Fixed point of nonself total asymptotically nonexpansive mappings in Banach spaces

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Abstract. In this work, a new iterative process is introduced for two nonself total asymptotically nonexpansive mappings. Weak and strong convergence theorems for computing common fixed points of two nonself total asymptotically nonexpansive mappings are established in the framework of Banach spaces. Finally, applicability of our theorems are shown by an example.

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1 Introduction

Let K be a nonempty subset of a real normed linear space E. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$. A mapping $T : K \to K$ is called

(i) nonexpansive if T satisfy $\|Tx - Ty\| \leq \|x - y\|$,

(ii) asymptotically nonexpansive if there exists a sequence $\{k_n\} \in [1, \infty)$ satisfying $\lim_{n\to\infty} k_n = 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \; n \geq 1, \tag{1.1}$$

(iii) uniformly L-Lipschitzian if there exists constant $L \geq 0$ such that

$$\|T^n x - T^n y\| \leq L \|x - y\|,$$

(iv) asymptotically quasi-nonexpansive if $F(T) \neq \varnothing$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - p\| \leq k_n \|x - p\|$$

for all $x, y \in K$ and $p \in F(T)$.

Note that an asymptotically nonexpansive mapping with a nonempty fixed point set is an asymptotically quasi-nonexpansive, but the converse of this statement is not

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true. As a generalization of nonexpansive mappings, Goebel and Kirk [7] introduced asymptotically nonexpansive mappings. They also showed that an asymptotically nonexpansive mapping defined on a nonempty closed bounded subset of a real uniformly convex Banach space has a fixed point.

Iterative techniques for asymptotically nonexpansive self-mapping in Banach spaces including modified Mann and modified Ishikawa iterations processes have been studied widely by varied authors. However, these iteration processes may fail to be well defined when the domain of T is a proper subset of E.

If there exists a continuous mapping $P : E \to K$ such that $Px = x$, for all $x \in K$, then a subset K of E is said to be a retract of E. As an example of a retract of any uniformly convex Banach space E, we can give a closed convex subset of E. The mapping P is called a retraction if it satisfy $P^2 = P$. It follows that, if a mapping P is a retraction, then $Py = y$ for all y in the range of P. Intercalarily, if K is closed convex and P satisfy $P(Px + t(x - Px)) = Px$ for all $x \in E$ and for all $t \geq 0$, then P is said to be sunny [3].

In 2003, Chidume et al. [6] defined a new concept of asymptotically nonexpansive self-mapping, which is defined as follows:

Definition 1.1. *[6]* Let K be a nonempty subset of a real normed space E and $P : E \to K$ be a nonexpansive retraction of E onto K. A nonself mapping $T : K \to E$ *is called asymptotically nonexpansive if there exists a sequence* $\{k_n\} \subset [1, \infty)$ *with* $\lim_{n\to\infty} k_n = 1$ *such that*

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x - y\| \tag{1.2}$$

for all $x, y \in K$ *and* $n \geq 1$. T *is called uniformly* L*-Lipschitzian if there exists a constant* $L > 0$ *such that*

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\|$$

for all $x, y \in K$ *and* $n \geq 1$.

Note that if T is a self-mapping, then P is an identity mapping.

To define a new and more general class of nonlinear mappings, Alber et al. [2] introduced total asymptotically nonexpansive mappings.

Definition 1.2. [2] Let K be a nonempty closed subset of a real normed linear space E. A mapping $T : K \to K$ is called total asymptotically nonexpansive if there exist nonnegative real sequences $\{\mu_n\}$, $\{\lambda_n\}$ with $\mu_n, \lambda_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$ such that

$$\|T^n x - T^n y\| \leq \|x - y\| + \mu_n \phi(\|x - y\|) + \lambda_n, \tag{1.3}$$

for all $x, y \in K$, $n \geq 1$.

Remark 1.3. If $\phi(\lambda) = \lambda$ and $\lambda_n = 0$ for all $n \geq 1$, then total asymptotically nonexpansive mappings coincide with asymptotically nonexpansive mappings. Thus an asymptotically nonexpansive mappings is a total asymptotically nonexpansive mappings.

The strongly convergence of iterative processes for a finite family of total asymptotically nonexpansive mappings in Banach spaces have been studied by Chidume and Ofoedu [4, 5] and they defined nonself total asymptotically nonexpansive mappings as follow:

Definition 1.4. [4] Let K be a nonempty subset of E. Let $P : E \to K$ be the nonexpansive retraction of E onto K. A nonself mapping $T : K \to E$ is called total asymptotically nonexpansive if there exist nonnegative real sequences $\{\mu_n\}$, $\{\lambda_n\}$ with $\mu_n, \lambda_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$ such that

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \leq \|x - y\| + \mu_n \phi(\|x - y\|) + \lambda_n, \tag{1.4}$$

for all $x, y \in K$, $n \geq 1$.

Remark 1.5. If $\phi(\lambda) = \lambda$ and $\lambda_n = 0$ for all $n \geq 1$, then nonself total asymptotically nonexpansive mappings coincide with nonself asymptotically nonexpansive mappings.

Remark 1.6. (*see* [21]) In case $T : K \to E$ is asymptotically nonexpansive and $P : E \to K$ is a nonexpansive retraction, then $PT : K \to K$ is asymptotically nonexpansive. In fact we have

$$\begin{aligned} \|(PT)^n x - (PT)^n y\| &= \left\|PT(PT)^{n-1}x - PT(PT)^{n-1}y\right\| \\ &\leq \left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \\ &\leq k_n \|x - y\|, \end{aligned} \tag{1.5}$$

for all x,$y \in K$ and $n \in \mathbb{N}$.

But the converse of this is not true. Therefore, Zhou et al. [21] introduced the following generalized definition recently.

Definition 1.7. [21] Let K be a nonempty subset of real normed linear space E. Let $P : E \to K$ be the nonexpansive retraction of E into K. A nonself mapping $T : K \to E$ is called asymptotically nonexpansive with respect to P if there exists sequences $\{k_n\} \in [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|(PT)^n x - (PT)^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in K, \ n \in \mathbb{N}. \tag{1.6}$$

T is said to be uniformly L-Lipschitzian with respect to P if there exists a constant $L \geq 0$ such that

$$\|(PT)^n x - (PT)^n y\| \leq L \|x - y\|, \quad \forall x, y \in K, \ n \in \mathbb{N}. \tag{1.7}$$

Incorporating the above definitions, Khan et al. [10] introduced the following more general definition of nonself total asymptotically nonexpansive mappings.

Definition 1.8. Let K be a nonempty subset of E. Let $P : E \to K$ be the nonexpansive retraction of E onto K. A nonself mapping $T : K \to E$ is called total asymptotically nonexpansive if there exist nonnegative real sequences $\{\mu_n\}$, $\{\lambda_n\}$ with $\mu_n, \lambda_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$ such that

$$\|(PT)^n x - (PT)^n y\| \leq \|x - y\| + \mu_n \phi(\|x - y\|) + \lambda_n, \text{for all } x, y \in K, \ n \geq 1. \tag{1.8}$$

Remark 1.9. We note that if $T : K \to E$ is a total asymptotically nonexpansive mapping defined in (1.4) and $P : E \to K$ is a nonexpansive retraction, then $PT : K \to K$ is total asymptotically nonexpansive defined (1.3). Actually using (1.8), we have

$$\begin{aligned}\|(PT)^n x - (PT)^n y\| &\leq \left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \\ &\leq \|x-y\| + \mu_n \phi\left(\|x-y\|\right) + \lambda_n, \text{for all } x, y \in K,\ n \geq 1.\end{aligned}$$

Conversely, it may not be true.

Remark 1.10. If $\phi(\lambda) = \lambda$ and $\lambda_n = 0$ for all $n \geq 1$, then (1.8) reduces to (1.3).

In this paper, we propose an iteration process for calculating common fixed points of two nonself total asymptotically nonexpansive mappings; and give convergence criteria for this iteration process for the mappings in Banach spaces. Some convergence criteria of the iteration process under some restrictions is also established in uniformly convex Banach spaces. We use the following iteration:

Let K be a nonempty closed convex subset of a real normed linear space E with retraction P. Let $T_1, T_2 : K \to E$ be two nonself asymptotically nonexpansive mappings with respect to P.

$$(1.9) \qquad \begin{cases} x_1 \in K \\ x_{n+1} = (1-\alpha_n)\left(PT_1\right)^n y_n + \alpha_n \left(PT_2\right)^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n \left(PT_1\right)^n x_n, \quad n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$.

2 Preliminaries

Now we recall some results and concepts.

Let E be a Banach space with its dimension equal or greater than 2. The function $\delta_E(\varepsilon) : (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \left\|\frac{1}{2}(x+y)\right\| : \|x\| = 1,\ \|y\| = 1,\ \varepsilon = \|x-y\|\right\}$$

is called modulus of E. If $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$ then E is called uniformly convex.

Let E be a Banach space and $S(E) = \{x \in E : \|x\| = 1\}$. If the limit $\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$ exists for all $x, y \in S(E)$, then E said to be smooth.

Let E be a Banach space. E is said to have the Opial property if, whenever $\{x_n\}$ is a sequence in E converging weakly to some $x_0 \in E$ and $x \neq x_0$, it follows that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

Let K be a nonempty subset of a Banach space E. For $x \in K$, the inward set of x is defined by $\{x + \lambda(u-x) : u \in K, \lambda \geq 1\}$ and it is indicated by $I_K(x)$. Let $cl[I_K(x)]$

denotes the closure of the inward set. A mapping $T : K \to E$ is called weakly inward if $Tx \in cl[I_K(x)]$ for all $x \in K$. We can give self-mappings as example of weakly inward mappings.

We denote domain and range of a mapping T in E by $D(T)$ and $R(T)$, respectively. T is said to be demiclosed at p if for any given sequence $\{x_n\}$ in $D(T)$, the conditions $x_n \rightharpoonup x_0$ and $Tx_n \to p$ imply $Tx_0 = p$, where $x_n \rightharpoonup x_0$ means that $\{x_n\}$ converges weakly to x_0.

Let $T : K \to K$ be a mapping. $T : K \to K$ is called

(i) completely continuous if for every bounded sequence $\{x_n\}$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{Tx_{n_j}\}$ converges to some element in $R(T)$,

(ii) demi-compact if any sequence $\{x_n\}$ in K satisfying $x_n - Tx_n \to 0$ as $n \to \infty$ has a convergent subsequence.

(iii) satisfy condition (A) [17] if $F(T) \neq \varnothing$ and there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0, \infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in K$, where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.

We use a modified version of condition (A) defined by Khan and Fukharuddin [9] for two mappings, as follows:

Two mappings T_1, $T_2 : K \to K$ are said to satisfy condition (A') [9] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0, \infty)$ such that

$$\frac{1}{2}(\|x - PT_1x\| + \|x - PT_2x\|) \geq f(d(x, F))$$

for all $x \in K$.

It is pointed out that condition (A') reduces to condition (A) when T_1 equal to T_2. Also, condition (A) is weaker than demicompactness or semicompactness of T, see [17].

For proving the main theorems of our paper, we need the following lemmas.

Lemma 2.1. *[19] If $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are three sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1 + b_n) a_n + c_n, \quad n \in \mathbb{N}$$

and the sum $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ are finite, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.2. *[16]* Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Also, suppose that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that

$$\limsup_{n\to\infty} \|x_n\| \leq r, \quad \limsup_{n\to\infty} \|y_n\| \leq r \text{ and } \lim_{n\to\infty} \|(1 - t_n) x_n + t_n y_n\| = r$$

hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.3. *[18] Let E be real smooth Banach space, let K be nonempty closed convex subset of E with P as a sunny nonexpansive retraction, and let $T : K \to E$ be a mapping satisfying weakly inward condition. Then $F(PT) = F(T)$.*

3 Main results

In this section, we shall prove the convergence of the iteration scheme defined (1.9) for two nonself total asymptotically nonexpansive mappings. We always assume $F = F(T_1) \cap F(T_2) = \{x \in K : T_1 x = T_2 x = x\} \neq \varnothing$. In order to prove our main results, we shall make use of following lemmas.

Lemma 3.1. *Let E be a real Banach space, K be a nonempty subset of E which as also a nonexpansive retract with retraction P. Let $T_1, T_2 : K \to E$ be two nonself total asymptotically nonexpansive mappings with sequences $\{\mu_n\}, \{\lambda_n\}$. Suppose that there exist $M, M^* > 0$, such that $\phi(\kappa) \leq M^* \kappa$ for all $\kappa \geq M$. Then for any $x, y \in K$ following inequalities hold;*

$$\|(PT_1)^n x - (PT_1)^n y\| \leq (1 + \mu_n M^*) \|x - y\| + \mu_n \phi(M) + \lambda_n, \tag{3.1}$$

$$\|(PT_2)^n x - (PT_2)^n y\| \leq (1 + \mu_n M^*) \|x - y\| + \mu_n \phi(M) + \lambda_n, \ n \geq 1. \tag{3.2}$$

Proof. Since $T_1, T_2 : K \to E$ are two nonself total asymptotically nonexpansive mappings with sequences $\{\mu_n\}, \{\lambda_n\}$, then we have for all $x, y \in K$,

$$\|(PT_1)^n x - (PT_1)^n y\| \leq \|x - y\| + \mu_n \phi(\|x - y\|) + \lambda_n, \tag{3.3}$$

$$\|(PT_2)^n x - (PT_2)^n y\| \leq \|x - y\| + \mu_n \phi(\|x - y\|) + \lambda_n, n \geq 1. \tag{3.4}$$

Since $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing continuous function with $\phi(0) = 0$, we get $\phi(\kappa) \leq \phi(\kappa)$ whenever $\kappa \leq M$ and $\phi(\kappa) \leq M^* \kappa$ if $\kappa \geq M$. In two case, we get

$$\phi(\kappa) \leq \phi(M_1) + M_1^* \kappa, \quad \phi(\sigma) \leq \phi(M_2) + M_2^* \sigma \tag{3.5}$$

for some $M, M^* \geq 0$. The (3.3) and (3.4) with (3.5) yield that

$$\|(PT_1)^n x - (PT_1)^n y\| \leq (1 + \mu_n M^*) \|x - y\| + \mu_n \phi(M) + \lambda_n,$$
$$\|(PT_2)^n x - (PT_2)^n y\| \leq (1 + \mu_n M^*) \|x - y\| + \mu_n \phi(M) + \lambda_n, \ n \geq 1.$$

This completes the proof. □

Lemma 3.2. *Let E be a real Banach space, K be a closed convex nonempty subset of E which as also a nonexpansive retract with retraction P. Let $T_1, T_2 : K \to E$ be two nonself total asymptotically nonexpansive mappings with sequences $\{\mu_n\}, \{\lambda_n\}$ such that $\sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} \lambda_n < \infty$. Suppose that there exist $M, M^* > 0$, such that $\phi(\kappa) \leq M^* \kappa$ for all $\kappa \geq M$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0, 1]$, $F \neq \varnothing$ and $\{x_n\}$ is defined by (1.9). Then the limits $\lim_{n\to\infty} \|x_n - p\|$ and $\lim_{n\to\infty} d(x_n, F)$ exists, where $d(x_n, F) = \inf_{p \in F} \|x_n - p\|$.*

Proof. Let $p \in F$. It follows from (1.9) and (3.1) that

$$\begin{aligned}
\|y_n - p\| &= \|(1 - \beta_n)x_n + \beta_n (PT_1)^n x_n - p\| \\
&\leq (1 - \beta_n) \|x_n - p\| + \beta_n \|(PT_1)^n x_n - p\| \\
&\leq (1 - \beta_n) \|x_n - p\| + \beta_n [(1 + \mu_n M^*) \|x - p\| + \mu_n \phi(M) + \lambda_n] \\
&\leq \|x_n - p\| + \beta_n \mu_n M^* \|x_n - p\| + \mu_n \phi(M) + \lambda_n \\
&\leq (1 + \mu_n M^*) \|x_n - p\| + \phi(M) \mu_n + \lambda_n. \qquad (3.6)
\end{aligned}$$

Similarly, from (1.9), (3.1) and (3.2) we have

$$\begin{aligned}
\|x_{n+1}-p\| &= \|(1-\alpha_n)(PT_1)^n y_n + \alpha_n (PT_2)^n y_n - p\| \\
&\le (1-\alpha_n)\|(PT_1)^n y_n - p\| + \alpha_n \|(PT_2)^n y_n - p\| \\
&\le (1-\alpha_n)[(1+\mu_n M^*)\|y_n - p\| + \mu_n \phi(M) + \lambda_n] \\
&+ \alpha_n[(1+\mu_n M^*)\|y_n - p\| + \mu_n \phi(M) + \lambda_n] \\
&\le (1+\mu_n M^*)\|y_n - p\| + \mu_n \phi(M) + \lambda_n \\
&\le (1+\mu_n M^*)[(1+\mu_n M^*)\|x_n - p\| + \mu_n \phi(M) + \lambda_n] \\
&\le \left(1 + 2\mu_n M^* + (\mu_n M^*)^2\right)\|x_n - p\| \\
&+ \phi(M) M^* \mu_n^2 + M^* \lambda_n \mu_n M^* + \mu_n \phi(M) + \lambda_n \\
&\le (1+b_n)\|x_n - p\| + c_n, \qquad (3.7)
\end{aligned}$$

where $b_n = 2\mu_n M_1^* + (\mu_n M_1^*)^2$ and $c_n = \phi(M_1) M_1^* \mu_n^2 + M_1^* \lambda_n \mu_n M_1^* + \mu_n \phi(M_1) + \lambda_n$. Since (3.7) is true for each p in F, we infer that

$$d(x_{n+1}, p) \le (1+b_n)\, d(x_n, p) + c_n. \qquad (3.8)$$

Since $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, from Lemma 2.1, we obtain that $\lim\limits_{n\to\infty} \|x_n - p\|$ and $\lim_{n\to\infty} d(x_n, F)$ exists. This completes the proof. □

Now we have enough knowledge to formulate and prove a criterion on strong convergence of $\{x_n\}$ given by (1.9).

Theorem 3.3. *Let E be a real Banach space, K be a nonempty closed convex subset of E which as also a nonexpansive retract with retraction P. Let $T_1, T_2 : K \to E$ be two continuous nonself total asymptotically nonexpansive mappings with sequences $\{\mu_n\}, \{\lambda_n\}$ such that $\sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} \lambda_n < \infty$. Suppose that there exist $M, M^* > 0$ such that $\phi(\kappa) \le M^* \kappa$ for all $\kappa \ge M$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1]$, $F \neq \varnothing$ and $\{x_n\}$ is defined by (1.9). Then the sequence $\{x_n\}$ strongly converges to a common fixed point of T_1 and T_2 if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

Proof. The necessity of the conditions is obvious. Therefore, we give the proof for sufficiency.
Assume that $\liminf_{n\to\infty} d(x_n, F) = 0$. From Lemma 3.2, $\lim_{n\to\infty} d(x_n, F)$ exists. Our hypothesis implies $\liminf_{n\to\infty} d(x_n, F) = 0$, therefore we get $\lim_{n\to\infty} d(x_n, F) = 0$.

Now we shall show that $\{x_n\}$ is a Cauchy sequence in E. As $1 + t \le \exp(t)$ for all $t > 0$, from (3.8), we obtain

$$\|x_{n+1} - p\| \le \exp b_n \left(\|x_n - p\| + c_n\right). \qquad (3.9)$$

Thus, for any given m, n, iterating (3.9), we obtain

$$\begin{aligned} \|x_{n+m} - p\| &\leq \exp b_{n+m-1} \left(\|x_{n+m-1} - p\| + c_{n+m-1} \right) \\ &\vdots \\ &\leq \exp(\sum_{i=n}^{n+m-1} b_i) \left(\|x_n - p\| + \sum_{i=n}^{n+m-1} c_i \right) \\ &\leq \exp(\sum_{i=n}^{\infty} b_i) \left(\|x_n - p\| + \sum_{i=n}^{\infty} c_i \right). \end{aligned}$$

Therefore,

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - p\| + \|x_n - p\| \\ &\leq \left[1 + \left(\exp(\sum_{i=n}^{\infty} b_i) \right) \right] \|x_n - p\| \\ &\quad + \exp(\sum_{i=n}^{\infty} b_i) \left(\sum_{i=n}^{\infty} c_i \right). \end{aligned}$$

So

$$\|x_{n+m} - x_n\| \leq D \|x_n - p\| + D c_i \tag{3.10}$$

for a real number $D > 0$. Taking infimum on $p \in F$ in (3.10), we get

$$\|x_{n+m} - x_n\| \leq D d(x_n, F) + D \left(\sum_{i=n}^{\infty} c_i \right).$$

For given $\epsilon > 0$, using $\lim_{n\to\infty} d(x_n, F) = 0$ and $\sum_{i=n}^{\infty} c_i < \infty$, there exists an integer $N_1 > 0$ such that for all $n \geq N_1$, $d(x_n, F) < \epsilon/2D$ and $\sum_{i=n}^{\infty} c_i < \epsilon/2D$. Consequently, from last inequality we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq D d(x_n, F) + D \left(\sum_{i=n}^{\infty} c_i \right) \\ &\leq D \frac{\epsilon}{2D} + D \frac{\epsilon}{2D} = \epsilon, \end{aligned}$$

which means that $\{x_n\}$ is a Cauchy sequence. Since the space E is complete, thus $\lim_{n\to\infty} x_n$ exists. Let $\lim_{n\to\infty} x_n = q$. Since T_1 and T_2 are continuous mappings, the set of common fixed points of T_1 and T_2 is closed. We now show that $q \in F$. Suppose that $q \notin F$. Since F is closed subset of E, we have that $d(q, F) > 0$. But, for all $p \in F$, we have

$$\|q - p\| \leq \|q - x_n\| + \|x_n - p\|.$$

This inequality gives

$$d(q, F) \leq \|q - x_n\| + d(x_n, F),$$

and so we get $d(q, F) = 0$ as $n \to \infty$, which contradicts the fact $d(q, F) > 0$. Hence, $q \in F$. This completes the proof of the theorem. □

For our next theorems, we start by proving the following lemma which will be used in the sequel.

Lemma 3.4. *Let E be a uniformly convex real Banach space, K be a nonempty closed convex subset of E which as also a nonexpansive retract with retraction P. Let $T_1, T_2 : K \to E$ be two uniformly L-Lipschitzian, nonself total asymptotically nonexpansive mappings with sequences $\{\mu_n\}, \{\lambda_n\}$ such that $\sum_{n=1}^{\infty}\mu_n < \infty, \sum_{n=1}^{\infty}\lambda_n < \infty$. Suppose that there exist $M, M^* > 0$, such that $\phi(\kappa) \leq M^*\kappa$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $[\varepsilon, 1-\varepsilon]$, for some $\varepsilon \in (0,1)$. Suppose that $\{x_n\}$ is generated iteratively by (1.9). Then*

$$\lim_{n\to\infty} \|x_n - T_1x_n\| = \lim_{n\to\infty} \|x_n - T_2x_n\| = 0.$$

Proof. For any given $p \in F$, by Lemma 3.2, we know that $\lim_{n\to\infty} \|x_n - p\|$ exists. Suppose $\lim_{n\to\infty} \|x_n - p\| = c$, for some $c \geq 0$. If $c = 0$, there is no anything to prove. Assume $c > 0$. Taking lim sup on (3.6), we have

$$\limsup_{n\to\infty} \|y_n - p\| \leq \limsup_{n\to\infty} (1 + \mu_n M^*) \|x_n - p\| + \phi(M)\mu_n + \lambda_n = c. \tag{3.11}$$

Therefore $\|(PT_1)^n y_n - p\| \leq (1+\mu_n M^*) \|y_n - p\| + \mu_n\phi(M) + \lambda_n$ for all $n \geq 1$ implies that

$$\limsup_{n\to\infty} \|(PT_1)^n y_n - p\| \leq c. \tag{3.12}$$

Similar way, we get

$$\limsup_{n\to\infty} \|(PT_2)^n y_n - p\| \leq \limsup_{n\to\infty} (1 + \mu_n M^*) \|y_n - p\| + \mu_n\phi(M) + \lambda_n \leq c. \tag{3.13}$$

In addition,

$$\begin{aligned} c &= \lim_{n\to\infty} \|x_{n+1} - p\| \\ &= \lim_{n\to\infty} \|(1-\alpha_n)(PT_1)^n y_n + \alpha_n (PT_2)^n y_n - p\| \\ &= \lim_{n\to\infty} \|(1-\alpha_n)((PT_1)^n y_n - p) + \alpha_n ((PT_2)^n y_n - p)\|, \end{aligned} \tag{3.14}$$

which gives that

$$\lim_{n\to\infty} \|(1-\alpha_n)((PT_1)^n y_n - p) + \alpha_n ((PT_2)^n y_n - p)\| = c. \tag{3.15}$$

Now using (3.12) with (3.13) and applying Lemma 2.2 to (3.15), we obtain

$$\lim_{n\to\infty} \|(PT_1)^n y_n - (PT_2)^n y_n\| = 0. \tag{3.16}$$

Noting that

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\alpha_n)(PT_1)^n y_n + \alpha_n (PT_2)^n y_n - p\| \\ &\leq \|(PT_1)^n y_n - p\| + \alpha_n \|(PT_2)^n y_n - (PT_1)^n y_n\| \\ &\leq (1 + \mu_n M^*) \|y_n - p\| + \mu_n\phi(M) + \lambda_n \\ &\quad + \alpha_n \|(PT_2)^n y_n - (PT_1)^n y_n\|, \end{aligned} \tag{3.17}$$

this implies that

$$c \leq \liminf_{n\to\infty} \|y_n - p\| . \tag{3.18}$$

From (3.11) and (3.18), we get

$$\lim_{n\to\infty} \|y_n - p\| = c. \tag{3.19}$$

Otherwise, $\|(PT_1)^n x_n - p\| \leq (1 + \mu_n M^*) \|x_n - p\| + \mu_n \phi(M) + \lambda_n$ for all $n \geq 1$ implies that

$$\limsup_{n\to\infty} \|(PT_1)^n x_n - p\| \leq c. \tag{3.20}$$

Consequently,

$$\begin{aligned} c &= \lim_{n\to\infty} \|y_n - p\| \\ &= \lim_{n\to\infty} \|(1-\beta_n)x_n + \beta_n (PT_1)^n x_n - p\| \\ &= \lim_{n\to\infty} \|(1-\beta_n)(x_n - p) + \beta_n ((PT_1)^n x_n - p)\| \end{aligned} \tag{3.21}$$

gives that

$$\lim_{n\to\infty} \|(1-\beta_n)(x_n - p) + \beta_n ((PT_1)^n x_n - p)\| = c. \tag{3.22}$$

Again using Lemma 2.2, we get

$$\lim_{n\to\infty} \|(PT_1)^n x_n - x_n\| = 0. \tag{3.23}$$

Moreover, from (1.9), we have

$$\begin{aligned} \|y_n - x_n\| &= \|(1-\beta_n)x_n + \beta_n (PT_1)^n x_n - x_n\| \\ &= \beta_n \|(PT_1)^n x_n - x_n\| \end{aligned} \tag{3.24}$$

Thus from (3.23)

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.25}$$

Also

$$\begin{aligned} \|(PT_2)^n y_n - x_n\| &\leq \|(PT_2)^n y_n - (PT_1)^n y_n\| + \|(PT_1)^n y_n - (PT_1)^n x_n\| \\ &\quad + \|(PT_1)^n x_n - x_n\| \\ &\leq \|(PT_2)^n y_n - (PT_1)^n y_n\| + (1 + \mu_n M^*) \|x_n - y_n\| \\ &\quad + \mu_n \phi(M) + \lambda_n + \|(PT_1)^n x_n - x_n\| \end{aligned} \tag{3.26}$$

yields from (3.16), (3.23) and (3.25) that

$$\lim_{n\to\infty} \|(PT_2)^n y_n - x_n\| = 0. \tag{3.27}$$

By (3.25) and (3.27), we get

$$\begin{aligned}\left\|(PT_2)^n x_n - x_n\right\| &\leq \left\|(PT_2)^n x_n - (PT_2)^n y_n\right\| + \left\|(PT_2)^n y_n - x_n\right\| \\ &\leq (1+\mu_n M^*)\left\|x_n - y_n\right\| + \mu_n \phi(M) + \lambda_n \\ &+ \left\|(PT_2)^n y_n - x_n\right\| \end{aligned} \tag{3.28}$$

and this implies

$$\lim_{n\to\infty} \left\|(PT_2)^n x_n - x_n\right\| = 0. \tag{3.29}$$

Then

$$\begin{aligned}\left\|(PT_1)^n y_n - x_n\right\| &\leq \left\|(PT_1)^n y_n - (PT_1)^n x_n\right\| + \left\|(PT_1)^n x_n - x_n\right\| \\ &\leq (1+\mu_n M^*)\left\|x_n - y_n\right\| + \mu_n \phi(M) + \lambda_n \\ &+ \left\|(PT_1)^n x_n - x_n\right\| \end{aligned} \tag{3.30}$$

gives

$$\lim_{n\to\infty} \left\|(PT_1)^n y_n - x_n\right\| = 0. \tag{3.31}$$

Using (3.16) and (3.31), we have

$$\begin{aligned}\left\|x_{n+1} - x_n\right\| &= \left\|(1-\alpha_n)(PT_1)^n y_n + \alpha_n (PT_2)^n y_n - x_n\right\| \\ &\leq \left\|(PT_1)^n y_n - x_n\right\| + \alpha_n \left\|(PT_1)^n y_n - (PT_2)^n y_n\right\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.32}$$

It follows (3.23) and (3.32) that

$$\begin{aligned}\left\|x_n - (PT_1)^{n-1} x_n\right\| &\leq \left\|x_n - x_{n-1}\right\| + \left\|x_{n-1} - (PT_1)^{n-1} x_{n-1}\right\| \\ &+ \left\|(PT_1)^{n-1} x_{n-1} - (PT_1)^{n-1} x_n\right\| \\ &\leq \left\|x_n - x_{n-1}\right\| + \left\|x_{n-1} - (PT_1)^{n-1} x_{n-1}\right\| \\ &+ (1+\mu_{n-1} M^*)\left\|x_n - x_{n-1}\right\| + \mu_{n-1}\phi(M) + \lambda_{n-1} \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.33}$$

Since T_1 is uniformly L-Lipschitzian, it follows from (3.33) that

$$\left\|(PT_1)^n x_n - (PT_1) x_n\right\| \leq L \left\|(PT_1)^{n-1} x_n - x_n\right\| \to 0 \text{ as } n \to \infty. \tag{3.34}$$

Thus, by (3.23) and (3.34), we get

$$\begin{aligned}\left\|x_n - (PT_1) x_n\right\| &\leq \left\|x_n - (PT_1)^n x_n\right\| + \left\|x_n - (PT_1)^n x_n\right\| \\ &+ \left\|(PT_1)^n x_n - (PT_1) x_n\right\| \end{aligned} \tag{3.35}$$

and so

$$\lim_{n\to\infty} \left\|x_n - (PT_1) x_n\right\| = 0. \tag{3.36}$$

In the same way, we obtain

$$\lim_{n\to\infty} \|x_n - (PT_2)\, x_n\| = 0. \tag{3.37}$$

This completes the proof of lemma. □

Theorem 3.5. *Let K be a nonempty closed convex subset of a real uniformly convex and smooth Banach space E with P as a sunny nonexpansive retraction. Let $T_1, T_2 : K \to E$ be two weakly inward and nonself uniformly L-Lipschitzian total asymptotically nonexpansive mappings with sequences $\{\mu_n\}, \{\lambda_n\}$ such that $\sum_{n=1}^{\infty}\mu_n < \infty, \sum_{n=1}^{\infty}\lambda_n < \infty$. Suppose that there exist $M, M^* > 0$, such that $\phi(\kappa) \le M^*\kappa$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $[\varepsilon, 1-\varepsilon]$, for some $\varepsilon \in (0,1)$. Suppose that $\{x_n\}$ is generated iteratively by (1.9). If one of the mappings T_1 and T_2 is completely continuous and $F \neq \varnothing$, then the sequence $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2.*

Proof. From 3.2, $\lim_{n\to\infty} \|x_n - p\|$ exists for any $p \in F$. We must show that $\{x_n\}$ has a subsequence which converges strongly to an element of fixed point set. From Lemma 3.4, we have $\lim_{n\to\infty} \|x_n - (PT_1)\, x_n\| = \lim_{n\to\infty} \|x_n - (PT_2)\, x_n\| = 0$. Assume that T_1 is completely continuous. By the nonexpansiveness of P, there exists subsequence $\{PT_1x_{n_j}\}$ of $\{PT_1x_n\}$ such that $PT_1x_{n_j} \to p$. Thus $\left\|x_{n_j} - p\right\| \le \left\|x_{n_j} - PT_1x_{n_j}\right\| + \left\|PT_1x_{n_j} - p\right\|$ gives $x_{n_j} \to p$ $(j \to \infty)$. Similarly $\lim_{j\to\infty} \left\|x_{n_j} - (PT_1)\, x_{n_j}\right\| = 0$ implies by continuity of P and T_1 that $p = PT_1p$. The same way $p = PT_2p$. By Lemma 2.3, $p = T_1p = T_2p$. Since F is closed, so $p \in F$. This completes the proof. □

Theorem 3.6. *Let K be a nonempty closed convex subset of a real uniformly convex and smooth Banach space E with P as a sunny nonexpansive retraction. Let $T_1, T_2 : K \to E$ be two weakly inward and nonself uniformly L-Lipschitzian total asymptotically nonexpansive mappings with sequences $\{\mu_n\}, \{\lambda_n\}$ such that $\sum_{n=1}^{\infty}\mu_n < \infty, \sum_{n=1}^{\infty}\lambda_n < \infty$. Suppose that there exist $M, M^* > 0$, such that $\phi(\kappa) \le M^*\kappa$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $[\varepsilon, 1-\varepsilon]$, for some $\varepsilon \in (0,1)$. Suppose that $\{x_n\}$ is generated iteratively by (1.9). If T_1 and T_2 satisfy condition (A') and $F \neq \varnothing$, then the sequence $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2.*

Proof. By Lemma 3.2, $\lim_{n\to\infty} d\,(x_n, F)$ exists for all $p \in F$. Also, by Lemma 3.4,

$$\lim_{n\to\infty} \|x_n - (PT_1)\, x_n\| = \lim_{n\to\infty} \|x_n - (PT_2)\, x_n\| = 0.$$

Using condition (A') and Lemma 2.3, we get

$$\lim_{n\to\infty} f\,(d\,(x_n, F)) \le \lim_{n\to\infty} \left(\frac{1}{2}\left(\|x_n - (PT_1)\, x_n\| + \|x_n - (PT_2)\, x_n\|\right)\right) = 0.$$

Since f is a nondecreasing function and $f(0) = 0$, so we have $\lim_{n\to\infty} d\,(x, F) = 0$. Now applying the theorem 3.3, we see $p \in F$. □

Our weak convergence theorem is as follows:

Theorem 3.7. *Let K be a nonempty closed convex subset of a real uniformly convex and smooth Banach space E satisfying Opial's condition with P as a sunny nonexpansive retraction. Let $T_1, T_2 : K \to E$ be two weakly inward and nonself uniformly L-Lipschitzian total asymptotically nonexpansive mappings with sequences $\{\mu_n\}, \{\lambda_n\}$ such that $\sum_{n=1}^{\infty}\mu_n < \infty, \sum_{n=1}^{\infty}\lambda_n < \infty$. Suppose that there exist $M, M^* > 0$, such that $\phi(\kappa) \leq M^*\kappa$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $[\varepsilon, 1-\varepsilon]$, for some $\varepsilon \in (0,1)$. Suppose that $\{x_n\}$ is generated iteratively by (1.9). If $I - T_2$ and $I - T_1$ are demiclosed at zero then the sequence $\{x_n\}$ converges weakly to a common fixed point of T_1 and T_2.*

Proof. Let $p \in F$. Then by Lemma 3.2, $\lim_{n\to\infty} \|x_n - p\|$ exists and $\{x_n\}$ is bounded. It is point out that PT_1 and PT_2 are self-mappings defined on K. We prove that $\{x_n\}$ converges subsequentially in F. Uniformly convexity of Banach space E implies that there exist two weakly convergent subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of bounded sequence $\{x_n\}$. Suppose $w_1 \in K$ and $w_2 \in K$ are weak limits of the $\{x_{n_i}\}$ and $\{x_{n_j}\}$, respectively. Using Lemma 3.4, we get $\lim_{n\to\infty} \|x_{n_i} - (PT_1)\, x_{n_i}\| = \lim_{n\to\infty} \|x_{n_i} - (PT_2)\, x_{n_i}\| = 0$. Since T_1 is demiclosed with respect to zero then we get $PT_1w_1 = w_1$. Similarly, $PT_2w_1 = w_1$. That is, $w_1 \in F$. In the same way, we have that $w_2 \in F$. Lemma 2.3 guarantee that $p_1, p_2 \in F$.

Next, we give uniqueness. For this aim, assume that $w_1 \neq w_2$. Using Opial's condition, we have

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - w_1\| &= \lim_{i\to\infty} \|x_{n_i} - w_1\| \\
&< \lim_{i\to\infty} \|x_{n_i} - w_2\| \\
&= \lim_{n\to\infty} \|x_n - w_2\| \\
&= \lim_{j\to\infty} \|x_{n_j} - w_2\| \\
&< \lim_{j\to\infty} \|x_{n_j} - w_1\| \\
&= \lim_{n\to\infty} \|x_n - w_1\|,
\end{aligned}$$

which is a contradiction. Thus $\{x_n\}$ converges weakly to a point of F. □

Finally, we give an example which show that our theorems are applicable.

Example 3.1. Let $\mathbb{R}$ be the real line with the usual norm $\|\cdot\|$ and let $K = [-1, 1]$. Define two mappings $T_1, T_2 : K \to K$ by

$$T_1(x) = \begin{cases} -2sin\frac{x}{2}, \ if \ x \in [0,1] \\ 2sin\frac{x}{2}, \ if \ x \in [-1,0) \end{cases} \quad \text{and } T_2(x) = \begin{cases} x, \ if \ x \in [0,1] \\ -x, \ if \ x \in [-1,0). \end{cases}$$

In [8], the authors show that above mappings are asymptotically nonexpansive mappings with common fixed point set $F = \{0\}$. Since T_1 and T_1 are asymptotically nonexpansive mappings, then they are uniformly L-Lipschitzian and total asymptotically nonexpansive mappings with $F \neq \varnothing$. Consequently, our theorems are applicable.

4 Conclusions

We have the following concluding remarks:

(i) We used the concept of nonself total asymptotically nonexpansive mapping which generalizes definition of some nonlinear mappings in existing literature.

(ii) Since the class of total asymptotically nonexpansive mappings includes asymptotically nonexpansive mappings, our results generalize results of Tukmen et al. [20] and Akbulut et al. [1].

(iii) Our results also improve and extend the corresponding ones studied by Ya.I. Alber et al. [2], Chidume and Ofoedu [4, 5] to a case of one mapping.

(iv) Our results generalize and extend the related results of Mukhamedov and Saburov [11, 12, 13, 14, 15] in view of more general class of mappings.

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