third order difference equation of the form

$$\Delta(a_n\Delta(b_n\Delta(x_n))) + c_n f^{\nu}(x_{n+1-\tau}) = 0, \;\; n \in N(a), \tag{1.1}$$

where $N(a) = \{a, a+1, a+2, ...\}$, $a = \max(1, \tau)$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are positive real sequences, τ is a nonnegative integer for $n \in N(a)$, ν is the ratio of odd positive integers such that $0 < \nu < 1$, Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$ and the continuous function $f : \mathbb{R} \to \mathbb{R}$ is nondecreasing in u such that $uf(u) > 0$ for $u \neq 0$.

By a solution of the equation (1.1), we mean a real sequence $\{x_n\}$ which is defined for all $n \in N(a)$ and satisfies the equation (1.1) for sufficiently large $n \in N(a)$. A nontrivial solution $\{x_n\}$ of the equation (1.1) is said to be an eventually positive or eventually negative if for every positive integer M there exists a positive integer $n \geq M$ such that $x_n x_{n+1} > 0$. A nontrivial solution $\{x_n\}$ of the equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. An equation is said to be oscillatory if all its solutions are oscillatory.

Some sufficient conditions were established for oscillation and nonoscillation of second order nonlinear neutral delay difference equation using superlinearity and sublinearity conditions, see [10]. Superlinearity and sublinearity conditions were used to study the oscillation criteria for some second order dynamic equations in time scale,

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see [3, 5, 6]. In [8], some sufficient conditions were studied for third order non-neutral difference equation of the form

$$\Delta\left(\frac{1}{a_n}\Delta^2\left(y_n\right)\right)+p_nf\left(y_{\sigma(n)}\right)=0,\ \ n\in\mathbb{N}, \tag{1.2}$$

where $\mathbb{N}=\{0,1,2,...\}$, $a_n>0$, $p_n>0$, $0<\sigma(n)\le n$ for $n\in\mathbb{N}$, $\lim\limits_{n\to\infty}\sigma(n)=\infty$, $R_n=\sum\limits_{s=n_1}^{n-1}a_s\to\infty$ as $n\to\infty$, Δ is the forward difference operator defined by $\Delta x_n=x_{n+1}-x_n$, $f:\mathbb{R}\to\mathbb{R}$ is a continuous function such that $xf(x)>0$, and $\frac{f(x)}{x}\ge L>0$ for all $x\ne 0$.

In [9], sublinearity conditions were used to discuss the oscillatory behavior of second order neutral delay difference equation of the form:

$$\Delta^2(x_n+p_nx_{n-\tau}-q_nx_{n-\sigma})+r_nf^{\nu}(x_{n-l})=0,\ \ n\in\mathbb{N}, \tag{1.3}$$

where $\mathbb{N}=\{0,1,2,...\}$, $p_n>0$, $q_n>0$, $r_n>0$ for $n\in\mathbb{N}$, $\tau\ge0$, $\sigma\ge0$, $l\le0$, $\tau<\sigma$, $l\in\{-s,...,0\}$, $s\ge\sigma$, ν is the ratio of odd positive integers such that $\nu<1$, Δ is the forward difference operator defined by $\Delta x_n=x_{n+1}-x_n$ and the continuous function $f:\mathbb{R}\to\mathbb{R}$ is nondecreasing in u such that $uf(u)>0$ for $u\ne0$.

In [4], it was shown that the following difference equations were considered:

$$\Delta(a_n\Delta(x_n-p_nx_{n-1}))+q_nf(x_{n-\tau})=0,\ \ n\in N(a)\ \text{ and} \tag{1.4}$$

$$\Delta(a_n\Delta(b_n\Delta(y_n)))+q_n^*g(y_{n+1-\tau})=0,\ \ n\in N(a), \tag{1.5}$$

where $N(a)=\{a,a+1,a+2,...\}$, $a=\max(1,\tau)$, $a_n>0$, $p_n>0$, $q_n>0$ for $n\in N(a)$, τ is a nonnegative integer, Δ is the forward difference operator defined by $\Delta x_n=x_{n+1}-x_n$ and the function $f:\mathbb{R}\to\mathbb{R}$ is continuous in u such that $uf(u)>0$ for $u\ne0$, $b_n=\prod\limits_{i=1}^{n}p_i$, $q_n^*=q_nb_n^*$, $f(b_{n-\tau}z)=b_n^*g(z)$, and $g:\mathbb{R}\to\mathbb{R}$ is a continuous function in z such that $zg(z)>0$ for $z\ne0$, they have studied trichotomy of nonoscillatory solutions of the second order neutral difference equation (1.4) by using the same category of the third order difference equation (1.5) under the transformation $y_{n+1}=x_n\prod\limits_{i=1}^{n}\frac{1}{p_i}$.

We observed that many authors have considered superlinearity and sublinearity conditions only for discussing the oscillatory behavior of second order difference equations. Motivated by the above observations, in this paper, we have extended the equation (1.5) to the equation (1.1) and considered superlinearity conditions to study the sufficient condition for oscillatory solutions of the third order difference equation (1.1).

The aim of this paper is to study the sufficient condition for oscillatory solutions of the equation (1.1) under the following conditions:

(C1) $\sum\limits_{s=n_1}^{\infty}\frac{1}{a_s}=\infty$ for $n_1\in N(a)$, (C2) $\sum_{s=n_1}^{\infty} \frac{1}{b_s} = \infty$ for $n_1 \in N(a)$,

(C3) $T_n = \sum_{s=n_1}^{n-1} \frac{1}{a_s} \sum_{t=s+1}^{n-1} \frac{1}{b_t} = \sum_{t=n_1}^{n-1} \frac{1}{b_t} \sum_{s=n_1}^{t-1} \frac{1}{a_s}$,

The result obtained here is an extension to some of those achieved in [3, 4, 5, 6, 8, 9, 10]. For the general theory of difference equations, one can refer to [1, 2]. Many references to some applications of the difference equations can be found in [7].

2 Preliminary results

We use the following Definition and Lemma to prove our main result.

Definition 2.1 (Superlinear function)**.** (see [10]) Let the function $f : \mathbb{R} \to \mathbb{R}$ be continuous and nondecreasing with $uf(u) > 0$ for $u \neq 0$. The function f is said to be superlinear if

$$0 < \int_{\epsilon}^{\infty} \frac{dx}{f(x)}, \int_{-\infty}^{-\epsilon} \frac{dx}{f(x)} < \infty \text{ for all } \epsilon > 0. \tag{2.1}$$

The above conditions (2.1) are called superlinearity conditions of $f(x)$, see [5].

Lemma 2.1 (see [4])**.** *Assume that the conditions (C1) and (C2) hold. Let $\{x_n\}$ be an eventually positive solution to the equation (1.5). Then exactly one of the following statements holds:*

(i) $x_n > 0$, $\Delta x_n > 0$, $\Delta(b_n \Delta x_n) > 0$;

(ii) $x_n > 0$, $\Delta x_n < 0$, $\Delta(b_n \Delta x_n) > 0$,

for sufficiently large n.

The proof of the above Lemma is trivial, and hence omitted.

3 The main result

This section provides the sufficient condition for oscillatory solutions of the equation (1.1).

Theorem 3.1. *In addition to the conditions (C1), (C2), (C3), assume that*

$$0 < \int_{\epsilon}^{\infty} \frac{dx}{f^{\nu}(x)}, \int_{-\infty}^{-\epsilon} \frac{dx}{f^{\nu}(x)} < \infty \text{ for all } \epsilon > 0 \tag{3.1}$$

holds. If

$$\sum_{t=n_1}^{n-1} \frac{1}{b_t} \sum_{s=n_1}^{t-1} \frac{1}{a_s} \sum_{r=s}^{s+1-\tau} c_r = \infty \text{ as } n \to \infty, \tag{3.2}$$

then the equation (1.1) is oscillatory.

Proof. Let $\{x_n\}$ be an eventually positive solution of the equation (1.1). Then there exists $n_1 \in N(a)$ such that $x_n > 0$ and $x_{n+1-\tau} > 0$ for $n \in N(n_1)$. Therefore from Lemma 2.1, we assume that $x_n > 0$, $\Delta x_n < 0$, $\Delta(b_n \Delta x_n) > 0$ for $n \in N(n_1)$. From equation (1.1) we see that

$$\Delta(a_n \Delta(b_n \Delta(x_n))) < 0 \text{ for } n \in N(n_1). \tag{3.3}$$

This implies that $\{a_n \Delta(b_n \Delta(x_n))\}$ is nonincreasing for $n \in N(n_1)$. By our assumption we have $\Delta(b_n \Delta x_n) > 0$ for $n \in N(n_1)$, which implies that $\{b_n \Delta x_n\}$ is increasing for $n \in N(n_1)$. Again by our assumption we have $\Delta x_n < 0$ for $n \in N(n_1)$. Thus, there exists $n_2 \geq n_1$ and $k > 0$ such that

$$\Delta x_n \leq -k \text{ for } n \in N(n_2).$$

By summing the above inequality from n_2 to $n-1$, we obtain

$$x_n \leq x_{n_2} - k(n - n_2) \text{ for } n \in N(n_2).$$

This implies that $x_n \to -\infty$ as $n \to \infty$, which is a contradiction to the fact that $x_n > 0$ for all $n \in N(n_1)$. Thus, the condition (i) of Lemma 2.1 holds.

Suppose that $K > 0$. Then from equation (1.1), we have

$$\Delta(a_n \Delta(b_n \Delta(x_n))) + \frac{c_n}{K} f^{\nu}(x_{n+1-\tau}) \leq 0 \text{ for } n \in N(n_1).$$

By summing the above inequality from n to ∞, we obtain

$$-\Delta(b_n \Delta(x_n)) + \frac{1}{a_n} \sum_{r=n}^{\infty} \frac{c_r}{K} f^{\nu}(x_{r+1-\tau}) \leq 0 \text{ for } n \in N(n_1),$$

and further, by summing the above inequality from n_1 to $n-1$, we infer that, for all $n \in N(n_1)$:

$$-\Delta x_n + \frac{1}{b_n} \sum_{s=n_1}^{n-1} \frac{1}{a_s} \sum_{r=s}^{\infty} \frac{c_r}{K} f^{\nu}(x_{r+1-\tau}) \leq 0,$$

that is:

$$\begin{cases} \frac{1}{b_n} \sum_{s=n_1}^{n-1} \frac{1}{a_s} \sum_{r=s}^{\infty} \frac{c_r}{K} f^{\nu}(x_{r+1-\tau}) \leq \Delta x_n, \\ \left(\frac{1}{b_n} \sum_{s=n_1}^{n-1} \frac{1}{a_s} \sum_{r=s}^{s+1-\tau} \frac{c_r}{K} \right) f^{\nu}(x_n) \leq \Delta x_n, \\ \frac{1}{b_n} \sum_{s=n_1}^{n-1} \frac{1}{a_s} \sum_{r=s}^{s+1-\tau} \frac{c_r}{K} \leq \frac{\Delta x_n}{f^{\nu}(x_n)}. \end{cases}$$

By summing the above inequality from n_1 to $n-1$, we have

$$\frac{1}{K} \sum_{t=n_1}^{n-1} \frac{1}{b_t} \sum_{s=n_1}^{t-1} \frac{1}{a_s} \sum_{r=s}^{s+1-\tau} c_r \leq \sum_{t=n_1}^{n-1} \frac{\Delta x_t}{f^{\nu}(x_t)} \leq \sum_{t=n_1}^{n-1} \int_{x_{t-1}}^{x_t} \frac{du}{f^{\nu}(u)} \leq \int_{n_1}^{n-1} \frac{du}{f^{\nu}(u)}.$$

Therefore, we obtain from (3.1) that

$$\frac{1}{K} \sum_{t=n_1}^{n-1} \frac{1}{b_t} \sum_{s=n_1}^{t-1} \frac{1}{a_s} \sum_{r=s}^{s+1-\tau} c_r < \infty \text{ as } n \to \infty,$$

which is a contradiction to the condition (3.2). Thus the equation (1.1) is oscillatory.

Similar arguments can be used for proving the result when $\{x_n\}$ is eventually negative and so we omit this proof. □

We give the following example to illustrate the result.

Example 3.1. Consider the third order difference equation:

$$\Delta^3 x_n + 8x_n^{\frac{1}{3}} = 0,$$

where $a_n = 1$, $b_n = 1$, $c_n = 8$, $\tau = 1$, $\nu = \frac{1}{3}$, $f^\nu(u) = u^\nu$. We note that all the conditions of Theorem 3.1 are satisfied, and hence all its solutions are oscillatory (e.g., $\{x_n\} = \{(-1)^n\}$ is one such solution).

References

[1] R. P. Agarwal, *Difference Equations and Inequalities, Theory, Methods and Applications*, Marcel Dekker, New York 2000.

[2] R. P. Agarwal, M. Bohner, S. R. Grace, D. O'Regan, *Discrete Oscillation Theory*, Hindawi, New York 2005.

[3] J. Baoguo, L. Erbe and A. Peterson, *Kiguradze-type oscillation theorems for second order superlinear dynamic equations on time scales*, Canad. Math. Bull. 54, 4 (2011), 580-592.

[4] A. Bezubik, M. Migda, M. Nockowska-Rosiak and E. Schmeidel, *Trichotomy of nonoscillatory solutions to second-order neutral difference equation with quasi-difference*, Adv. Differ. Equ. 2015, 192 (2015), 1-14. DOI 10.1186/s13662-015-0531-6

[5] T. S. Hassan, *Oscillation criteria for second order nonlinear dynamic equations*, Adv. Differ. Equ. 2012, 171 (2012), 1-13. DOI 10.1186/1687-1847-2012-171

[6] T. S. Hassan, L. Erbe and A. Peterson, *Oscillation of second order superlinear dynamic equations with damping on time scales*, Comput. Math. Appl. 59, 1 (2010), 550-558.

[7] W. G. Kelley and A. C. Peterson, *Difference Equations - An Introduction with Applications*, Academic Press, San Diego 2001.

[8] B. Selvaraj and I. Mohammed Ali Jaffer, *On the oscillation of the solution to third order nonlinear difference equations*, J. Comp. Math. Sci. 1, 7 (2010), 873-876.

[9] B. Selvaraj and M. Raju, *Asymptotic and oscillatory behavior of certain second order neutral delay difference equations*, International Journal of Mathematical Analysis(IJMA). 9, 20(2015), 965-977. DOI 10.12988/ijma.2015.5390

[10] E. Thandapani and P. Mohan Kumar, *Oscillation and nonoscillation of nonlinear neutral delay difference equations*, Tamkang J. Math. 38, 4 (2007), 323-333.

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