Strong insertion of a contra-continuous function between two comparable real-valued functions

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Abstract. Necessary and sufficient conditions in terms of lower cut sets are given for the strong insertion of a contra-continuous function between two comparable real-valued functions on such topological spaces that kernel of sets are open.

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1 Introduction

The concept of a $C-$open set in a topological space was introduced by E. Hatir, T. Noiri and S. Yksel in [12]. The authors define a set S to be a $C-$*open* set if $S = U \cap A$, where U is open and A is semi-preclosed. A set S is a $C-$*closed* set if its complement (denoted by S^c) is a $C-$open set or equivalently if $S = U \cup A$, where U is closed and A is semi-preopen. The authors show that a subset of a topological space is open if and only if it is an $\alpha-$open set and a $C-$open set or equivalently a subset of a topological space is closed if and only if it is an $\alpha-$closed set and a $C-$closed set. This enables them to provide the following decomposition of continuity: a function is continuous if and only if it is $\alpha-$continuous and $C-$continuous or equivalently a function is contra-continuous if and only if it is contra-$\alpha-$continuous and contra-$C-$continuous.

Recall that a subset A of a topological space (X, τ) is called $\alpha-$*open* if A is the difference of an open and a nowhere dense subset of X. A set A is called $\alpha-$*closed* if its complement is $\alpha-$open or equivalently if A is the union of a closed and a nowhere dense set. Sets which are dense in some regular closed subspace are called *semi-preopen or* $\beta-$*open*. A set is *semi-preclosed or* $\beta-$*closed* if its complement is semi-preopen or $\beta-$open.

In [7] it was shown that a set A is $\beta-$open if and only if $A \subseteq Cl(Int(Cl(A)))$. A generalized class of closed sets was considered by Maki in [20]. He investigated the sets that can be represented as union of closed sets and called them $V-$sets. Complements of $V-$sets, i.e., sets that are intersection of open sets are called $\Lambda-$sets [20].

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Recall that a real-valued function f defined on a topological space X is called $A-$continuous [25] if the preimage of every open subset of $\mathbb{R}$ belongs to A, where A is a collection of subsets of X. Most of the definitions of function used throughout this paper are consequences of the definition of $A-$continuity. However, for unknown concepts the reader may refer to [4, 11]. In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.

J. Dontchev in [5] introduced a new class of mappings called contra-continuity.S. Jafari and T. Noiri in [13, 14] exhibited and studied among others a new weaker form of this class of mappings called contra-$\alpha-$continuous. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 3, 8, 9, 10, 23].

Hence, a real-valued function f defined on a topological space X is called *contra-continuous* (resp. *contra-$C-$continuous* , *contra-$\alpha-$continuous*) if the preimage of every open subset of $\mathbb{R}$ is closed (resp. $C-$closed , $\alpha-$closed) in X[5].

Results of Katětov [15, 16] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give a necessary and sufficient conditions for the insertion of a contra-continuous function between two comparable real-valued functions on such topological spaces that $\Lambda-$sets or kernel of sets are open [20].

If g and f are real-valued functions defined on a space X, we write $g \leq f$ in case $g(x) \leq f(x)$ for all x in X.

The following definitions are modifications of conditions considered in [17].

A property P defined relative to a real-valued function on a topological space is a *$cc-$property* provided that any constant function has property P and provided that the sum of a function with property P and any contra-continuous function also has property P. If P_1 and P_2 are $cc-$properties, the following terminology is used:(i) A space X has the *weak $cc-$insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f, g$ has property P_1 and f has property P_2, then there exists a contra-continuous function h such that $g \leq h \leq f$.(ii) A space X has the *strong $cc-$insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f, g$ has property P_1 and f has property P_2, then there exists a contra-continuous function h such that $g \leq h \leq f$ and if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$.

In this paper, for a topological space whose $\Lambda-$sets or kernel of sets are open, is given a sufficient condition for the weak $cc-$insertion property. Also for a space with the weak $cc-$insertion property, we give necessary and sufficient conditions for the space to have the strong $cc-$insertion property. Several insertion theorems are obtained as corollaries of these results. In addition, the insertion of a contra-continuous function between two comparable contra-precontinuous real-valued functions has also recently considered by the author in [21].

2 The main result

Before giving a sufficient condition for insertability of a contra-continuous function, the necessary definitions and terminology are stated.

The abbreviations *cc*, *cαc* and *cCc* are used for contra-continuous, contra-$\alpha-$continuous and contra-$C-$continuous, respectively.

Definition 2.1. Let A be a subset of a topological space (X, τ). We define the subsets A^Λ and A^V as follows:
$A^\Lambda = \cap\{O : O \supseteq A, O \in (X, \tau)\}$ and $A^V = \cup\{F : F \subseteq A, F^c \in (X, \tau)\}$.
In [6, 19, 22], A^Λ is called the *kernel* of A.

The family of all $\alpha-$open, $\alpha-$closed, $C-$open and $C-$closed will be denoted by $\alpha O(X, \tau)$, $\alpha C(X, \tau)$, $CO(X, \tau)$ and $CC(X, \tau)$, respectively.

We define the subsets $\alpha(A^\Lambda), \alpha(A^V), C(A^\Lambda)$ and $C(A^V)$ as follows:
$\alpha(A^\Lambda) = \cap\{O : O \supseteq A, O \in \alpha O(X, \tau)\}$,
$\alpha(A^V) = \cup\{F : F \subseteq A, F \in \alpha C(X, \tau)\}$,
$C(A^\Lambda) = \cap\{O : O \supseteq A, O \in CO(X, \tau)\}$ and
$C(A^V) = \cup\{F : F \subseteq A, F \in CC(X, \tau)\}$.
$\alpha(A^\Lambda)$ (resp. $C(A^\Lambda)$) is called the $\alpha - kernel$ (resp. $C - kernel$) of A.

The following first two definitions are modifications of conditions considered in [15, 16].

Definition 2.2. If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x\ \bar{\rho}\ y$ if and only if $y\ \rho\ v$ implies $x\ \rho\ v$ and $u\ \rho\ x$ implies $u\ \rho\ y$ for any u and v in S.

Definition 2.3. A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:

1) If $A_i\ \rho\ B_j$ for any $i \in \{1, \ldots, m\}$ and for any $j \in \{1, \ldots, n\}$, then there exists a set C in $P(X)$ such that $A_i\ \rho\ C$ and $C\ \rho\ B_j$ for any $i \in \{1, \ldots, m\}$ and any $j \in \{1, \ldots, n\}$.

2) If $A \subseteq B$, then $A\ \bar{\rho}\ B$.

3) If $A\ \rho\ B$, then $A^\Lambda \subseteq B$ and $A \subseteq B^V$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

Definition 2.4. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number ℓ, then $A(f, \ell)$ is called a *lower indefinite cut set* in the domain of f at the level ℓ.

We now give the following main result:

Theorem 2.1. Let g and f be real-valued functions on the topological space X, in which kernel of sets are open, with $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1)\ \rho\ A(g, t_2)$, then there exists a contra-continuous function h defined on X such that $g \leq h \leq f$.

Proof. Let g and f be real-valued functions defined on the X such that $g \leq f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f,t)$ and $A(g,t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f,t_1)\ \rho\ A(g,t_2)$.

Define functions F and G mapping the rational numbers $\mathbb{Q}$ into the power set of X by $F(t) = A(f,t)$ and $G(t) = A(g,t)$. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1)\ \bar{\rho}\ F(t_2), G(t_1)\ \bar{\rho}\ G(t_2)$, and $F(t_1)\ \rho\ G(t_2)$. By Lemmas 1 and 2 of [16] it follows that there exists a function H mapping $\mathbb{Q}$ into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1)\ \rho\ H(t_2), H(t_1)\ \rho\ H(t_2)$ and $H(t_1)\ \rho\ G(t_2)$.

For any x in X, let $h(x) = \inf\{t \in \mathbb{Q} : x \in H(t)\}$.

We first verify that $g \leq h \leq f$: If x is in $H(t)$ then x is in $G(t')$ for any $t' > t$; since x is in $G(t') = A(g,t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence $g \leq h$. If x is not in $H(t)$, then x is not in $F(t')$ for any $t' < t$; since x is not in $F(t') = A(f,t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence $h \leq f$.

Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have $h^{-1}(t_1,t_2) = H(t_2)^V \setminus H(t_1)^\Lambda$. Hence $h^{-1}(t_1,t_2)$ is closed in X, i.e., h is a contra-continuous function on X. ■

The above proof used the technique of theorem 1 in [15].

If a space has the strong *cc*-insertion property for (P_1,P_2), then it has the weak *cc*-insertion property for (P_1,P_2).The following result uses lower cut sets and gives a necessary and sufficient condition for a space satisfies that weak *cc*-insertion property to satisfy the strong *cc*-insertion property.

Theorem 2.2. Let P_1 and P_2 be $cc-$property and X be a space that satisfies the weak $cc-$insertion property for (P_1,P_2). Also assume that g and f are functions on X such that $g \leq f, g$ has property P_1 and f has property P_2. The space X has the strong $cc-$insertion property for (P_1,P_2) if and only if there exist lower cut sets $A(f-g,2^{-n})$ and there exists a sequence $\{F_n\}$ of subsets of X such that (i) for each n, F_n and $A(f-g,2^{-n})$ are completely separated by contra-continuous functions, and (ii)$\{x \in X : (f-g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$.

Proof. Suppose that there is a sequence $(A(f-g,2^{-n}))$ of lower cut sets for $f-g$ and suppose that there is a sequence (F_n) of subsets of X such that

$$\{x \in X : (f-g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$$

and such that for each n, there exists a contra-continuous function k_n on X into $[0,2^{-n}]$ with $k_n = 2^{-n}$ on F_n and $k_n = 0$ on $A(f-g,2^{-n})$. The function k from X into $[0,1/4]$ which is defined by

$$k(x) = 1/4\sum_{n=1}^{\infty} k_n(x)$$

is a contra-continuous function by the Cauchy condition and the properties of contra-continuous functions, (1) $k^{-1}(0) = \{x \in X : (f-g)(x) = 0\}$ and (2) if $(f-g)(x) > 0$ then $k(x) < (f-g)(x)$: In order to verify (1), observe that if $(f-g)(x) = 0$, then $x \in A(f-g,2^{-n})$ for each n and hence $k_n(x) = 0$ for each n. Thus $k(x) = 0$.

Conversely, if $(f-g)(x)>0$, then there exists an n such that $x \in F_n$ and hence $k_n(x)=2^{-n}$. Thus $k(x) \neq 0$ and this verifies (1). Next, in order to establish (2), note that

$$\{x \in X : (f-g)(x)=0\} = \bigcap_{n=1}^{\infty} A(f-g, 2^{-n})$$

and that $(A(f-g,2^{-n}))$ is a decreasing sequence. Thus if $(f-g)(x)>0$ then either $x \notin A(f-g,1/2)$ or there exists a smallest n such that $x \notin A(f-g,2^{-n})$ and $x \in A(f-g,2^{-j})$ for $j=1,\ldots,n-1$.
In the former case,

$$k(x) = 1/4 \sum_{n=1}^{\infty} k_n(x) \leq 1/4 \sum_{n=1}^{\infty} 2^{-n} < 1/2 \leq (f-g)(x),$$

and in the latter,

$$k(x) = 1/4 \sum_{j=n}^{\infty} k_j(x) \leq 1/4 \sum_{j=n}^{\infty} 2^{-j} < 2^{-n} \leq (f-g)(x).$$

Thus $0 \leq k \leq f-g$ and if $(f-g)(x)>0$ then $(f-g)(x)>k(x)>0$. Let $g_1 = g+(1/4)k$ and $f_1 = f-(1/4)k$. Then $g \leq g_1 \leq f_1 \leq f$ and if $g(x)<f(x)$ then

$$g(x) < g_1(x) < f_1(x) < f(x).$$

Since P_1 and P_2 are $cc-$properties, then g_1 has property P_1 and f_1 has property P_2. Since by hypothesis X has the weak $cc-$insertion property for (P_1,P_2) , then there exists a contra-continuous function h such that $g_1 \leq h \leq f_1$. Thus $g \leq h \leq f$ and if $g(x)<f(x)$ then $g(x)<h(x)<f(x)$. Therefore X has the strong $cc-$insertion property for (P_1,P_2). (The technique of this proof is by Lane [17].)

Conversely, assume that X satisfies the strong $cc-$insertion for (P_1,P_2). Let g and f be functions on X satisfying P_1 and P_2 respectively such that $g \leq f$. Thus there exists a contra-continuous function h such that $g \leq h \leq f$ and such that if $g(x)<f(x)$ for any x in X, then $g(x)<h(x)<f(x)$. We follow an idea contained in Powderly [24]. Now consider the functions 0 and $f-h$.0 satisfies property P_1 and $f-h$ satisfies property P_2. Thus there exists a contra-continuous function h_1 such that $0 \leq h_1 \leq f-h$ and if $0<(f-h)(x)$ for any x in X, then $0<h_1(x)<(f-h)(x)$. We next show that

$$\{x \in X : (f-g)(x)>0\} = \{x \in X : h_1(x)>0\}.$$

If x is such that $(f-g)(x)>0$, then $g(x)<f(x)$. Therefore $g(x)<h(x)<f(x)$. Thus $f(x)-h(x)>0$ or $(f-h)(x)>0$. Hence $h_1(x)>0$. On the other hand, if $h_1(x)>0$, then since $(f-h) \geq h_1$ and $f-g \geq f-h$, therefore $(f-g)(x)>0$. For each n, let $A(f-g,2^{-n}) = \{x \in X : (f-g)(x) \leq 2^{-n}\}$,
$F_n = \{x \in X : h_1(x) \geq 2^{-n+1}\}$ and
$k_n = \sup\{\inf\{h_1, 2^{-n+1}\}, 2^{-n}\} - 2^{-n}$.
Since $\{x \in X : (f-g)(x)>0\} = \{x \in X : h_1(x)>0\}$, it follows that

$$\{x \in X : (f-g)(x)>0\} = \bigcup_{n=1}^{\infty} F_n.$$

We next show that k_n is a contra-continuous function which completely separates F_n and $A(f-g,2^{-n})$. From its definition and by the properties of contra-continuous functions, it is clear that k_n is a contra-continuous function. Let $x \in F_n$. Then, from the definition of $k_n, k_n(x) = 2^{-n}$. If $x \in A(f-g,2^{-n})$, then since $h_1 \leq f-h \leq f-g, h_1(x) \leq 2^{-n}$. Thus $k_n(x) = 0$, according to the definition of k_n. Hence k_n completely separates F_n and $A(f-g,2^{-n})$. ■

Theorem 2.3. Let P_1 and P_2 be $cc-$properties and assume that the space X satisfied the weak $cc-$insertion property for (P_1,P_2). The space X satisfies the strong $cc-$insertion property for (P_1,P_2) if and only if X satisfies the strong $cc-$insertion property for (P_1,cc) and for (cc,P_2).
Proof. Assume that X satisfies the strong $cc-$insertion property for (P_1,cc) and for (cc,P_2). If g and f are functions on X such that $g \leq f, g$ satisfies property P_1, and f satisfies property P_2, then since X satisfies the weak $cc-$insertion property for (P_1,P_2) there is a contra-continuous function k such that $g \leq k \leq f$. Also, by hypothesis there exist contra-continuous functions h_1 and h_2 such that $g \leq h_1 \leq k$ and if $g(x) < k(x)$ then $g(x) < h_1(x) < k(x)$ and such that $k \leq h_2 \leq f$ and if $k(x) < f(x)$ then $k(x) < h_2(x) < f(x)$. If a function h is defined by $h(x) = (h_2(x)+h_1(x))/2$, then h is a contra-continuous function, $g \leq h \leq f$, and if $g(x) < f(x)$ then $g(x) < h(x) < f(x)$. Hence X satisfies the strong $cc-$insertion property for (P_1,P_2).

The converse is obvious since any contra-continuous function must satisfy both properties P_1 and P_2. (The technique of this proof is by Lane [18].) □

3 Applications

Before stating the consequences of Theorems 2.1, 2.2 and 2.3 we suppose that X is a topological space whose kernel of sets are open.

Corollary 3.1. If for each pair of disjoint $\alpha-$open (resp. $C-$open) sets G_1, G_2 of X , there exist closed sets F_1 and F_2 of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$ then X has the weak $cc-$insertion property for $(c\alpha c, c\alpha c)$ (resp. (cCc, cCc)).
Proof. Let g and f be real-valued functions defined on X, such that f and g are $c\alpha c$ (resp. cCc), and $g \leq f$.If a binary relation ρ is defined by $A\ \rho\ B$ in case $\alpha(A^\Lambda) \subseteq \alpha(B^V)$ (resp. $C(A^\Lambda) \subseteq C(B^V)$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f,t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g,t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is an $\alpha-$open (resp. $C-$open) set and since $\{x \in X : g(x) < t_2\}$ is an $\alpha-$closed (resp. $C-$closed) set, it follows that $\alpha(A(f,t_1)^\Lambda) \subseteq \alpha(A(g,t_2)^V)$ (resp. $C(A(f,t_1)^\Lambda) \subseteq C(A(g,t_2)^V)$). Hence $t_1 < t_2$ implies that $A(f,t_1)\ \rho\ A(g,t_2)$. The proof follows from Theorem 2.1. ■

Corollary 3.2. If for each pair of disjoint $\alpha-$open (resp. $C-$open) sets G_1, G_2, there exist closed sets F_1 and F_2 such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$ then every contra-$\alpha-$continuous (resp. contra-$C-$continuous) function is contra-continuous.
Proof. Let f be a real-valued contra-$\alpha-$continuous (resp. contra-$C-$continuous)

function defined on X. Set $g = f$, then by Corollary 3.1, there exists a contra-continuous function h such that $g = h = f$.$\blacksquare$

Corollary 3.3. If for each pair of disjoint $\alpha-$open (resp. $C-$open) sets G_1, G_2 of X, there exist closed sets F_1 and F_2 of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$ then X has the strong $cc-$insertion property for $(c\alpha c, c\alpha c)$ (resp. (cCc, cCc)).
Proof. Let g and f be real-valued functions defined on the X, such that f and g are $c\alpha c$ (resp. cCc), and $g \leq f$. Set $h = (f+g)/2$, thus $g \leq h \leq f$ and if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$. Also, by Corollary 3.2, since g and f are contra-continuous functions hence h is a contra-continuous function.$\blacksquare$

Corollary 3.4. If for each pair of disjoint subsets G_1, G_2 of X, such that G_1 is $\alpha-$open and G_2 is $C-$open, there exist closed subsets F_1 and F_2 of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$ then X have the weak $cc-$insertion property for $(c\alpha c, cCc)$ and $(cCc, c\alpha c)$.
Proof. Let g and f be real-valued functions defined on X, such that g is $c\alpha c$ (resp. cCc) and f is cCc (resp. $c\alpha c$), with $g \leq f$.If a binary relation ρ is defined by $A\ \rho\ B$ in case $C(A^\Lambda) \subseteq \alpha(B^V)$ (resp. $\alpha(A^\Lambda) \subseteq C(B^V)$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a $C-$open (resp. $\alpha-$open) set and since $\{x \in X : g(x) < t_2\}$ is an $\alpha-$closed (resp. $C-$closed) set, it follows that $C(A(f,t_1)^\Lambda) \subseteq \alpha(A(g,t_2)^V)$ (resp. $\alpha(A(f,t_1)^\Lambda) \subseteq C(A(g,t_2)^V)$). Hence $t_1 < t_2$ implies that $A(f,t_1)\ \rho\ A(g,t_2)$. The proof follows from Theorem 2.1. $\blacksquare$

Before stating consequences of Theorem 2.2, we state and prove the necessary lemmas.
Lemma 3.1. The following conditions on the space X are equivalent:

(i) For each pair of disjoint subsets G_1, G_2 of X, such that G_1 is $\alpha-$open and G_2 is $C-$open, there exist closed subsets F_1, F_2 of X such that $G_1 \subseteq F_1, G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$.

(ii) If G is a $C-$open (resp. $\alpha-$open) subset of X which is contained in an $\alpha-$closed (resp. $C-$closed) subset F of X, then there exists a closed subset H of X such that $G \subseteq H \subseteq H^\Lambda \subseteq F$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $G \subseteq F$, where G and F are $C-$open (resp. $\alpha-$open) and $\alpha-$closed (resp. $C-$closed) subsets of X, respectively. Hence, F^c is an $\alpha-$open (resp. $C-$open) and $G \cap F^c = \varnothing$.

By (i) there exists two disjoint closed subsets F_1, F_2 such that $G \subseteq F_1$ and $F^c \subseteq F_2$. But

$$F^c \subseteq F_2 \Rightarrow F_2^c \subseteq F,$$

and

$$F_1 \cap F_2 = \varnothing \Rightarrow F_1 \subseteq F_2^c$$

hence

$$G \subseteq F_1 \subseteq F_2^c \subseteq F$$

and since F_2^c is an open subset containing F_1, we conclude that $F_1^\Lambda \subseteq F_2^c$, i.e.,

$$G \subseteq F_1 \subseteq F_1^\Lambda \subseteq F.$$

By setting $H = F_1$, condition (ii) holds.

(ii) $\Rightarrow$ (i) Suppose that G_1, G_2 are two disjoint subsets of X, such that G_1 is $\alpha-$open and G_2 is $C-$open.

This implies that $G_2 \subseteq G_1^c$ and G_1^c is an $\alpha-$closed subset of X. Hence by (ii) there exists a closed set H such that $G_2 \subseteq H \subseteq H^\Lambda \subseteq G_1^c$.
But

$$H \subseteq H^\Lambda \Rightarrow H \cap (H^\Lambda)^c = \varnothing$$

and

$$H^\Lambda \subseteq G_1^c \Rightarrow G_1 \subseteq (H^\Lambda)^c.$$

Furthermore, $(H^\Lambda)^c$ is a closed subset of X. Hence $G_2 \subseteq H, G_1 \subseteq (H^\Lambda)^c$ and $H \cap (H^\Lambda)^c = \varnothing$. This means that condition (i) holds.■

Lemma 3.2. Suppose that X is a topological space. If each pair of disjoint subsets G_1, G_2 of X, where G_1 is $\alpha-$open and G_2 is $C-$open, can be separated by closed subsets of X then there exists a contra-continuous function $h : X \to [0, 1]$ such that $h(G_2) = \{0\}$ and $h(G_1) = \{1\}$.

Proof. Suppose G_1 and G_2 are two disjoint subsets of X, where G_1 is $\alpha-$open and G_2 is $C-$open. Since $G_1 \cap G_2 = \varnothing$, hence $G_2 \subseteq G_1^c$. In particular, since G_1^c is an $\alpha-$closed subset of X containing the $C-$open subset G_2 of X,by Lemma 3.1, there exists a closed subset $H_{1/2}$ such that

$$G_2 \subseteq H_{1/2} \subseteq H_{1/2}^\Lambda \subseteq G_1^c.$$

Note that $H_{1/2}$ is also an $\alpha-$closed subset of X and contains G_2, and G_1^c is an $\alpha-$closed subset of X and contains the $C-$open subset $H_{1/2}^\Lambda$ of X. Hence, by Lemma 3.1, there exists closed subsets $H_{1/4}$ and $H_{3/4}$ such that

$$G_2 \subseteq H_{1/4} \subseteq H_{1/4}^\Lambda \subseteq H_{1/2} \subseteq H_{1/2}^\Lambda \subseteq H_{3/4} \subseteq H_{3/4}^\Lambda \subseteq G_1^c.$$

By continuing this method for every $t \in D$, where $D \subseteq [0, 1]$ is the set of rational numbers that their denominators are exponents of 2, we obtain closed subsets H_t with the property that if $t_1, t_2 \in D$ and $t_1 < t_2$, then $H_{t_1} \subseteq H_{t_2}$. We define the function h on X by $h(x) = \inf\{t : x \in H_t\}$ for $x \notin G_1$ and $h(x) = 1$ for $x \in G_1$.

Note that for every $x \in X, 0 \leq h(x) \leq 1$, i.e., h maps X into [0,1]. Also, we note that for any $t \in D, G_2 \subseteq H_t$; hence $h(G_2) = \{0\}$. Furthermore, by definition, $h(G_1) = \{1\}$. It remains only to prove that h is a contra-continuous function on X. For every $\alpha \in \mathbb{R}$, we have if $\alpha \leq 0$ then $\{x \in X : h(x) < \alpha\} = \varnothing$ and if $0 < \alpha$ then $\{x \in X : h(x) < \alpha\} = \cup\{H_t : t < \alpha\}$, hence, they are closed subsets of X. Similarly, if $\alpha < 0$ then $\{x \in X : h(x) > \alpha\} = X$ and if $0 \leq \alpha$ then $\{x \in X : h(x) > \alpha\} = \cup\{(H_t^\Lambda)^c : t > \alpha\}$ hence, every of them is a closed subset. Consequently h is a contra-continuous function. ■

Lemma 3.3. Suppose that X is a topological space. If each pair of disjoint subsets G_1, G_2 of X, where G_1 is $\alpha-$open and G_2 is $C-$open, can separate by closed subsets of X, and G_1 (resp. G_2) is a closed subsets of X, then there exists a contra-continuous function $h : X \to [0,1]$ such that, $h^{-1}(0) = G_1$ (resp. $h^{-1}(0) = G_2$) and $h(G_2) = \{1\}$ (resp. $h(G_1) = \{1\}$).

Proof. Suppose that G_1 (resp. G_2) is a closed subset of X. By Lemma 3.2, there exists a contra-continuous function $h : X \to [0,1]$ such that, $h(G_1) = \{0\}$ (resp. $h(G_2) = \{0\}$) and $h(X \setminus G_1) = \{1\}$ (resp. $h(X \setminus G_2) = \{1\}$). Hence, $h^{-1}(0) = G_1$ (resp. $h^{-1}(0) = G_2$) and since $G_2 \subseteq X \setminus G_1$ (resp. $G_1 \subseteq X \setminus G_2$), therefore $h(G_2) = \{1\}$ (resp. $h(G_1) = \{1\}$). ■

Lemma 3.4. Suppose that X is a topological space such that every two disjoint $C-$open and $\alpha-$open subsets of X can be separated by closed subsets of X. The following conditions are equivalent:

(i) For every two disjoint subsets G_1 and G_2 of X, where G_1 is $\alpha-$open and G_2 is $C-$open, there exists a contra-continuous function $h : X \to [0,1]$ such that, $h^{-1}(0) = G_1$ (resp. $h^{-1}(0) = G_2$) and $h^{-1}(1) = G_2$ (resp. $h^{-1}(1) = G_1$).

(ii) Every $\alpha-$open (resp. $C-$open) subset of X is a closed subsets of X.

(iii) Every $\alpha-$closed (resp. $C-$closed) subset of X is an open subsets of X.

Proof. (i) $\Rightarrow$ (ii) Suppose that G is an $\alpha-$open (resp. $C-$open) subset of X. Since $\varnothing$ is a $C-$open (resp. $\alpha-$open) subset of X, by (i) there exists a contra-continuous function $h : X \to [0,1]$ such that, $h^{-1}(0) = G$. Set $F_n = \{x \in X : h(x) < \frac{1}{n}\}$. Then for every $n \in \mathbb{N}$, F_n is a closed subset of X and $\bigcap_{n=1}^{\infty} F_n = \{x \in X : h(x) = 0\} = G$.

(ii) $\Rightarrow$ (i) Suppose that G_1 and G_2 are two disjoint subsets of X, where G_1 is $\alpha-$open and G_2 is $C-$open. By Lemma 3.3, there exists a contra-continuous function $f : X \to [0,1]$ such that, $f^{-1}(0) = G_1$ and $f(G_2) = \{1\}$. Set $G = \{x \in X : f(x) < \frac{1}{2}\}$, $F = \{x \in X : f(x) = \frac{1}{2}\}$, and $H = \{x \in X : f(x) > \frac{1}{2}\}$. Then $G \cup F$ and $H \cup F$ are two open subsets of X and $(G \cup F) \cap G_2 = \varnothing$. By Lemma 3.3, there exists a contra-continuous function $g : X \to [\frac{1}{2}, 1]$ such that, $g^{-1}(1) = G_2$ and $g(G \cup F) = \{\frac{1}{2}\}$. Define h by $h(x) = f(x)$ for $x \in G \cup F$, and $h(x) = g(x)$ for $x \in H \cup F$. Then h is well-defined and a contra-continuous function, since $(G \cup F) \cap (H \cup F) = F$ and for every $x \in F$ we have $f(x) = g(x) = \frac{1}{2}$. Furthermore, $(G \cup F) \cup (H \cup F) = X$, hence h defined on X and maps to $[0,1]$. Also, we have $h^{-1}(0) = G_1$ and $h^{-1}(1) = G_2$.

(ii) $\Leftrightarrow$ (iii) By De Morgan law and noting that the complement of every open subset of X is a closed subset of X and complement of every closed subset of X is an open subset of X, the equivalence is hold. ■

Corollary 3.5. If for every two disjoint subsets G_1 and G_2 of X, where G_1 is $\alpha-$open (resp. $C-$open) and G_2 is $C-$open (resp. $\alpha-$open), there exists a contra-continuous function $h : X \to [0,1]$ such that, $h^{-1}(0) = G_1$ and $h^{-1}(1) = G_2$ then X has the strong $cc-$insertion property for $(c\alpha c, cCc)$ (resp. $(cCc, c\alpha c)$).

Proof. Since for every two disjoint subsets G_1 and G_2 of X, where G_1 is $\alpha-$open (resp. $C-$open) and G_2 is $C-$open (resp. $\alpha-$open), there exists a contra-continuous function $h : X \to [0,1]$ such that, $h^{-1}(0) = G_1$ and $h^{-1}(1) = G_2$, define $F_1 = \{x \in X : h(x) < \frac{1}{2}\}$ and $F_2 = \{x \in X : h(x) > \frac{1}{2}\}$. Then F_1 and F_2 are two disjoint closed subsets of X that contain G_1 and G_2, respectively. Hence by Corollary 3.4, X has the weak $cc-$insertion property for $(c\alpha c, cCc)$ and $(cCc, c\alpha c)$. Now, assume that g and f are functions on X such that $g \leq f, g$ is $c\alpha c$ (resp. cCc) and f is cc. Since $f - g$ is $c\alpha c$ (resp. cCc), therefore the lower cut set $A(f - g, 2^{-n}) = \{x \in X : (f - g)(x) \leq 2^{-n}\}$

is an $\alpha-$open (resp. $C-$open) subset of X. Now setting $H_n = \{x \in X : (f-g)(x) > 2^{-n}\}$ for every $n \in \mathbb{N}$, then by Lemma 3.4, H_n is an open subset of X and we have $\{x \in X : (f-g)(x) > 0\} = \bigcup_{n=1}^{\infty} H_n$ and for every $n \in \mathbb{N}, H_n$ and $A(f-g, 2^{-n})$ are disjoint subsets of X. By Lemma 3.2, H_n and $A(f-g, 2^{-n})$ can be completely separated by contra-continuous functions. Hence by Theorem 2.2, X has the strong $cc-$insertion property for $(c\alpha c, cc)$ (resp. (cCc, cc)).

By an analogous argument, we can prove that X has the strong $cc-$insertion property for (cc, cCc) (resp. $(cc, c\alpha c)$). Hence, by Theorem 2.3, X has the strong $cc-$insertion property for $(c\alpha c, cCc)$ (resp. $(cCc, c\alpha c)$).■

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