$\Psi-$asymptotic stability of solutions of a nonlinear Lyapunov matrix differential equation

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Abstract. In this paper, we obtain some new sufficient conditions for $\Psi-$asymptotic stability of trivial solution of a nonlinear Lyapunov matrix differential equation with integral term as right-hand side.

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Key words: Ψ-stability; Ψ-asymptotic stability; nonlinear Lyapunov matrix differential equations.

1 Introduction

The Lyapunov matrix differential equations occur in many branches of control theory such as optimal control and stability analysis.

Recent works for $\Psi-$ stability, $\Psi-$ asymptotic stability, $\Psi-$ boundedness, $\Psi-$ instability, controllability, dichotomy and conditioning for Lyapunov matrix differential equations have been given in many papers. See [3], [4], [5], [6], [7], [8], [10], [14], [15], [16] and the references therein.

In this paper are presented a several new sufficient conditions for $\Psi-$ asymptotic stability of the trivial solution to the nonlinear Lyapunov matrix differential equation with integral term as right-hand side:

$$Z' = A(t)Z + ZB(t) + \int_0^t F(t,s,Z(s))ds. \tag{1.1}$$

These conditions can be expressed in the terms of a fundamental matrices of the matrix differential equations

$$X' = A(t)X \tag{1.2}$$

$$Y' = YB(t) \tag{1.3}$$

and on the function F.

Here, Ψ is a matrix function whose introduction permits to obtain a mixed asymptotic behavior for the components of solutions.

The main tool used in this paper is the technique of Kronecker product of matrices (combined with the variation of constants formula), which has been successfully applied in various fields of matrix theory, group theory and particle physics. See, for example, the cited papers and the references cited therein.

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2 Preliminaries

In this section we present some basic notations, definitions, hypotheses and results which are useful later on.

Let R^d be the Euclidean $d-$ dimensional space. For $x = (x_1, x_2, ..., x_d)^T \in R^d$, let $\| x \| = \max\{| x_1|, | x_2|, ..., | x_d|\}$ be the norm of x (here, ^{T} denotes transpose).

Let $\mathbb{M}_{d\times d}$ be the linear space of all real $d \times d$ matrices.

For $A = (a_{ij}) \in \mathbb{M}_{d\times d}$, we define the norm $| A |$ by formula $| A | = \sup\limits_{\|x\|\leq 1} \| Ax \|$. It is well-known that $| A | = \max\limits_{1\leq i\leq d}\{\sum\limits_{j=1}^{d} | a_{ij}|\}$.

By a solution of the equation (1.1) we mean a continuous differentiable $d \times d$ matrix function satisfying the equation (1.1) for all $t \in \mathbb{R}_+ = [0, \infty)$

In equation (1.1), we assume that A and B are continuous $d\times d$ matrices on $\mathbb{R}_+$ and $F : D \longrightarrow \mathbb{M}_{d\times d}$, $D = \{(t, s, Z) \mid 0 \leq s \leq t < +\infty, Z \in \mathbb{M}_{d\times d}\}$, is continuous function such that $F(t, s, O_d) = O_d$. It is well-known that these conditions ensure the local existence of a solution of (1.1) passing through any given point $(t_0, Z_0) \in \mathbb{R}_+ \times \mathbb{M}_{d\times d}$, but it does not guarantee that the solution is unique or that it can be continued for large values of $t \in \mathbb{R}_+$.

Let Ψ_i :$\mathbb{R}_+ \longrightarrow (0, \infty)$, $i = 1, 2, ..., d$, be continuous functions and

$$\Psi = \text{diag }[\Psi_1,\Psi_2, \cdots \Psi_d].$$

In this paper, we will admit the following hypothesis:

(H) For all $t_0 \in \mathbb{R}_+$, $Z_0 \in \mathbb{M}_{d\times d}$ and $\rho > 0$, if $| \Psi(t_0)Z_0 | < \rho$, then any solution $Z(t)$ of (1.1) which satisfies the equality $Z(t_0) = Z_0$ exists on $\mathbb{R}_+$ and satisfies the inequality $| \Psi(t)Z(t) | \leq \rho$ for all $t \in [0, t_0]$.

This is a natural hypothesis in studying $\Psi-$ stability of trivial solution of (1.1). See, for example, [12], [11], [9].

Definition 2.1. ([3], [4]) (i) The trivial solution of the equation (1.1) is said to be $\Psi-$ stable over $\mathbb{R}_+$ if for each $\varepsilon > 0$ and each $t_0 \in \mathbb{R}_+$ there is a a corresponding $\delta = \delta(\varepsilon, t_0) > 0$ such that any solution $Z(t)$ of (1.1) which satisfies the inequality $| \Psi(t_0)Z(t_0) | < \delta$, exists and satisfies the inequality $| \Psi(t)Z(t) | < \varepsilon$ for all $t \geq t_0$.
(ii) The trivial solution of the equation (1.1) is said to be $\Psi-$ *asymptotically* stable over $\mathbb{R}_+$ if it is $\Psi-$ stable over $\mathbb{R}_+$ and in addition, for each $t_0 \in \mathbb{R}_+$, there is a corresponding $\delta_0 = \delta_0(t_0) > 0$ such that any solution $Z(t)$ of (1.1) which satisfies the inequality $| \Psi(t_0)Z(t_0) | < \delta_0$, satisfies the condition $\lim\limits_{t\to\infty} \Psi(t)Z(t) = O_d$.

Remark 2.2. 1. The Definition extends the definition of (asymptotic) stability from (vector) differential equations to matrix differential equations.
2. For $\Psi = I_d$, one obtain the notion of classical stability (see [2]).
3. It is easy to see that if Ψ and Ψ^{-1} are bounded on $\mathbb{R}_+$, then the $\Psi-$ stability is equivalent with the classical stability.

Definition 2.3. ([1]) Let $A = (a_{ij}) \in M_{m\times n}$ and $B = (b_{ij}) \in M_{p\times q}$. The Kronecker product of A and B, written $A \otimes B$, is defined to be the partitioned matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

Obviously, $A \otimes B \in \mathbb{M}_{mp\times nq}$.

The important rules of calculation of the Kronecker product are given in [1], [13], Chapter 2 and Lemma 1, [3].

Definition 2.4. ([13]) The application $\mathcal{V}ec : \mathbb{M}_{m\times n} \longrightarrow R^{mn}$, *defined by*

$$\mathcal{V}ec(A) = (a_{11}, a_{21}, \cdots, a_{m1}, a_{12}, a_{22}, \cdots, a_{m2}, \cdots, a_{1n}, a_{2n}, \cdots, a_{mn})^T,$$

where $A = (a_{ij}) \in \mathbb{M}_{m\times n}$, *is called the vectorization operator.*

For important properties and rules of calculation of the $\mathcal{V}ec$ operator, see Lemmas 2, 3, 4, [3].

For "*corresponding Kronecker product system associated with (1.1)*", see Lemma 5, [3].

The Lemmas 6 and 8, [3], play an important role in the proofs of main results of present paper.

3 Main results

Theorem 3.1. *Suppose that:*
(1) the hypothesis (H) is fulfilled;
(2) the fundamental matrices $X(t)$ and $Y(t)$ for the linear Lyapunov matrix differential equations (1.2) and (1.3) respectively satisfy the condition

$$\int_0^t \varphi(s) \left| \left(Y^T(t)(Y^T)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s)\right) \right| ds \le M, \ \textit{for all } t \ge 0,$$

where M is a positive constant and φ is a continuous function $\varphi : \mathbb{R}_+ \to (0, \infty)$ such that $\int_0^\infty \varphi(t)dt = +\infty$;
(3) the matrix function $F(t, s, Z)$ satisfies the inequality

$$| \Psi(t)F(t, s, Z) | \le f(t, s) \mid \Psi(s)Z \mid,$$

for $(t, s, Z) \in D$, where $f(t, s)$ is a continuous nonnegative function for $t \ge s \ge 0$ such that

$$K \ = \ \sup_{t\ge 0} \int_0^t \frac{f(t,s)}{\varphi(t)} ds < \frac{1}{Md}, \qquad \lim_{s\to\infty} \int_0^t \frac{f(s,u)}{\varphi(s)} du = 0, \ \textit{for all } t > 0.$$

Then, the trivial solution of (1.1) is Ψ-asymptotically stable over $\mathbb{R}_+$.

Proof. We will use Definition of $\Psi-$*asymptotic stability.* For a given $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, we choose $\delta = \delta(\varepsilon, t_0)$ such that $0 < \delta < \min\{\frac{\varepsilon}{2}, \frac{1-dKM}{dKM+NN_0}\frac{\varepsilon}{2d}\}$, where N and N_0 are positive constants that will be specified below.
Let $Z(t)$ be any solution of equation (1.1) which satisfies the inequality $\mid \Psi(t_0)Z(t_0) \mid < \delta$. From hypothesis (1), $Z(t)$ exists on $\mathbb{R}_+$ and satisfies the inequality $\mid \Psi(t)Z(t) \mid \leq \delta$, for all $t \in [0, t_0]$.
We may reason by reduction ad absurdum.
Suppose that there exists $\tau > t_0$ such that $\mid \Psi(\tau)Z(\tau) \mid = \varepsilon$ and one can consider $\mid \Psi(t)Z(t) \mid < \varepsilon$ for $t \in [t_0, \tau)$.
Then, Z(t) is a solution of the equation

$$Z' = A(t)Z + ZB(t) + \int_0^t F(t, s, Z(s))ds, \;\; t \in [t_0, \tau].$$

From Lemma 5, [3], the vector function $z(t) = \mathcal{V}ec(Z(t))$ is a solution of the corresponding Kronecker product system associated with this equation, i.e. of the system

$$z' = \left(I_d \otimes A(t) + B^T(t) \otimes I_d\right) z + \int_0^t f(t, s, z(s))ds \tag{3.1}$$

where $f(t, s, z) = \mathcal{V}ec\left(F(t, s, Z)\right)$, on the same interval $[t_0, \tau]$.
From Lemma 8, [3], we know that if $X(t)$ and $Y(t)$ are fundamental matrices for equations (1.2) and (1.3) respectively, then, the matrix $U(t) = Y^T(t) \otimes X(t)$ is a fundamental matrix for the linear homogeneous system associated with system (3.1), i.e. for the differential system

$$z' = \left(I_d \otimes A(t) + B^T(t) \otimes I_d\right) z. \tag{3.2}$$

Therefore, by the variation of constants formula (see [2], Chapter II, section 2 (8))

$$z(t) = U(t)U^{-1}(t_0)z_0 + \int_{t_0}^t U(t)U^{-1}(s)\int_0^s f(s, u, z(u))duds,$$

for $t \in [t_0, \tau]$, where $z_0 = \mathcal{V}ec(Z(t_0))$.
From Lemma 1, [3],

$$\begin{aligned} U(t)U^{-1}(t_0) &= \left(Y^T(t) \otimes X(t)\right)\left(Y^T(t_0) \otimes X(t_0)\right)^{-1} = \\ &= \left(Y^T(t)\left(Y^T\right)^{-1}(t_0)\right) \otimes \left(X(t)X^{-1}(t_0)\right), \; t \geq t_0, \end{aligned}$$

and then, for $t \in [t_0, \tau]$

$$\begin{aligned} z(t) &= \left[\left(Y^T(t)\left(Y^T\right)^{-1}(t_0)\right) \otimes \left(X(t)X^{-1}(t_0)\right)\right] z_0 + \\ &+ \int_{t_0}^t \left[\left(Y^T(t)\left(Y^T\right)^{-1}(s)\right) \otimes \left(X(t)X^{-1}(s)\right)\right] \int_0^s f(s, u, z(u))duds. \end{aligned} \tag{3.3}$$

We define the vector function $w(t)$ by setting $w(t) = (I_d \otimes \Psi(t))\, z(t)$, for $t \in [t_0, \tau]$ and the matrix function $\Phi(t,s) = \left(Y^T(t)\left(Y^T\right)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s)\right)$, for $t \geq s \geq 0$. Multiplying through $I_d \otimes \Psi(t)$, we have (see Lemma 1, [3])

$$w(t) = \Phi(t,t_0)w(t_0) + \int_{t_0}^{t} \Phi(t,s) \int_0^s (I_d \otimes \Psi(s))\, f(s,u,z(u))duds,\ t \in [t_0,\tau].$$

As a result, for $t \in [t_0, \tau]$,
(3.4)

$$\| w(t) \| \leq | \Phi(t,t_0)| \| w(t_0) \| + \int_{t_0}^{t} | \Phi(t,s) | \int_0^s \| (I_d \otimes \Psi(s))\, f(s,u,z(u)) \| \,duds.$$

From hypotheses and Lemma 6, [3],

$$\begin{aligned} \| (I_d \otimes \Psi(s))\, f(s,u,z(u)) \| &= \| (I_d \otimes \Psi(s))\, \mathcal{V}ec\,(F(s,u,Z(u))) \| \leq \\ \leq \ | \Psi(s)F(s,u,Z(u)) | &\leq f(s,u) \mid \Psi(u)Z(u) \mid \leq \\ \leq d \cdot f(s,u) \,\| (I_d \otimes \Psi(u))\, \mathcal{V}ec(Z(u)) \| &= d \cdot f(s,u) \,\| w(u) \|, \end{aligned}$$

for $s \in [t_0, \tau]$, $s \geq u \geq 0$.
From hypothesis (2), Theorems 1 and 2, [3] and Theorem 2, [4], there exists a constant $N > 0$ such that $| Y^T(t) \otimes \Psi(t)X(t) | \leq Ne^{-M^{-1}\int_0^t \varphi(s)ds}$, for $t \geq 0$.
As a result, for $t \in [t_0, \tau]$,

$$\begin{aligned} | \Phi(t,t_0) | &= \left| \left(Y^T(t)\left(Y^T\right)^{-1}(t_0)\right) \otimes \left(\Psi(t)X(t)X^{-1}(t_0)\Psi^{-1}(t_0)\right) \right| = \\ &= \left| \left(Y^T(t) \otimes \Psi(t)X(t)\right) \cdot \left(\left(Y^T\right)^{-1}(t_0) \otimes X^{-1}(t_0)\Psi^{-1}(t_0)\right) \right| \leq \\ &\leq | Y^T(t) \otimes \Psi(t)X(t) | \left| \left(Y^T\right)^{-1}(t_0) \otimes X^{-1}(t_0)\Psi^{-1}(t_0) \right| \leq NN_0, \end{aligned}$$

where $N_0 = \left| \left(Y^T\right)^{-1}(t_0) \otimes X^{-1}(t_0)\Psi^{-1}(t_0) \right|$.
Therefore, from (3.4) and the above results, for $t \in [t_0, \tau]$,
$\| w(t) \| \leq | \Phi(t,t_0) | \| w(t_0) \| + \int_{t_0}^t | \Phi(t,s) | \int_0^s \| (I_d \otimes \Psi(s))\, f(s,u,z(u)) \| \,duds \leq$
$\leq NN_0 \| w(t_0) \| + \int_{t_0}^t \varphi(s) \mid \Phi(t,s) \mid \left(\int_0^s \frac{d \cdot f(s,u)}{\varphi(s)} \| w(u) \| du\right) ds \leq$
$\leq NN_0\delta + d\int_{t_0}^t \varphi(s) \mid \Phi(t,s) \mid \left(\int_0^{t_0} \frac{f(s,u)}{\varphi(s)} \| w(u) \| du + \int_{t_0}^s \frac{f(s,u)}{\varphi(s)} \| w(u) \| du\right) ds \leq$
$\leq NN_0\delta + d\delta\int_{t_0}^t \varphi(s) \mid \Phi(t,s) \mid \left(\int_0^{t_0} \frac{f(s,u)}{\varphi(s)} du\right) ds +$
$+ d\int_{t_0}^t \varphi(s) \mid \Phi(t,s) \mid \left(\int_{t_0}^s \frac{f(s,u)}{\varphi(s)} \| w(u) \| du\right) ds \leq$
$\leq NN_0\delta + d\delta K\int_{t_0}^t \varphi(s) \mid \Phi(t,s) \mid ds +$
$+ d\int_{t_0}^t \varphi(s) \mid \Phi(t,s) \mid \left[\int_{t_0}^s \frac{f(s,u)}{\varphi(s)} \left(\sup_{t\in[t_0,\tau]} \| w(t) \|\right) du\right] ds \leq$
$\leq \delta(NN_0 + dKM) + dKM\left(\sup_{t\in[t_0,\tau]} \| w(t) \|\right).$

It follows that

$$\left(\sup_{t\in[t_0,\tau]} \parallel w(t) \parallel\right) \leq \frac{NN_0+dKM}{1-dKM}\delta \tag{3.5}$$

and then, $\parallel w(t) \parallel\leq \frac{\varepsilon}{2d}$, for $t\in[t_0,\tau]$.
From Lemma 6, [3], it follows that$\mid \Psi(\tau)Z(\tau) \mid\leq \frac{\varepsilon}{2}$. This is a clear contradiction of the fact that $\mid \Psi(\tau)Z(\tau) \mid=\varepsilon$.
From Definition, it follows that the trivial solution of (1.1) is $\Psi-$ stable over $\mathbb{R}_+$.
To prove that the trivial solution of (1.1) is $\Psi-$ asymptotically stable over $\mathbb{R}_+$, we must show furter that for each $t_0\in\mathbb{R}_+$, there exists $\delta_0=\delta_0(t_0)>0$ such that if $\mid \Psi(t_0)Z(t_0) \mid<\delta_0$, then $\lim\limits_{t\to\infty}\Psi(t)Z(t)=O_d$.
From the above result, it follows that for $\varepsilon=1$ and $t_0\geq 0$, there exists $\delta_0=\delta(1,t_0)<1$ such that if $Z_0\in\mathbb{M}_{d\times d}$ and $\mid \Psi(t_0)Z_0 \mid<\delta_0$, then any solution $Z(t)$ of (1.1) which satisfies the equality $Z(t_0)=Z_0$, satisfies the condition $\mid \Psi(t)Z(t) \mid<1$, for $t\geq 0$. Then, $\parallel w(t) \parallel<1$ for $t\geq 0$.
We will show that $\lim\limits_{t\to\infty}\Psi(t)Z(t)=O_d$, by showing $\lim\limits_{t\to\infty}\parallel w(t) \parallel=0$.
Suppose that $\limsup\limits_{t\to\infty}\parallel w(t) \parallel=\ell>0$. Let θ be such that $dKM<\theta<1$. It follows that there exists $t_1\geq t_0$ such that $\mid \Psi(t)Z(t) \mid\leq\frac{\ell}{\theta}$, for $t\geq t_1$. From hypothesis (3), it follows that there exists $s_0>t_1$ such that

$$\int_0^{t_1}\frac{f(s,u)}{\varphi(s)}du<\ell\frac{\theta-dKM}{2d\theta M},\ \text{for } s\geq s_0.$$

From (3.4), we obtain an estimate for $\parallel w(t) \parallel$, for $t>s_0$:
$\parallel w(t) \parallel\leq\mid \Phi(t,t_0) \mid\parallel w(t_0) \parallel+\int_{t_0}^t\mid \Phi(t,s) \mid\int_0^s\parallel (I_d\otimes\Psi(s))\,f(s,u,z(u)) \parallel duds<$
$<\mid Y^T(t)\otimes\Psi(t)X(t) \mid\mid \left(Y^T\right)^{-1}(t_0)\otimes X^{-1}(t_0)\Psi^{-1}(t_0) \mid\delta_0+$
$+\int_{t_0}^{s_0}\varphi(s)\mid \Phi(t,s) \mid\left(\int_0^s\frac{df(s,u)}{\varphi(s)}\parallel w(u) \parallel du\right)ds+$
$+\int_{s_0}^{t}\varphi(s)\mid \Phi(t,s) \mid\left(\int_0^{t_1}\frac{df(s,u)}{\varphi(s)}\parallel w(u) \parallel du\right)ds+$
$+\int_{s_0}^{t}\varphi(s)\mid \Phi(t,s) \mid\left(\int_{t_1}^{s}\frac{df(s,u)}{\varphi(s)}\parallel w(u) \parallel du\right)ds\leq$
$\leq(N_0\delta_0+P_0)\mid Y^T(t)\otimes\Psi(t)X(t) \mid+d\ell\frac{\theta-dKM}{2d\theta M}M+dMK\frac{\ell}{\theta}$,
where $P_0=\int_{t_0}^{s_0}\varphi(s)\mid \left(Y^T\right)^{-1}(s)\otimes X^{-1}(s)\Psi^{-1}(s) \mid\left(\int_0^s\frac{df(s,u)}{\varphi(s)}du\right)ds$.
From $\lim\limits_{t\to\infty}\mid Y^T(t)\otimes\Psi(t)X(t) \mid=0$, it follows that there exists $t_2>s_0$ such that

$$\mid Y^T(t)\otimes\Psi(t)X(t) \mid<\frac{\ell-\frac{\ell}{\theta}\frac{\theta+dKM}{2}}{2\left(N_0\delta_0+P_0\right)},\ \text{for all } t>t_2.$$

Thus, for $t>t_2$, we have

$$\parallel w(t) \parallel\leq(N_0\delta_0+P_0)\frac{\ell-\frac{\ell}{\theta}\frac{\theta+dKM}{2}}{2\left(N_0\delta_0+P_0\right)}+d\ell\frac{\theta-dKM}{2d\theta M}M+dMK\frac{\ell}{\theta}=\frac{\ell}{4}\left(3+\frac{dKM}{\theta}\right).$$

From this and definition of ℓ, we obtain $\ell\leq\frac{\ell}{4}\left(3+\frac{dKM}{\theta}\right)<\ell$, which is contradictory.
Therefore, $\lim\limits_{t\to\infty}\parallel w(t) \parallel=0$ and then (see Lemma 6, [3])

$$\lim_{t\to\infty}\Psi(t)Z(t)=O_d.$$

Thus, the trivial solution of equation (1.1) is $\Psi-$ asymptotically stable over $\mathbb{R}_+$. $\square$

Theorem 3.2. *Suppose that:*
(1) the hypothesis (H) is fulfilled;
(2) the fundamental matrix $X(t)$ for the linear matrix differential equation (1.2) satisfies the condition

$$\int_0^t \varphi(s)\left|\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s)\right| ds \leq M, \text{ for all } t \geq 0,$$

where M is a positive constant and φ is a continuous function $\varphi : \mathbb{R}_+ \to (0,\infty)$ such that $\int_0^\infty \varphi(t)dt = +\infty$;
(3) the matrix function $F(t,s,Z)$ satisfies the inequality

$$| \Psi(t)F(t,s,Z) | \leq f(t,s) \mid \Psi(s)Z \mid,$$

for $(t,s,Z) \in D$, where $f(t,s)$ is a continuous nonnegative function for $t \geq s \geq 0$ such that

$$K = \sup_{t\geq 0}\left[\frac{| B(t) |}{\varphi(t)} + \int_0^t \frac{f(t,s)}{\varphi(t)}ds\right] < \frac{1}{M}, \quad \lim_{s\to\infty}\int_0^t \frac{f(s,u)}{\varphi(s)}du = 0, \text{ for all } t > 0.$$

Then, the trivial solution of (1.1) is $\Psi-$ asymptotically *stable over* $\mathbb{R}_+$.

Proof. We will use Definition of $\Psi-$ asymptotic stability. For a given $\varepsilon > 0$ and $t_0 \in R_+$, we choose $\delta = \delta(\varepsilon, t_0)$ such that

$$0 < \delta < \frac{1-KM}{NN_0 + KM}\varepsilon,$$

where N and N_0 are positive constants that will be specified below.
Let $Z(t)$ be any solution of (1.1) which satisfies the inequality $\mid \Psi(t_0)Z(t_0) \mid < \delta$. From the first assumption of the Theorem, the solution Z(t) is defined on $\mathbb{R}_+$ and $\mid \Psi(t)Z(t) \mid \leq \delta$ for all $t \in [0, t_0]$. The solution $Z(t)$ satisfies the equation (1.1) written in the form

$$Z' = A(t)Z + \left[ZB(t) + \int_0^t F(t,s,Z(s))ds\right], \ t \in R_+.$$

From the variation of constants formula (see [2], Chapter II, Section 2 (8))

$$Z(t) = X(t)X^{-1}(t_0)Z(t_0) + \int_{t_0}^t X(t)X^{-1}(s)\left[Z(s)B(s) + \int_0^s F(s,u,Z(u))du\right]ds, \tag{3.6}$$

for all $t \geq t_0$.
From hypothesis (2), Theorems 1 and 2, [3] and Theorem 2, [4], there exists a constant $N > 0$ such that

$$\mid \Psi(t)X(t) \mid \leq Ne^{-M^{-1}\int_0^t \varphi(s)ds}, \text{ for } t \geq 0.$$

We define the matrix function $\Phi(t,s)$ by $\Phi(t,s) = \Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s)$, for $t \geq s \geq 0$ and $N_0 = \mid X^{-1}(t_0)\Psi^{-1}(t_0) \mid$.
For an arbitrary $T \in (t_0, \infty)$, let $W(T)$ be defined by $W(T) = \sup\limits_{t\in[t_0,T]} \mid \Psi(t)Z(t) \mid$.

It follows from (3.6) that, for $t \in [t_0, T]$,
$\mid \Psi(t)Z(t) \mid \leq \mid \Psi(t)X(t)X^{-1}(t_0)Z(t_0) \mid +$
$+ \int_{t_0}^{t} \mid \Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s) \mid\mid \Psi(s)Z(s)B(s) + \int_0^s \Psi(s)F(s,u,Z(u))du \mid ds \leq$
$\leq \mid \Psi(t)X(t)X^{-1}(t_0)\Psi^{-1}(t_0)\Psi(t_0)Z(t_0) \mid +$
$+ \int_{t_0}^{t} \mid \Phi(t,s) \mid \left[\mid \Psi(s)Z(s) \mid\mid B(s) \mid + \int_0^s \mid \Psi(s)F(s,u,Z(u)) \mid du\right] ds \leq$
$\leq \mid \Psi(t)X(t) \mid\mid X^{-1}(t_0)\Psi^{-1}(t_0) \mid\mid \Psi(t_0)Z(t_0) \mid +$
$+ \int_{t_0}^{t} \mid \Phi(t,s) \mid \left[\mid \Psi(s)Z(s) \mid\mid B(s) \mid + \int_0^s f(s,u) \mid \Psi(u)Z(u) \mid du\right] ds \leq$
$\leq NN_0\delta + \int_{t_0}^{t} \varphi(s) \mid \Phi(t,s) \mid \left[\mid \Psi(s)Z(s) \mid \frac{|B(s)|}{\varphi(s)} + \int_0^s \frac{f(s,u)}{\varphi(s)} \mid \Psi(u)Z(u) \mid du\right] ds \leq$
$\leq NN_0\delta + \int_{t_0}^{t} \varphi(s) \mid \Phi(t,s) \mid [\mid \Psi(s)Z(s) \mid \frac{|B(s)|}{\varphi(s)} + \int_0^{t_0} \frac{f(s,u)}{\varphi(s)} \mid \Psi(u)Z(u) \mid du +$
$+ \int_{t_0}^{s} \frac{f(s,u)}{\varphi(s)} \mid \Psi(u)Z(u) \mid du]ds \leq$
$\leq NN_0\delta + \int_{t_0}^{t} \varphi(s) \mid \Phi(t,s) \mid \left[W(T)\frac{|B(s)|}{\varphi(s)} + \delta \int_0^{t_0} \frac{f(s,u)}{\varphi(s)} du + \int_{t_0}^{s} \frac{f(s,u)}{\varphi(s)} W(T) du\right] ds \leq$
$\leq NN_0\delta + \delta KM + W(T) \int_{t_0}^{t} \varphi(s) \mid \Phi(t,s) \mid \left[\frac{|B(s)|}{\varphi(s)} + \int_{t_0}^{s} \frac{f(s,u)}{\varphi(s)} du\right] ds \leq$
$\leq \delta\,(NN_0 + KM) + KMW(T).$
It follows that

$$W(T) \leq \frac{NN_0 + KM}{1 - KM}\delta < \varepsilon. \tag{3.7}$$

From this,

$$\mid \Psi(t)Z(t) \mid < \varepsilon, \text{ for all } t \geq t_0.$$

Thus, we proved that for a given $\varepsilon > 0$ and for each $t_0 \in R_+$, there exists a $\delta \in \left(0, \frac{1-KM}{NN_0+KM}\varepsilon\right)$ such that any solution Z(t) of (1.1) which satisfies the inequality $\mid \Psi(t_0)Z(t_0) \mid < \delta$, exists and satisfies the inequality $\mid \Psi(t)Z(t) \mid < \varepsilon$, for all $t \geq t_0$.
From Definition, it follows that the trivial solution of (1.1) is $\Psi-$ stable over $\mathbb{R}_+$.
To prove that the trivial solution of (1.1) is $\Psi-$ asymptotically stable over $\mathbb{R}_+$, we must show furter that for each $t_0 \in \mathbb{R}_+$, there exists $\delta_0 = \delta_0(t_0) > 0$ such that if $\mid \Psi(t_0)Z(t_0) \mid < \delta_0$, then $\lim\limits_{t\to\infty} \Psi(t)Z(t) = O_d$.
From the above result, it follows that for $\varepsilon = 1$ and $t_0 \geq 0$, there exists $\delta_0 = \delta(1, t_0) < 1$ such that if $Z_0 \in \mathbb{M}_{d\times d}$ and $\mid \Psi(t_0)Z_0 \mid < \delta_0$, then any solution $Z(t)$ of (1.1) which satisfies the equality $Z(t_0) = Z_0$, satisfies the condition $\mid \Psi(t)Z(t) \mid < 1$, for $t \geq 0$.
We will show that $\lim\limits_{t\to\infty} \Psi(t)Z(t) = O_d$.
Let $\ell = \limsup\limits_{t\to\infty} \mid \Psi(t)Z(t) \mid$ and choose θ so that $KM < \theta < 1$.
If $\ell > 0$, then $\mid \Psi(t)Z(t) \mid \leq \frac{\ell}{\theta}$, for $t \geq t_1 > t_0$ say.
From hypothesis (3), we can choose $s_0 > t_1$ so large that

$$\int_0^{t_1} \frac{f(s,u)}{\varphi(s)} du < \ell\frac{\theta - KM}{2\theta M}, \text{ for } s \geq s_0.$$

Now, from (3.6) and the above results, for $t > s_0$,
$\mid \Psi(t)Z(t) \mid \leq \mid \Psi(t)X(t)X^{-1}(t_0)Z(t_0) \mid +$
$+ \int_{t_0}^{t} \mid \Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s) \mid\mid \Psi(s)Z(s)B(s) + \int_0^s \Psi(s)F(s,u,Z(u))du \mid ds \leq$
$\leq \mid \Psi(t)X(t) \mid \cdot \left[\mid X^{-1}(t_0)\Psi^{-1}(t_0) \mid\mid \Psi(t_0)Z(t_0) \mid +\right.$
$\left.+ \int_{t_0}^{t_1} \mid X^{-1}(s)\Psi^{-1}(s) \mid\mid \Psi(s)Z(s)B(s) + \int_0^s \Psi(s)F(s,u,Z(u))du \mid ds\right] +$

$+\int_{t_1}^{t}\varphi(s)\mid\Phi(t,s)\mid\left[\mid\Psi(s)Z(s)\mid\frac{|B(s)|}{\varphi(s)}+\int_0^s\frac{f(s,u)}{\varphi(s)}\mid\Psi(u)Z(u)\mid du\right]ds\leq$
$\leq\mid\Psi(t)X(t)\mid\cdot\left[\mid X^{-1}(t_0)\Psi^{-1}(t_0)\mid\mid\Psi(t_0)Z(t_0)\mid+\right.$
$\left.+\int_{t_0}^{t_1}\mid X^{-1}(s)\Psi^{-1}(s)\mid\mid\Psi(s)Z(s)B(s)+\int_0^s\Psi(s)F(s,u,Z(u))du\mid ds\right]+$
$+\int_{t_1}^{t}\varphi(s)\mid\Phi(t,s)\mid\left[\frac{\ell}{\theta}\frac{|B(s)|}{\varphi(s)}+\int_0^{t_1}\frac{f(s,u)}{\varphi(s)}du+\int_{t_1}^{s}\frac{f(s,u)}{\varphi(s)}\frac{\ell}{\theta}du\right]ds\leq$
$\leq\mid\Psi(t)X(t)\mid\cdot\left[\mid X^{-1}(t_0)\Psi^{-1}(t_0)\mid\mid\Psi(t_0)Z(t_0)\mid+\right.$
$\left.+\int_{t_0}^{t_1}\mid X^{-1}(s)\Psi^{-1}(s)\mid\mid\Psi(s)Z(s)B(s)+\int_0^s\Psi(s)F(s,u,Z(u))du\mid ds\right]+$
$+M\ell\frac{\theta-KM}{2\theta M}+\frac{\ell}{\theta}KM=$
$\leq\mid\Psi(t)X(t)\mid\cdot\left[\mid X^{-1}(t_0)\Psi^{-1}(t_0)\mid\mid\Psi(t_0)Z(t_0)\mid+\right.$
$\left.+\int_{t_0}^{t_1}\mid X^{-1}(s)\Psi^{-1}(s)\mid\mid\Psi(s)Z(s)B(s)+\int_0^s\Psi(s)F(s,u,Z(u))du\mid ds\right]+$
$+\ell\frac{\theta+KM}{2\theta}$.
Letting $t\to\infty$, we get

$$\ell\leq\ell\frac{\theta+KM}{2\theta}<\ell,$$

which is impossible.
Therefore, $\ell=0$.
From Definition of ℓ, it results that $\lim\limits_{t\to\infty}\Psi(t)Z(t)=O_d$.
Thus, the trivial solution of (1.1) is $\Psi-$ asymptotically stable over $\mathbb{R}_+$. □

Remark 3.1. The Theorem 3.2 is no longer true if we require that the above equation (1.2) is only $\Psi-$ asymptotically stable over $\mathbb{R}_+$ instead of the sufficient condition (2) of Theorem for $\Psi-$ asymptotic stability of equation (1.2) (see Theorems 1 and 2, [4]). To see this, we show the following simple example, adapted from Example 4, [4].

Example 3.2. Consider the equation (1.1) with

$$A(t)=O_2, B(t)=\begin{pmatrix}0 & 0\\ 0 & \frac{a}{\sqrt{t+1}}\end{pmatrix}, F(t,s,Z)=O_2,$$

where a ia a positive constant.
A fundamental matrix X(t) for the matrix differential equation (1.2) is $X(t)=I_2$.
Consider $\Psi(t)=\begin{pmatrix}\frac{1}{t^2+1} & 0\\ 0 & \frac{1}{t+1}\end{pmatrix}$.
From $\lim\limits_{t\to\infty}\Psi(t)X(t)=O_2$ and Theorem 1, [4], it follows that the equation (1.2) is $\Psi-$ asymptotically stable over $\mathbb{R}_+$.
It is easy to see (by reduction to absurdity) that there is not a function φ to satisfy the condition of Theorem 2.
Now, the equation (1.1) becomes $Z'=ZB(t)$ and have the general solution

$$Z(t)=C\begin{pmatrix}1 & 0\\ 0 & e^{2a\sqrt{t+1}}\end{pmatrix},$$

where C is a real 2×2 constant matrix.
A particular solution of the equation is

$$Z_0(t)=c\begin{pmatrix}1 & 0\\ 0 & e^{2a\sqrt{t+1}}\end{pmatrix},$$

where c is a real constant.
From $\lim\limits_{t\to\infty} \mid \Psi(t)Z_0(t) \mid = +\infty$, it is easy to see that the equation is not $\Psi-$ asymptotically stable over $\mathbb{R}_+$.

Theorem 3.3. *Suppose that:*
(1) the hypothesis (H) is fulfilled;
(2) the fundamental matrix $Y(t)$ for the linear matrix differential equations (1.3) satisfies the condition

$$\int_0^t \varphi(s) \left| \left(Y^T(t)(Y^T)^{-1}(s)\right) \otimes \left(\Psi(t)\Psi^{-1}(s)\right) \right| ds \leq M, \ \textit{for all } t \geq 0,$$

where M is a positive constant and φ is a continuous function $\varphi : R_+ \longrightarrow (0, \infty)$ such that $\int_0^\infty \varphi(t)dt = +\infty$;
(3) the matrix function $F(t, s, Z)$ satisfies the inequality

$$\mid \Psi(t)F(t, s, Z) \mid \leq f(t, s) \mid \Psi(s)Z \mid,$$

for $(t, s, Z) \in D$, where $f(t, s)$ is a continuous nonnegative function for $t \geq s \geq 0$ such that

$$K \ = \ \sup_{t\geq 0} \left[\frac{\mid \Psi(t)A(t)\Psi^{-1}(t) \mid}{\varphi(t)} + d\int_0^t \frac{f(t,s)}{\varphi(t)} ds \right] < \frac{1}{M}$$

and

$$\lim_{s\to\infty} \int_0^t \frac{f(s,u)}{\varphi(s)} du = 0, \ \textit{for all } t > 0.$$

Then, the trivial solution of (1.1) is $\Psi-$ asymptotically *stable over* $\mathbb{R}_+$.

Proof. We see equation (1.1) in the form

$$Z' = ZB(t) + \left[A(t)Z + \int_0^t F(t, s, Z(s))ds \right], \ t \geq 0.$$

With a similar proof as in Theorem 3.1, we have the conclusion. □

Remark 3.3. 1. The above Theorem is no longer true if we require that the above equation (3) is only $\Psi-$asymptotically stable over $\mathbb{R}_+$, instead of the condition (2) of Theorem. A simple example can shows this.
For Theorem 3.1, we have a similar situation.
2. On the other hand, if the assumption " $\lim\limits_{s\to\infty} \int_0^t \frac{f(s,u)}{\varphi(s)} du = 0$, for all $t > 0$" is omitted in Theorems, the trivial solution of (1.1) cannot be $\Psi-$asymptotically stable over $\mathbb{R}_+$.

Remark 3.4. The above Theorems generalize the results from Theorem 4.2, [12], Theorem 4.1, [9], and Theorem 5, [11], from differential systems to Lyapunov matrix differential equations.

Remark 3.5. The above Theorems have very useful corollaries in the particular cases when $f(t, s) = h(t)g(s)$ or $f(t, s) = k(t - s)$.

Remark 3.6. The function φ in the above Theorems can serve to weaken the required hypotheses on the fundamental matrices X and Y or function F.

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