# Neimark-Sacker bifurcation control for delayed Nicholson′s equation

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**Abstract.** In this article, for delayed Nicholson′s blowflies equation, we propose a PD control nonstandard finite-difference (NSFD) scheme in which state feedback and parameter perturbation are used to control the Neimark-Sacker bifurcation. A PD control numerical algorithm is introduced to generate the Neimark-Sacker bifurcation at a desired point. Finally, numerical simulation results confirm that the control strategy is efficient in controlling the Neimark-Sacker bifurcation.

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**Key words**: PD control; nonstandard finite-difference scheme; Neimark-Sacker bifurcation; Nicholson′s blowflies equation; delay.

## 1 Introduction

The delay differential equation (DDE)

$$\frac{d}{dt}x(t) = ax(t-\tau)e^{-bx(t-\tau)} - cx(t) \tag{1.1}$$

which is one of the important ecological systems, describes the dynamics of Nicholson′s blowflies equation. Here $x(t)$ is the size of the population at time $t$, a is the maximum per capita daily egg production rate, $\frac{1}{b}$ is the size at which the population reproduces at the maximum rate, $c$ is the per capita daily adult death rate, and $\tau$ is the generation time[3], the positive equilibrium $x_* = (\frac{1}{b})\ln(\frac{a}{c})$. Equation (1.1) has been extensively studied in the literature. The majority of the results on (1.1) deal with the global attractiveness of the positive equilibrium and oscillatory behaviors of solutions [5,6].For experimental or computational purposes, it is common to discretize the continuous-time system corresponding to (1.1). It is desired that the discrete-time model is dynami-cally consistent′ with the continuous-time model. The aim of bifurcation control is to delay(advance) the onset of an inherent bifurcation, change the parameter value of an existing bifurcation point, stabilize a bifurcated solution or branch,etc.[7, 8]. In [9, 10, 11], the hybrid control strategy is used to control the bifurcation. We consider the delay differential equation(1.1) and $\ln(\frac{a}{c}) > 1$.

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Let $u(t) = x(\tau t)$ then $\frac{d}{dt}u(t) = \frac{d}{dt}x(\tau t) = \tau[ax(\tau t - \tau)e^{-bx(\tau t-\tau)} - cx(\tau t)]$ and equation(1.1) can be rewritten as $\frac{d}{dt}u(t) = a\tau u(t-1)e^{-bu(t-1)} - c\tau u(t)$ . We consider the time delay as the bifurcation parameter and $1 \leq \tau \leq n$, The critical point in the system is $u_* = (\frac{1}{b})\ln(\frac{a}{c})$ and With the change of variables $v(t) = u(t) - u_*$ we have $\frac{d}{dt}v(t) = a\tau(v(t-1)+u_*)e^{-b(v(t-1)+u_*)} - c\tau(v(t)+u_*)$.

# 2 Stabilization of NSFD PD control system

In this section, we mainly discuss the stability and bifurcation of the numerical discrete PD control system. We implement the PD control strategy[4] the equation becomes

$$\frac{d}{dt}v(t) = a\tau(v(t-1)+u_*)e^{-b(v(t-1)+u_*)} - c\tau(v(t)+u_*) + k_p v(t) + k_d\frac{d}{dt}v(t) \tag{2.1}$$

where $k_p$ is proportional control parameter and $k_d$ is derivative control parameter and $0 < k_p < 1; 0 < k_d < 1$. The differential equation $\frac{d}{dt}v(t) = -\alpha\beta c\tau v(t)$ that $\alpha = \frac{1}{1-k_d}, \beta = (1 - \frac{k_p}{c\tau})$ has the general solution $v(t) = Ae^{-\alpha\beta c\tau t}$. We consider step-size of the form $h = \frac{1}{m}$ where $m \in Z_+$. The solution can be written as $\frac{v(t+h)-v(t)}{\frac{1-e^{-\alpha\beta c\tau h}}{\alpha\beta c\tau}} = -\alpha\beta c\tau v(t)$. Employ the NSFD scheme[1] to equation (2.1) and choose the denominator function′ $\phi(h) = \frac{1-e^{-\alpha\beta c\tau h}}{\alpha\beta c\tau}$. It yields the difference equation

$$v_{n+1} = e^{-\alpha\beta c\tau h}v_n + (\frac{e^{-\alpha\beta c\tau h}-1}{\beta})u_* + (\frac{1-e^{-\alpha\beta c\tau h}}{\beta})(v_{n-m}+u_*)e^{-bv_{n-m}} \tag{2.2}$$

Introducing a new variable $V_n = (v_n, v_{n-1}, ..., v_{n-m})^T$. we can rewrite (2.2) as $V_{n+1} = F(V_n, \tau)$ where $F = (F_0, F_1, ..., F_m)^T$ and

$$F_k = \begin{cases} e^{-\alpha\beta c\tau h}v_{n-k} + (\frac{e^{-\alpha\beta c\tau h}-1}{\beta})u_* & \\ +(\frac{1-e^{-\alpha\beta c\tau h}}{\beta})(v_{n-m-k}+u_*)e^{-bv_{n-m-k}} & if k = 0 \\ v_{n-k+1} & if 1 \leqslant k \leqslant m \end{cases} \tag{2.3}$$

the linear part of $V_{n+1} = F(V_n, \tau)$ is $V_{n+1} = \Phi V_n$. Here

$$\Phi = \begin{bmatrix} e^{-\alpha\beta c\tau h} & 0 & \cdots & 0 & 0 & (\frac{1-e^{-\alpha\beta c\tau h}}{\beta})(1-bu_*) \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \tag{2.4}$$

The characteristic equation of $\Phi$ is

$$\lambda^{m+1} - e^{-\alpha\beta c\tau h}\lambda^m - (\frac{1-e^{-\alpha\beta c\tau h}}{\beta})(1-bu_*) = 0 \tag{2.5}$$

**Lemma 2.1.** *if* $\frac{k_p}{1-k_d} < \frac{\ln(1+h)}{h}$, *then all roots of (2.5) have modulus less than one for sufficiently small* $\tau > 0$.

*Proof.* For $\tau = 0$, (2.5) becomes $\lambda^{m+1} - \lambda^m = 0$. The equation has an m-fold root $\lambda = 0$ and a simple root $\lambda = 1$. Consider the root $\lambda(\tau)$ such that $|\lambda(0)| = 1$. This root is a $C^1$ function of $\tau$. For equation (2.5) we have $\frac{d|\lambda|^2}{d\tau} = \lambda\frac{d\bar{\lambda}}{d\tau} + \bar{\lambda}\frac{d\lambda}{d\tau}$ and $\frac{d|\lambda|^2}{d\tau}|_{\lambda=1,\tau=0} = \frac{\alpha hce^{\alpha hk_p} + \frac{c(1-e^{\alpha hk_p})(1-bu_*)}{k_p}}{me^{\alpha hk_p}-(m+1)} < 0$ □

A Neimark-Sacker bifurcation occurs when two roots of the characteristic equation (2.5) cross the unit circle. We have to find values of $\tau$ such that there exist roots on the unit circle. The roots on the unit circle are given by $e^{i\omega}$, $\omega \in (-\pi, \pi]$. Since we are dealing with a real polynomial, complex roots occur in complex conjugate pairs and we have only to look for $\omega \in (0, \pi]$. For $\omega \in (0, \pi]$, $e^{i\omega}$ is a root of (2.5) if and only if $e^{i\omega} - e^{-\alpha\beta c\tau h} - (\frac{1-e^{-\alpha\beta c\tau h}}{\beta})(1-bu_*)e^{-im\omega} = 0$. Separating the real and imaginary parts

$$(2.6) \qquad \begin{cases} cos\omega - e^{-\alpha\beta c\tau h} - (\frac{1-e^{-\alpha\beta c\tau h}}{\beta})(1-bu_*)cosm\omega = 0 \\ sin\omega + (\frac{1-e^{-\alpha\beta c\tau h}}{\beta})(1-bu_*)sinm\omega = 0 \end{cases}$$

We obtain

$$(2.7) \quad \omega_k = \arccos\left(\frac{1 + e^{-2(\frac{c\tau-k_p}{1-k_d})h} - \frac{(1-e^{-(\frac{c\tau-k_p}{1-k_d})h})^2}{(1-\frac{k_p}{c\tau})^2}(1-bu_*)^2}{2e^{-(\frac{c\tau-k_p}{1-k_d})h}}\right) + 2k\pi, \quad k = 0, 1, 2, ...$$

It is clear that there exists a sequence of the time delay parameters $\tau_k$ satisfying equation. (2.6) according to $\omega = \omega_k$.

**Lemma 2.2.** *For any step size h, if the following conditions are established, let* $\lambda_k(\tau) = r_k(\tau)e^{i\omega_k(\tau)}$ *be a root of equation (2.5) near* $\tau = \tau_k$ *satisfying* $r_k(\tau_k) = 1$ *and* $\omega_k(\tau_k) = \omega_k$, *then* $\frac{d{r^2}_k(\tau)}{d\tau}|_{\tau=\tau_k,\omega=\omega_k} > 0$.

$$(2.8) \qquad \begin{cases} (I)\frac{c-k_p}{mk_p(1-k_d)} > e^{\frac{cn}{m(1-k_d)}} \\ (II) 1 + e^{-2(\frac{c\tau-k_p}{1-k_d})h} < \frac{(1-e^{-(\frac{c\tau-k_p}{1-k_d})h})^2}{(1-\frac{k_p}{c\tau})^2}(1-bu_*)^2 \end{cases}$$

*Proof.* From equation (2.5) and for $\phi = \frac{-k_p}{\tau(c\tau-k_p)} + \frac{\alpha he^{-(\frac{c\tau-k_p}{1-k_d})h}}{1-e^{-(\frac{c\tau-k_p}{1-k_d})h}}$, we get

$$\frac{d{r^2}_k(\tau)}{d\tau}|_{\tau=\tau_k,\omega=\omega_k} = \frac{\frac{c}{1-k_d}e^{-2(\frac{c\tau-k_p}{1-k_d})h} + \phi me^{-2(\frac{c\tau-k_p}{1-k_d})h} + \phi(m+1)}{[me^{-(\frac{c\tau-k_p}{1-k_d})h} - (m+1)\cos\omega_k]^2 + [(m+1)sin\omega_k]^2} +$$

$$\frac{-cos\omega_k[\frac{c(m+1)h}{1-k_d}e^{-(\frac{c\tau-k_p}{1-k_d})h} + \phi me^{-(\frac{c\tau-k_p}{1-k_d})h} + \phi(m+1)e^{-(\frac{c\tau-k_p}{1-k_d})}]}{[me^{-(\frac{c\tau-k_p}{1-k_d})h} - (m+1)cos\omega_k]^2 + [(m+1)sin\omega_k]^2} > 0.$$

□

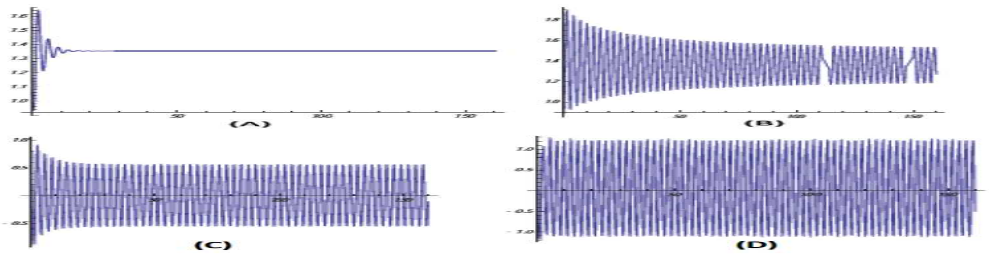


Figure 1: the numerical solution of non-controlled and controlled model corresponding to $a = 30, b = c = 2$ (A): $\tau = 0.4$, (B):$\tau = 0.8$, (C):$\tau = 0.4, k_d = 0.5, k_p = 0.1$ and (D):$\tau = 0.8, k_d = 0.5, k_p = 0.1$.

Figure 1 shows the behavior of the system before and after the control. By applying the appropriate controller, we bring the system to a stable state.

**Theorem 2.3.** *If (I),(II) are established, equation (2.2) undergoes a Neimark-Sacker bifurcation at* $u = u_*$ *when* $\tau = \tau_k$.

See references [12, 13] for more details.

# 3 Conclusions

In this paper, we have developed a controller, in view of controlling the Neimark-Sacker bifurcation in delayed Nicholson′s equation. By applying PD control, Nicholson′s blowflies equation with delay, we obtain the Hopf bifurcation. Compared with the results the PD control strategy can delay the onset of an inherent bifurcation when such a bifurcation is undesired (desired) by choosing an appropriate control parameter. For any step-size, we obtain the consistent dynamical results of the corresponding continuous time model. We obtain that the NSFD control scheme is better than the Previous controller methods. There are lots of good prospects in bifurcation and control area. In the future, we can further design a better controller to control the bifurcation of Nicholson′s blowflies equation with delay.

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