Symmetry group classification for generalized reaction-diffusion-convection equation

M. Nadjafikhah and S. Dodangeh

Abstract. In this paper, the symmetry group classification is given for the non-linear generalized reaction-diffusion-convection equation admitting an extension by one of the principal Lie algebra of the equation under consideration using the preliminary group classification approach. Using the adjoint representation G_ε of on its Lie algebra $\mathbf{g}_6$, the implementation of optimal systems to find its optimal one dimensional subalgebras is performed. The result of the work is a wide class of equations summarized in table 3.

M.S.C. 2010: 53A55, 34C14, 76M60.
Key words: Preliminary group classification; reaction-diffusion-convection equation; equivalence transformation; optimal system.

1 Introduction

The well-known class of the non-linear reaction-diffusion-convection RDC equation $u_t = (D(u)u_x)_x + g(u)$, where t is time, x is space, $u = u(x,t)$ is an unknown function, $D(u) \neq 0$ and $g(u)$ are arbitrary differentiable functions, are used to model the various processes in biology, ecology, physics and chemistry. For a specific form of functions $D(u)$ and $g(u)$, it describes species propagation in natural habitat, membrane ion transport, nerve impulse propagation, spread of chemical concentration waves, self-organization in biochemical systems, formation of apex zones of plants, non-linear effects in plasma, and so on [5]. (Find more information in [1, 2] and given references).

In this paper we deal with the following equation:

$$\text{GRDC}: \quad u_t = f(x,u)u_{xx} + g(x,u,u_x), \tag{1.1}$$

where $u(x,t)$ is an unknown function and $f(x,u) \neq 0$ and $g(x,u,u_x)$ are arbitrary functions with specified arguments. By comparing this equations, we find the generality of later equation w.r.t. equation RDC. Hence, we can consider the equation RDC as the "Generalized Reaction-Diffusion-Convection" equation (or GRDC-equation). Also, the equation GRDC can be considered as generalization of several the well known second-order evolution equations which have been pointed with some applications and physical information in the following Table:

Applied Sciences, Vol.20, 2018, pp. 139-147.

Equation	Some of its applications
Reaction-diffusion-convection $u_t = (D(u)u_x)_x + F(u)$	Oil and water flow in petroleum-reservoir
Fisher equation $u_t = \alpha u_{xx} + F(u)$	Population in a one-dimensional habitat
Fin equation $u_t = (f(u)u_x)_x + g(x)u_x$	Increase the heat transfer from surfaces
Non-linear heat equation $u_t = (f(u)u_x)_x$	Nonlinear problems of heat and mass transfer and flows in porous media
Reaction-diffusion equation $u_t = (u^\alpha u_x)_x + g(x)u_x$	Model for population growth

Physical applications of this type of equations have been a motivation for the several accounts (see e.g. [3, 10, 2]). In this paper by using preliminary group classification method (See [4, 7]) a symmetry analysis of the equation GRDC as well as similarity solutions have been considered. Indeed, this paper is organized as follows. In section 2. We will discuss about the symmetry analysis of model GRDC using the Lie classical symmetry method. In section 3. Some discussions about the equivalence transformations using infinitesimal criterion method to find a continuous subgroup ε_c of the set of all equivalence transformations which is denoted by ε, will be considered. Section 4 is devoted to preliminary group classification of equation GRDC which is defined as the most of extensions of the principal Lie algebra admitted by the equation under consideration are taken from the equivalence algebra $\mathbf{g}_\varepsilon$, by considering finite subalgebras of the obtained symmetry algebra in the pervious section. Also, discussion about the optimal system and applications of the preliminary group classifications have been given in this section.

We remind the reader that there are also some other papers which had used symmetry analysis approach for the investigation about some special class of the equation GRDC. For instance in [1], Lie symmetries and form-preserving transformations of the reaction-diffusion-convection equations $u_t = (A(u)u_x)_x + B(u)u_x + c(u)$, or in [10] Lie symmetry analysis via classical and non-classical symmetries of the remarkable Fin equation $u_t = (D(u)u_x)_x + h(x)u$, have been discussed. Also, Ibragimov, N.H. and et al, in [4] investigate about the preliminary group classification for the following equation:

$$u_{tt} = f(x, u_x)u_{xx} + g(x, u_x). \tag{1.2}$$

2 Classical symmetries

The symmetry group method plays an important role in the analysis of differential equations. The history of group classification methods goes back to *Sophus Lie* [6]. Roughly speaking, a symmetry group of a system of differential equations is a group which transforms solutions of the system to other solutions. In the classical framework of Lie, these groups consist of geometric transformations on the space of independent and dependent variables for the system, and act on solutions by transforming their graphs. To find symmetry group using infinitesimal criterion approach is popular and rather convenient ([8, 9]). In this section, we will have an attempt to apply such approach for the equation GRDC.

Let the following one-parameter group:

$$(\overline{x},\overline{t},\overline{u}) = \big(x+\varepsilon\xi(x,t,u), t+\varepsilon\eta(x,t,u), u+\varepsilon\varphi(x,t,u)\big) + O(\varepsilon^2),$$

with $X := \xi(x,t,u)\partial_x + \eta(x,t,u)\partial_t + \varphi(x,t,u)\partial_u$ as infinitesimal generator. Transformation group RDC is a symmetry group of the equation GRDC, if and only if the following condition (infinitesimal criterion) is hold:

$$X^{(2)}\Delta = 0, \qquad \text{whenever} \qquad \Delta = 0. \tag{2.1}$$

Where $X^{(2)}$ is second prolongation of X given as

$$X^{(2)} = X + \varphi^x\partial_{u_x} + \varphi^t\partial_{u_t} + \varphi^{xx}\partial_{xx} + \varphi^{xt}\partial_{u_{xt}} + \varphi^{tt}\partial_{u_{tt}},$$

and φ^x, φ^t, φ^{xx}, φ^{xt}, φ^{tt} are respectively

$$\varphi^x = D_xQ + \xi u_{xx} + \eta u_{xt}, \qquad \cdots \qquad \varphi^{tt} = D_t^2Q + \xi u_{xtt} + \eta u_{ttt}. \tag{2.2}$$

Here D_x, D_t are total derivative w.r.t specified variables and $Q = \varphi - \xi u_x - \eta u_t$ is the corresponding Lie characteristic of X.

By using infinitesimal criterion (2.1), we find

$$\Big[\varphi^t - (f_xu_{xx} + g_x)\xi - (f_uu_{xx} + g_u)\varphi - g_{u_x}\varphi^x - f\varphi^{xx}\Big]_{\text{GRDC}} = 0.$$

In the case of arbitrary $f(x,u)$ and $g(x,u,u_x)$, it follows $\xi = \varphi = \varphi^x = \varphi^{xx} = \varphi^t = 0$. So, for the coefficients of X, we have $\xi = 0$, $\eta = c$, $\varphi = 0$, where c is an arbitrary constant. Hence, the equation GRDC admit the one-dimensional Lie algebra $\mathbf{g}_1$ with basis: $X^1 = \partial_t$. Now, $\mathbf{g}_1$ called principle Lie algebra for equation GRDC.

In continuation, using a partial group classification of GRDC we obtain appropriate f and g such that desired equation GRDC admit extension symmetry algebra which respect to $\mathbf{g}_1$. Usually, the group classification is obtained by inspecting the determining equation. Here we do not solve the determining equation but, instead, we use the so-called preliminary group classification method [4, 7].

3 Equivalence transformations

An equivalence transformation is a non-degenerate change of the variables t, x and u taking any equation of the form GRDC into an equation of the same form, generally speaking, with different $f(x,u)$ and $g(x,u,u_x)$. The set of all equivalence transformations forms an equivalence group ε (Find more information in [4]). Therefore, we must find a continuous subgroup ε_c of it making use of the infinitesimal method. Let

$$\begin{aligned} Y := {}& \xi^1(x,t,u,f,g)\partial_x + \xi^2(x,t,u,f,g)\partial_t + \varphi(x,t,u,f,g)\partial_u \\ & +\mu(x,t,u,f,g)\partial_f + \nu(x,t,u,u_x,u_t,f,g)\partial_g, \end{aligned} \tag{3.1}$$

be an operator of the group ε_c. So from invariance conditions of GRDC, we have

$$u_t - f(x,u)u_{xx} - g(x,u,u_x) = 0, \qquad f_t = g_t = g_{u_t} = 0, \tag{3.2}$$

The invariance conditions of the system (3.2) are

$$Y^{(2)}[u_t - f u_{xx} - g] = 0, \qquad Y^{(2)}(f_t) = Y^{(2)}(g_t) = Y^{(2)}(g_{u_t}) = 0, \tag{3.3}$$

where $Y^{(2)}$ is second order prolongation of operator (3.1), given as

$$\begin{aligned} Y^{(2)} &= Y + \varphi^x u_x + \varphi^t u_t + \varphi^{xx} u_{xx} + \varphi^{xt} u_{xt} \\ &\quad + \varphi^{tt} u_{tt} + \mu^t \partial_{f_t} + \nu^t \partial_{g_t} + \nu^{u_t} \partial_{g_{u_t}}, \end{aligned} \tag{3.4}$$

where the coefficients $\varphi^x, \varphi^t, \varphi^{xx}, \varphi^{xt}$ and φ^{tt} given in (2.2) and for remaining coefficients we have

$$\begin{aligned} \mu^t &= \mu_t - f_x \xi^1_t - f_u \varphi_t \\ \nu^t &= \nu_t - g_x \xi^1_t - g_u \varphi_t - g_{u_x} (\varphi^x)_t, \\ \nu^{u_t} &= \nu_{u_t} - g_{u_x} (\varphi^x)_{u_t}, \end{aligned} \tag{3.5}$$

So, invariance condition (3.3) give $\mu^t = \nu^t = \nu^{u_t} = 0$. As a result we have

$$\begin{aligned} \mu_t - f_x \xi^1_t - f_u \varphi_t &= 0, \\ \nu_t - g_x \xi^1_t - g_u \varphi_t - g_{u_x} (\varphi^x)_t &= 0, \\ \nu_{u_t} - g_{u_x} (\varphi^x)_{u_t} &= 0. \end{aligned} \tag{3.6}$$

Since above equality hold for arbitrary functions f and g, we find

$$\mu_t = \xi^1_t = \varphi_t = \nu_t = \nu_{u_t} = (\varphi^x)_t = (\varphi^x)_{u_t} = 0, \tag{3.7}$$

Moreover with substituting (3.4) into (3.3), we obtain

$$\varphi^t - f \varphi^{xx} - \mu u_{xx} - \nu = 0 \tag{3.8}$$

By substituting (2.2) into (3.8), and introducing the relation $u_t = f(x,u)u_{xx} + g(x,u,u_x)$ to eliminate u_t we are left with a polynomial equation involving the various derivatives of $u(x,t)$ whose coefficients are certain derivatives of ξ^1, ξ^2, φ, μ and ν. We can equate the individual coefficients to zero, leading to the complete set of determining equations.

$$\begin{aligned} &\xi^1 = \xi^1(x), \qquad \xi^2 = c, \qquad \varphi = \varphi(x,u), \qquad \varphi_{xx} = 0, \\ &\nu = \varphi_u g - 2\varphi_{xu} u_x f + \xi^1_{xx} u_x f - \varphi_{uu} u_x^2 f, \qquad \mu = 2\xi^1_x f, \end{aligned}$$

where these system of equations can be simplified as

$$\begin{aligned} &\xi^1 = \alpha(x), \qquad \xi^2 = c, \qquad \varphi = \beta(u)x + \gamma(u), \qquad \mu = 2\alpha'(x) f, \\ &\nu = (\beta'(u)x + \gamma'(u))g - 2\beta'(u)u_x f + \alpha''(x) u_x f - (x\beta''(u) + \gamma''(u))u_x^2 f, \end{aligned}$$

whit arbitrary constant c and arbitrary functions $\alpha(x)$, $\beta(u)$ and $\gamma(u)$ with specified arguments. Therefore, the class of Eq. GRDC has an infinite continuous group of equivalence transformations generated by the following infinitesimal operators:

$$\begin{aligned} Y &:= \alpha(x)\partial_x + c\partial_t + (\beta(u)x + \gamma(u))\partial_u + 2\alpha'(x) f \partial_f \\ &\quad + \Big[(\beta'(u)x + \gamma'(u))g + (\alpha''(x) - 2\beta'(u))u_x f - (x\beta''(u) + \gamma''(u))u_x^2 f\Big]\partial_g, \end{aligned}$$

and the symmetry algebra of the GRDC equation GRDC is spanned by the vector fields

$$
\begin{aligned}
&Y_1 = \alpha(x)\partial_x + 2\alpha'(x)f\partial_f + \alpha''(x)u_x f\partial_g, \qquad Y_2 = \partial_t,\\
&Y_3 = \beta(u)x\partial_u + \big(x\beta'(u)g - 2\beta'(u)u_x f - x\beta''(u)u_x^2 f\big)\partial_g,\\
&Y_4 = \gamma(u)\partial_u + \big(g\gamma'(u) - \gamma''(u)u_x^2 f\big)\partial_g.
\end{aligned} \tag{3.9}
$$

Moreover, in the group of equivalence transformations there are included also discrete transformations, e.g. $(x,t,u,f,g) \mapsto -(x,t,u,f,g)$.

4 Preliminary group classification

The principal group classification is ε-extension which is defined as the most of extensions of the principal Lie algebra admitted by the equation under consideration are taken from the equivalence algebra $\mathbf{g}_\varepsilon$.

So, we can take any finite-dimensional subalgebra of an infinite-dimensional algebra with basis (3.9) and use it for a preliminary group classification. Hence, we select the subalgebra $\mathbf{g}_6$ spanned on the following operators:

$$
\begin{aligned}
&Y_1 = \partial_x, \quad Y_2 = \partial_t, \quad Y_3 = \partial_u,\\
&Y_4 = x\partial_u, \quad Y_5 = x\partial_x + 2f\partial_f, \quad Y_6 = u\partial_u + g\partial_g.
\end{aligned} \tag{4.1}
$$

The communication relations between these vector fields is given in the following Table, where the entry in the i^{th} row and j^{th} column is defined as $[Y_i, Y_j] = Y_i.Y_j - Y_j.Y_i$, $i,j = 1,\cdots,6$.

[,]	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
Y_1	0	0	0	Y_3	Y_1	0
Y_2	0	0	0	0	0	0
Y_3	0	0	0	0	0	Y_3
Y_4	$-Y_3$	0	0	0	$-Y_4$	Y_4
Y_5	$-Y_1$	0	0	Y_4	0	0
Y_6	0	0	$-Y_3$	$-Y_4$	0	0

Table 1: Commutation relations satisfied by infinitesimal generators

Since any linear combination of infinitesimal generators is also an infinitesimal generator, there are infinitely many different subgroups for a differential equation. In order to complete understanding the invariant solutions, it is necessary and significant to determine which subgroups would give essentially different types of solutions. However, a well known standard procedure [8, 7] allows us to classify all the one-dimensional subalgebras into subsets of conjugate subalgebras. This involves constructing the adjoint representation group, which introduces a conjugate relation in the set of all one-dimensional subalgebras. In fact, for one-dimensional subalgebras, the classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation. If we take only one representative from each family of equivalent subalgebras, an optimal set of subalgebras is created. The corresponding set of invariant solutions is then the minimal list from which we can get all other invariant solutions of one-dimensional subalgebras simply via transformations.

Each Y_i, $i = 1, \cdots, 6$, of the basis symmetries generates an adjoint representation (or interior automorphism) $\mathrm{Ad}(\exp(\varepsilon Y_i))$ defined by the Lie series

$$\mathrm{Ad}(\exp(\varepsilon.Y_i).Y_j) = Y_j - \varepsilon.[Y_i, Y_j] + \frac{\varepsilon^2}{2}.[Y_i, [Y_i, Y_j]] - \cdots, \tag{4.2}$$

where $[Y_i, Y_j]$ is the commutator for the Lie algebra, ε is a parameter, and $i, j = 1, \cdots, 6$ ([8, 7]). In Table 2, we give all the adjoint representations of the GRDC equation Lie group, with the (i, j) the entry indicating $\mathrm{Ad}(\exp(\varepsilon Y_i))Y_j$.

Ad	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
Y_1	Y_1	Y_2	Y_3	$Y_4 - \varepsilon Y_3$	$Y_5 - \varepsilon Y_1$	Y_6
Y_2	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
Y_3	Y_1	Y_2	Y_3	Y_4	Y_5	$Y_6 - \varepsilon Y_3$
Y_4	$Y_1 + \varepsilon Y_3$	Y_2	Y_3	Y_4	$Y_5 + \varepsilon Y_4$	$Y_6 - \varepsilon Y_4$
Y_5	$e^{\varepsilon} Y_1$	Y_2	Y_3	$e^{-\varepsilon} Y_4$	Y_5	Y_6
Y_6	Y_1	Y_2	$e^{\varepsilon} Y_3$	$e^{\varepsilon} Y_4$	Y_5	Y_6

Table 2: Adjoint representation of infinitesimal symmetries of GRDC equation

Theorem 4.1. *An optimal system of one-dimensional Lie subalgebras of generalized reaction-diffusion-convection equation GRDC is provided by those generated by*

$$\begin{aligned}
&Y^1 = Y_1 = \partial_x, \quad Y^2 = Y_2 = \partial_t, \quad Y^3 = Y_3 = \partial_u,\\
&Y^4 = Y_4 = x\partial_u, \quad Y^5 = Y_5 = x\partial_x + 2f\partial_f, \quad Y^6 = Y_6 = u\partial_u + g\partial_g,\\
&Y^7 = Y_2 - Y_1 = \partial_t - \partial_x, \quad Y^8 = Y_2 + Y_1 = \partial_x + \partial_t,\\
&Y^9 = Y_4 - Y_2 = x\partial_u - \partial_t, \quad Y^{10} = Y_4 + Y_2 = x\partial_u + \partial_t,\\
&Y^{11} = Y_5 + Y_2 = x\partial_x + 2f\partial_f + \partial_t, \quad Y^{12} = Y_5 - Y_2 = x\partial_x + 2f\partial_f - \partial_t,\\
&Y^{13} = Y_6 + Y_5 = u\partial_u + g\partial_g + x\partial_x + 2f\partial_f,\\
&Y^{14} = Y_6 - Y_5 = u\partial_u + g\partial_g - x\partial_x - 2f\partial_f,\\
&Y^{15} = Y_6 + Y_2 = u\partial_u + g\partial_g + \partial_t, \quad Y^{16} = Y_6 - Y_2 = u\partial_u + g\partial_g - \partial_t,\\
&Y^{17} = Y_2 + Y_5 + Y_6 = \partial_t + x\partial_x + 2f\partial_f + u\partial_u + g\partial_g,\\
&Y^{18} = Y_2 - Y_5 + Y_6 = \partial_t - x\partial_x - 2f\partial_f + u\partial_u + g\partial_g,\\
&Y^{19} = -Y_2 + Y_5 + Y_6 = -\partial_t + x\partial_x + 2f\partial_f + u\partial_u + g\partial_g,\\
&Y^{20} = -Y_2 - Y_5 + Y_6 = -\partial_t - x\partial_x - 2f\partial_f + u\partial_u + g\partial_g.
\end{aligned} \tag{4.3}$$

Proof. Let $\mathbf{g}_6$ is the symmetry algebra of equation GRDC with adjoint representation determined in Table 2 and $Y := a_1Y_1 + \cdots + a_6Y_6$, is a nonzero vector field of $\mathbf{g}_6$. We will simplify as many of the coefficients $a_i; i = 1; \cdots; 6$, as possible through proper adjoint applications on Y. We follow our aim in the below easy cases:

I. At first, assume that $a_6 \neq 0$. Scaling Y if necessary, we can assume that $a_6 = 1$ and so we get $Y' := a_1Y_1 + \cdots + a_5Y_5 + Y_6$, Using the table of adjoint (Table 2), if we act on Y' with $\mathrm{Ad}(\exp(a_3/a_4Y_1))$, the coefficient of Y_3 can be vanished. So, we find: $Y'' := a_1Y_1 + a_2Y_2 + a_4Y_4 + a_5Y_5 + Y_6$. Again, if we act on Y'' with $\mathrm{Ad}(\exp(a_1/a_4Y_1))$, the coefficient of Y_1 can be vanished. So, we find: $Y''' := a_2Y_2 + a_4Y_4 + a_5Y_5 + Y_6$. If we act on Y''' with $\mathrm{Ad}(\exp(a_4/(1-a_5)Y_4))$, the coefficient of Y_4 can be vanished. So, we find: $\tilde{Y} := a_2Y_2 + a_5Y_5 + Y_6$,

II. In this case we consider $a_6 = 0$ and $a_5 \neq 0$. So we assume that: $Y' = a_1Y_1 + a_2Y_2 + a_3Y_3 + a_4Y_4 + Y_5$, which after simplifying turn $\tilde{Y} = a_2Y_2 + Y_5$,

III. In this case we consider $a_6 = 0$, $a_5 = 0$ and $a_4 \neq 0$. So we assume that: $Y' = a_1Y_1 + aY_2 + a_3Y_3 + Y_4$, which after simplifying turn $\tilde{Y} = a_2Y_2 + Y_4$,

IV. In this case we consider $a_6 = 0$, $a_5 = 0$, $a_4 = 0$ and $a_3 \neq 0$. So we assume that: $Y' = a_1Y_1 + a_2Y_2 + Y_3$, which after simplifying turn $\tilde{Y} = a_1Y_1 + Y_2$ or $\tilde{Y} = Y_1$,

There is not any more possible case for studying and to end proof set a_is in any given $\tilde{Y}$ equal to ±1, 0 and list the resulted vector fields which are the same given in (4.3).

The coefficients f, g of GRDC depend on the variables x, u and x, u, u_x, respectively. Therefore, we take their optimal system's projections on the space (x, u, u_x, f, g). The nonzero in x-axis, u-axis or u_x-axis projections of (4.3) are

$$\begin{aligned}
&Z^1 = Y^1 = Y^7 = Y^8 = \partial_x, \quad Z^2 = Y^3 = \partial_u,\\
&Z^3 = Y^4 = Y^9 = Y^{10} = x\partial_u, \quad Z^4 = Y^5 = Y^{11} = Y^{12} = x\partial_x + 2f\partial_f,\\
&Z^5 = Y^6 = Y^{15} = Y^{16} = u\partial_u + g\partial_g,\\
&Z^6 = Y^{13} = Y^{17} = Y^{19} = u\partial_u + g\partial_g + x\partial_x + 2f\partial_f,\\
&Z^7 = Y^{14} = Y^{18} = Y^{20} = u\partial_u + g\partial_g - x\partial_x - 2f\partial_f.
\end{aligned} \tag{4.4}$$

□

Theorem 4.2. *Let $\mathbf{g}_m$ be an m-dimensional subalgebra with the basis Y^i $(i = 1, \cdots, m)$ of infinite-dimensional algebra $\mathbf{g}_1$. Denote by Y^i $(i = 1, \cdots, s,\ 0 < s \leq m,\ s \in \mathbb{N}^+)$ an optimal system of one-dimensional subalgebras of $\mathbf{g}_m$ and by Z^i $(i = 1, \cdots, t,$ $0 < t \leq s,\ t \in \mathbb{N}^+)$ the projections of Y^i, i.e. $Z^i = pr(Y^i)$. If equations $f = \varphi(x, u)$, $g = \psi(x, u, u_x)$, are invariant with respect to the optimal system Z^i then the equation*

$$u_t = \varphi(x, u)u_{xx} + \psi(x, u, u_x), \tag{4.5}$$

admits the operators X^i = project of Y^i on (t, x, u).

Proof. Let Eq. (4.5) and the equation

$$u_t = \varphi'(x, u)u_{xx} + \psi'(x, u, u_x) \tag{4.6}$$

be constructed according to Theorem 1 via optimal systems Z^i and Z'^i, respectively. If the subalgebras spanned on the optimal systems Z^i and Z'^i, respectively, are similar in $\mathbf{g}_m$, then Eqs. (4.5) and (4.6) are equivalent with respect to the equivalence group G_m, generated by $\mathbf{g}_m$. □

Now we apply Theorem 1 to the optimal system (4.3) and obtain all nonequivalent Eq. GRDC admitting ε-extensions of the principal Lie algebra g, by one dimension, i.e., equations of the form GRDC such that they admit, together with the one basic operators ∂_t of $\mathbf{g}_1$, also a second operator X_2. For every case, when this extension occurs, we indicate the corresponding coefficients f, g and the additional operator X_2.

We clarify the algorithm of passing from operators (4.1) to f, g and X_2 via the following examples. We take the last operator from (4.4): $Z^7 = u\partial_u + g\partial_g - x\partial_x - 2f\partial_f$. Invariants are found from the equations $dx/(-x) = du/u = df/(-2f) = dg/g$, and can be taken in the form $I_1 = xu$, $I_2 = f/x^2$ and $I_3 = xg$.

From the invariance equations taken in the form $I_2 = \varphi(I_1)$ and $I_3 = \psi(I_1)$. It follows that: $f = x^2\varphi(\lambda)$, $g = \psi(\lambda)/x$, where $\lambda = I_1$. From Theorem 1 applied to the operator Z^7 we obtain the additional operator $u\partial_u - x\partial_x \pm \partial_t$, $u\partial_u - x\partial_x$.

After similar calculations applied to all operators (4.4) we obtain the following result (Table 3) for the preliminary group classification of Eq. GRDC admitting an extension $\mathbf{g}_2$ of the principal Lie algebra $\mathbf{g}_1$.

Z	λ	Equation	Additional Operator X^2
Z^1	u	$u_t = \varphi u_{xx} + \psi$	$\partial_x, \partial_t + \partial_x, \partial_x - \partial_t$
Z^2	x	$u_t = \varphi u_{xx} + \psi$	∂_u
Z^3	x	$u_t = \varphi u_{xx} + \psi$	$x\partial_u, \partial_t + x\partial_u, -\partial_t + x\partial_u$
Z^4	u	$u_t = x^2\varphi u_{xx} + \psi$	$\partial_t + x\partial_x, -\partial_t + x\partial_x$
Z^5	x	$u_t = \varphi u_{xx} + u/\psi$	$\partial_t + u\partial_u, -\partial_t + u\partial_u, u\partial_u$
Z^6	u/x	$u_t = x^2\varphi u_{xx} + x\psi$	$u\partial_u + x\partial_x + \partial_t, u\partial_u + x\partial_x - \partial_t, u\partial_u + x\partial_x$
Z^7	xu	$u_t = x^2\varphi u_{xx} + \psi/x$	$u\partial_u - x\partial_x + \partial_t, u\partial_u - x\partial_x - \partial_t, u\partial_u - x\partial_x,$

Table 3: The result of the classification

5 Conclusions

In this paper, symmetry group classification and the algebraic structure of the symmetry groups for the generalized reaction-diffusion-convection equation using the preliminary group classification are obtained. Indeed, this classification is stated by constructing an optimal system with the aid of Theorems 1 and 2. The result of this work is summarized in Table 3.

References

[1] R. Cherniha, and O. Pliukhin, *New conditional symmetries and exact solutions of nonlinear reaction-diffusion-convection equations*, J. Phys. A: Math. Theor. 40 (2007), 10049–10070.

[2] R. Cherniha, M. Serov and I. Rassokha, *Lie symmetries and form-preserving transformations of reaction-diffusion-convection equations*, Journal of Mathematical Analysis and Applications, 342, 2 (2008), 1363–1379.

[3] R. Conte, and M. Musette, *The Painlevé Handbook*, Springer Science and Business Media B.V, 2008.

[4] N.H. Ibragimov, M.Torrisi and A. Valenti, *Preliminary group classification of equations*, J. Math Phys 32(11) (1991), 2988–2995.

[5] I.B. Kovalenko, and A.G. Kushner, *The non linear diffusion and thermal conductivity equation: Group classification and exact solutions*, Regular and Chaotic Dynamics, 8, 2 (2003).

[6] S. Lie, *Über die Integration durch bestimmte Integrale von einer Klasse linearer partieller Differentialgleichungen*, Arch. Math. 6 (1881), 328-368.

[7] M. Nadjafikhah and R. Bakhshandeh-Ch., *Symmetry group classification for general Burger's equation*, Commun. Nonlinear Sci. Numer. Simulat. 15 (2010), 2303–2310.

[8] P.J. Olver, *Applications of Lie Groups to Differential Equations*, New York, Springer, 1986.

[9] P.J. Olver, *Equivalence, Invariants and Symmetry*, Cambridge University Press, 1995.

[10] R.O. Popovych, C. Sophocleous, and O.O. Vaneeva, *Exact solutions of a remarkable fin equation*, Applied Mathematics Letters 21 (2008), 209–214.

[11] J. Satsuma, *Exact solutions of Burgers equation with reaction terms, Topics in soliton theory and exact solvable nonlinear equations*, World Scientific, Singapore, 1987; 255–262.

Authors' addresses:

Mehdi Nadjafikhah
Department of Pure Mathematics, School of Mathematics,
Iran University of Science and Technology,
Narmak, Tehran, 16846-13114, Iran.
E-mail: m_nadjafikhah@iust.ac.ir

Saeed Dodangeh
Department of Pure Mathematics, School of Mathematics,
Iran University of Science and Technology,
Narmak, Tehran, 16846-13114, Iran.
E-mail: saeed136541@yahoo.com