Generalized Wijsman rough Weierstrass statistical six dimensional triple geometric difference sequence spaces of fractional order defined by Musielak-Orlicz function of interval numbers

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Abstract. We generalized the concepts in probability of Wijsman rough lacunary statistical by introducing the interval numbers of Weierstrass of fractional order, where α is a proper fraction and $\gamma = (\gamma_{mnk})$ is any fixed sequence of nonzero real or complex numbers. We study some properties of this operator involving Wijsman rough lacunary sequence θ of interval numbers and arbitrary sequence $p = (p_{rst})$ of strictly positive real numbers and investigate the topological structures of related six dimensional triple geometric difference sequence spaces of interval numbers. In this study, we consider a generalization for Weierstrass rough six dimensional triple geometric difference sequence of these metric spaces by taking a ψ function, satisfying the following conditions. Let $\psi_{m,n,k}$ be a positive function for all $m, n, k \in \mathbb{N}$ such that (i) $\lim_{m,n,k\to\infty} \psi_{mnk} = 0$, (ii) $\Delta^3\psi_{mnk} = \psi_{mnk} - \psi_{m,n+1,k} - \psi_{m,n,k+1} + \psi_{m,n+1,k+1} - \psi_{m+1,n,k} + \psi_{m+1,n+1,k} + \psi_{m+1,n,k+1} - \psi_{m+1,n+1,k+1} \geq 0$. or $\psi_{mnk} = 1$. Therefore, according to class of functions which satisfying the conditions (i) and (ii) with metric spaces of six dimensional triple geometric difference sequence spaces of interval numbers defined by a Musielak-Orlicz function.

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1 Introduction

The theory of statistical convergence has been discussed in trigonometric series, summability theory, measure theory, turnpike theory, approximation theory, fuzzy set theory and so on.

The idea of rough convergence was introduced by Phu [14], who also introduced the concepts of rough limit points and roughness degree. The idea of rough convergence

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occurs very naturally in numerical analysis and has interesting applications. Aytar [15] extended the idea of rough convergence into rough statistical convergence using the notion of natural density just as usual convergence was extended to statistical convergence. Pal et al. [16] extended the notion of rough convergence using the concept of ideals which automatically extends the earlier notions of rough convergence and rough statistical convergence.

Let (X, ρ) be a metric space. For any non empty closed subsets $A, A_{mnk} \subset X\,(m, n, k \in \mathbb{N})$, we say that the triple sequence (A_{mnk}) is Wijsman statistical convergent to A is the triple sequence $(d\,(A, A_{mnk}))$ is statistically convergent to $d\,(A, A)$, i.e., for $\epsilon > 0$ and for each $A \in X$

$$\lim_{rst} \frac{1}{rst} \left|\{m \leq r, n \leq s, k \leq t : |d\,(A, A_{mnk}) - d\,(A, A)| \geq \epsilon\}\right| = 0.$$

In this case, we write $St - \lim_{mnk} A_{mnk} = A$ or $A_{mnk} \longrightarrow A\,(WS)$. The triple sequence (A_{mnk}) is bounded if $\sup_{mnk} d\,(A, A_{mnk}) < \infty$ for each $A \in X$.

In this paper, we introduce the notion of Wijsman rough statistical convergence of triple sequences. Defining the set of Wijsman rough statistical limit points of a triple sequence, we obtain to Wijsman statistical convergence criteria associated with this set. Later, we prove that this set of Wijsman statistical cluster points and the set of Wijsman rough statistical limit points of a triple sequence.

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}\,(\mathbb{C})$, where $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by *Sahiner et al.* [11, 12], *Esi et al. [1, 2, 3, 4]*, *Dutta et al.* [5], *Subramanian et al. [13]*, *Debnath et al.* [6], *Savas and Esi.* [10] and many others.

A triple sequence $x = (x_{mnk})$ is said to be triple analytic if

$$\sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The space of all triple analytic sequences are usually denoted by Λ^3. A triple sequence $x = (x_{mnk})$ is called triple chi sequence if

$$((m + n + k)!\,|x_{mnk}|)^{\frac{1}{m+n+k}} \rightarrow 0 \text{ as } m, n, k \rightarrow \infty.$$

A set of consisting of a closed interval of real numbers x such that $a \leq x \leq b$ is called an interval number. A real interval can also be considered as a set. Thus we can investigate some properties of interval numbers, for instance arithmetic properties or analytical properties. We denote the set of all real valued closed intervals by $\mathbb{R}^3$. Any elements of $\mathbb{R}^3$ is a closed interval and denoted by $\bar{A}$. That is $\bar{A} = \left\{x \in \mathbb{R}^3 : a \leq x \leq b\right\}$. An interval number $\bar{A}$ is closed subset of real numbers. Let x_r and x_s be first and last points of $\bar{A}$ interval numbers respectively. For $\bar{A}, \bar{B} \in \mathbb{R}^3$, we have

$$\bar{A} = \bar{B} \Longleftrightarrow x_{1r} = x_{2r}, x_{1s} = x_{2s}, \bar{A} + \bar{B} = \left\{x \in \mathbb{R}^3 : x_{1r} + x_{2r} \leq x \leq x_{1s} + x_{2s}\right\},$$

and if $\alpha \geq 0$, then

$$\alpha\bar{A} = \left\{x \in \mathbb{R}^3 : \alpha x_{1r} \leq x \leq \alpha x_{1s}\right\}$$

and if $\alpha < 0$, then

$$\alpha \bar{A} = \left\{ x \in \mathbb{R}^3 : \alpha x_{1r} \le x \le \alpha x_{1s} \right\},$$

$$\bar{A} \cdot \bar{B} = \{\min\{x_{1r} \cdot x_{2r}, x_{1r} \cdot x_{2s}, x_{1s} \cdot x_{2r}, x_{1s} \cdot x_{2s}\} \le x \le \max\{x_{1r} \cdot x_{2r}, x_{1r} \cdot x_{2s}, x_{1s} \cdot x_{2r}, x_{1s} \cdot x_{2s}\}\}.$$

The set of all interval numbers $\mathbb{R}^3$ is a complete metric space defined by

$$d\left(\bar{A}, \bar{B}\right) = \max\{|x_{1r} - x_{2r}|, |x_{1s} - x_{2s}|\}.$$

In the special case $\bar{A} = [a, a]$ and $\bar{B} = [b, b]$, we obtain usual metric of $\mathbb{R}^3$. Recently sequence spaces of interval numbers are studied by several authors, for example one may refer to *Esi and Catalbaş [17], Esi [18], Esi [19, 20, 21, 22, 23, 24] and Esi et al. [25, 26, 27].*

Let X and Y be two nonempty subsets of the space w of complex sequences. Let $A = \left(a^{ij\ell}_{mnk}\right), (m, n, k = 1, 2, 3, \ldots)$ be an six dimensional infinite matrix of complex numbers. We write $Ax = (A(x))$ if

$$A(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a^{ij\ell}_{mnk} x_{mnk} \tag{1.1}$$

converges. If $x = (x_{mnk}) \in X \Rightarrow Ax = (A(x)) \in Y$. We say that A defines a matrix transformation from $X \to Y$ and we denote it by $A : X \to Y$.

In the area of non-Newtonian calculus pioneering work was carried out by Grossman and Katz which we call as multiplicative calculus. The operations of multiplicative calculus are called as multiplicative derivative and multiplicative integral of different types of non-Newtonian calculi and its applications. An extension of multiplicative calculus to functions of complex variables.

Now a days geometric calculus is an alternative to the usual calculus of Newton and Leibnitz. It provides differentiation and integration tools based on multiplication instead of addition. Almost all properties in Newtonian calculus has an analog in multiplicative calculus. Generally speaking multiplicative caluculus is a methodology that allows one to have a different look at problems which can be investigated via calculus. In some cases, mainly problems of price elasticity, multiplicative growth etc. the use of multiplicative calculus is advocated instead of a traditional Newtonian one. Top know better about Non-Newtonian calculus, we must have idea about different types of arithmetics and their generators.

1.1 $\alpha-$ generator and geometric real field

A generator is a one-to-one function whose domain is $\mathbb{R}$ (the set of all real numbers) and range is a set $A \subset \mathbb{R}$. Each generator generates exactly one arithmetic and each arithmetic is generated by exactly one generator. For example, the identity function generates classical arithmetic, and exponential function generates geometric arithmetic. As a generator, we choose the function α such that whose basic algebraic operations are defined as follows:

(i) $\alpha-$ addition $x \dot{+} y = \alpha\left[\alpha^{-1}(x) + \alpha^{-1}(y)\right]$

(ii) $\alpha-$ subtraction $x \dot{-} y = \alpha\left[\alpha^{-1}(x) - \alpha^{-1}(y)\right]$

(iii) $\alpha-$ multiplication $x \dot{\times} y = \alpha\left[\alpha^{-1}(x) \times \alpha^{-1}(y)\right]$

(iv) $\alpha-$ division $x \dot{/} y = \alpha\left[\alpha^{-1}(x) / \alpha^{-1}(y)\right]$

(v) $\alpha-$ order $x \dot{<} y \Leftrightarrow \alpha^{-1}(x) < \alpha^{-1}(y)$ for $x, y \in A$, where A is a range of the function α.

If we choose exp as an $\alpha-$ generator defined by $\alpha(z) = Inz$ and $\alpha-$ arithmetic turns out to geometric arithmetic

(i) $\alpha-$ addition $x \oplus y = \alpha\left[\alpha^{-1}(x) + \alpha^{-1}(y)\right] = e^{[Inx+Iny]} = x.y$, geometric addition.

(ii) $\alpha-$ subtraction $x \ominus y = \alpha\left[\alpha^{-1}(x) - \alpha^{-1}(y)\right] = e^{[Inx-Iny]} = x \div y, y \neq 0$, geometric subtraction.

(iii) $\alpha-$ multiplication $x \odot y = \alpha\left[\alpha^{-1}(x) \times \alpha^{-1}(y)\right] = e^{[Inx \times Iny]} = x^{Iny}$, geometric multiplication.

(iv) $\alpha-$ division $x \oslash y = \alpha\left[\alpha^{-1}(x) / \alpha^{-1}(y)\right] = e^{[Inx \div Iny]} = x^{\frac{1}{Iny}}, y \neq 1$ geometric division.

Itis obvious that $In(x) < In(y)$ if $x < y$ for $x, y \in \mathbb{R}^{+}$. That is, $x < y \Leftrightarrow \alpha^{-1}(x) < \alpha^{-1}(y)$. So, with out loss of generality, we use $x < y$ instead of the geometric order $x \dot{<} y$.

Defined the sets of geometric integers, geometric real numbers and geometric complex numbers $\mathbb{Z}(G), \mathbb{R}(G)$ and $\mathbb{C}(G)$, respectively, as follows.

(i) $\mathbb{Z}(G) = \{e^{x} : x \in \mathbb{Z}\}$.

(ii) $\mathbb{R}(G) = \{e^{x} : x \in \mathbb{R}\} = \mathbb{R}^{+} \setminus \{0\}$.

(iii) $\mathbb{C}(G) = \{e^{z} : z \in \mathbb{C}\} = \mathbb{C} \setminus \{0\}$.

Remark 1.1. $(\mathbb{R}(G), \oplus, \odot)$ is a field with geometric zero 1 and geometric identity e, since

(i) $(\mathbb{R}(G), \oplus, \odot)$ is a geometric additive abelian group with geometric zero 1.

(ii) $(\mathbb{R}(G) \setminus 1, \odot)$ is a geometric multiplicative abelian group with geometric identity e.

(iii) $\odot$ is distributive over $\oplus$.

But $(\mathbb{C}(G), \oplus, \odot)$ is not a field, however, geometric binary operation $\odot$ is not associative in $\mathbb{C}(G)$.

1.2 Geometric limit

Geometric limit of a positive valued function defined in a positive interval is same to the ordinary limit. Here, we defined geometric limit of a function with the help of geometric arithmetic as follows:

A function f, which is positive in a given positive interval, is said to tend to the limit $l > 0$ as x tends to $a \in \mathbb{R}$, if, corresponding to any arbitrary chosen number $\epsilon > 1$ and r be a positive real number, however samml (but greater than 1), there exists a positive number $\delta > 1$, such that

$$1 < |f(x) \ominus l|^G < \epsilon$$

for all values of x for which $1 < |x \ominus a|^G < \delta$. We write

$$G \lim_{x \to a} f(x) = l \text{ or } f(x)\, Gl.$$

Here,

$$\begin{aligned} |x \ominus a|^G < \delta \Rightarrow & \left|\frac{x}{a}\right|^G < \delta \\ \Rightarrow & \frac{1}{\delta} < \frac{x}{a} < \delta \\ \Rightarrow & \frac{a}{\delta} < x < a\delta. \end{aligned}$$

Similarly $|f(x) \ominus l|^G < \epsilon \Rightarrow \frac{l}{\epsilon} < f(x) < l\epsilon$.

Thus, $f(x)\, Gl$. means that for any given positive real number $\epsilon > 1$, no matter however closer to 1, $\exists$ a finite number $\delta > 1$ such that $f(x)\left(\frac{l}{\epsilon}, l\epsilon\right)$ for every $x \in \left(\frac{a}{\delta}, a\delta\right)$. It is to be note that lengths of the open intervals $\left(\frac{a}{\delta}, a\delta\right)$ and $\left(\frac{l}{\epsilon}, l\epsilon\right)$ decreases as δ and ϵ respectively decreases to 1, $f(x)$ becomes closer and closer to l, as well as x becomes closer and closer to a as δ decreases to 1. Hence, l is also the ordinary limit of $f(x)$. i.e. $f(x)\, Gl \Rightarrow f(x) \to l$.

In other words, we say that $G-$ limit and ordinary limit are same for bipositive functions whose functional values as well as arguments are positive in the given interval only difference is that in geometric calculus we approach the limit geometrically, but in ordinary calculus we approach the limit linearly.

A function f is said to rough tend to limit l as x tends to a from the left, if for each $\epsilon > 1$ and r be a positive number (however small), there exists $\delta > 1$ such that $|f(x) \ominus l|^G < r + \epsilon$ when $\frac{a}{\delta} < x < a$. In symbols, we then write

$$G \lim_{x \to a} f(x) = l \text{ or } f(a-1) = l.$$

Similarly, a function f is said to rough tend to limit l as x tends to a from the right, if for each $\epsilon > 1$ (however small), there exists $\delta > 1$ such that $|f(x) \ominus l|^G < r+\epsilon$ when $a < x < a\delta$. In symbols, we then write

$$G \lim_{x \to a+} f(x) = l \text{ or } f(a+1) = l.$$

If $f(x)$ is negative valued in a given interval, it will be said to rough tend to a limit $l < 0$ if for $\epsilon > 1, \exists \delta > 1$ such that $f(x) \in \left(l\epsilon, \frac{l}{\epsilon}\right)$ whenever $x \in \left(\frac{a}{\delta}, a\delta\right)$.

1.3 Geometric continuity

A function f is said to be geometric continuous at $x = a$ if

(i) $f(a)$ i.e., the value of $f(x)$ at $x = a$, is a definite number,

(ii) the geometric-limit of the function $f(x)$ as xGa exists and is equal to $f(a)$.

Alternatively, a function f is said to rough Geometric-continuous at $x = a$, if for arbitrarily chosen $\epsilon > 1$, however small, there exists a number $\delta > 1$ such that

$$\lim_{x \to a} \frac{f(x)}{f(a)} = 1.$$

It is easy to prove that

$$w(G) = \{(x_{mnk}) : x_{mnk} \in \mathbb{R}(G) \text{ for all } m, n, k \in \mathbb{N}\}$$

is a vector space over $\mathbb{R}(G)$ with respect to the algebraic operations $\oplus$ addition and $\odot$ multiplication

$$\begin{aligned} \oplus :& w(G) \times w(G) \to w(G) \\ & (x, y) \to x \oplus y = (x_{mnk}) \oplus (y_{mnk}) = (x_{mnk} y_{mnk}) \\ \odot :& \mathbb{R}(G) \times w(G) \to w(G) \\ & (\alpha y) \to \alpha \odot y = \alpha \odot (y_{mnk}) = \left(\alpha^{Iny_{mnk}}\right), \end{aligned}$$

where $x = (x_{mnk}), y = (y_{mnk}) \in w(G)$.

2 Definitions and Preliminaries

Definition 2.1. An Orlicz function (see [7]) is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called modulus function.

Lindenstrauss and Tzafriri ([8]) used the idea of Orlicz function to construct Orlicz sequence space.

A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = \sup\{|v| u - (f_{mnk})(u) : u \geq 0\}, m, n, k = 1, 2, \ldots$$

is called the complementary function of a Musielak-Orlicz function f. For a given Musielak-Orlicz function f, (see [9]) the Musielak-Orlicz sequence space t_f is defined as follows

$$t_f = \left\{x \in w^3 : I_f(|x_{mnk}|)^{1/m+n+k} \to 0 \,\text{as}\, m, n, k \to \infty\right\},$$

where I_f is a convex modular defined by

$$I_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk}(|x_{mnk}|)^{1/m+n+k}, x = (x_{mnk}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} f_{mnk}\left(\frac{|x_{mnk}-y_{mnk}|^{1/m+n+k}}{mnk}\right).$$

3 Some new Wijsman rough six dimensional triple geometric difference sequence spaces of Weierstrass fractional order of lacunary statistical convergence

Let $\Gamma(\alpha)$ denote the Euler gamma function of a real number α. Using the definition $\Gamma(\alpha)$ can be expressed as an improper integral as follows: $\Gamma(\alpha)=\int_0^\infty e^{-x}x^{\alpha-1}dx$, where α is a proper fraction. We have defined the generalized Weierstrass fractional six dimensional triple geometric difference sequence spaces of operator

$$\Gamma_\gamma^\alpha(G,x)=\frac{e^{-\gamma\alpha}}{\alpha}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left\{a_{mnk}^{ij\ell}\left(1+\frac{\alpha}{mnk}\right)^{-1}e^{\frac{\alpha}{mnk}}\right\}_G \Delta x_{mnk},(\alpha\in\mathbb{N}), \tag{3.1}$$

where $\mathbb{N}$ is the set of complex numbers and γ denotes Euler-Mascheroni constant.

Now we determine the new classes of six dimensional triple geometric difference sequence spaces $\Gamma_\gamma^\alpha(G,x)$ as follows:

$$\Gamma_\gamma^\alpha(G,x)=\left\{x:(x_{mnk})\in w^3:\left(\Gamma_{\gamma G}^\alpha\Delta x\right)\in X\right\}, \tag{3.2}$$

where

$$\Gamma_\gamma^\alpha(G,x)=\frac{e^{-\gamma\alpha}}{\alpha}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left\{a_{mnk}^{ij\ell}\left(1+\frac{\alpha}{mnk}\right)^{-1}e^{\frac{\alpha}{mnk}}\right\}_G \Delta x_{mnk},(\alpha\in\mathbb{N})$$

and

$$X\in\chi_{f\psi}^{3\Gamma}(G,x)=\chi_{f\psi}^{3}\left(a_{mnk}^{ij\ell}\Delta_\gamma^\alpha x_{mnk}\right)=\mu_{mnk}\left(\Gamma_\gamma^\alpha G,x\right)=$$
$$\psi_{mnk}\left[f_{mnk}\left(a_{mnk}^{ij\ell}\left((m+n+k)!\left|\Gamma_\gamma^\alpha\Delta x\right|^G\right)^{\frac{1}{m+n+k}},\bar{0}\right)\right].$$

Proposition 3.1. *(i) For a proper fraction α, $\Gamma^\alpha: W\times W\times W\to W\times W\times W$ defined by equation of (3.1) is a linear operator.*

(ii) For $\alpha,\beta>0$, $\Gamma^\alpha\left(\Gamma^\beta(x_{mnk})\right)=\Gamma^{\alpha+\beta}(x_{mnk})$ and $\Gamma^\alpha\left(\Gamma^{-\alpha}(x_{mnk})\right)=x_{mnk}$.

Proof. Omitted. □

Proposition 3.2. *For a proper fraction α and Musielak-Orlicz function f, if $\chi_{f\psi}^3(G,x)$ is a linear space, then $\chi_{f\psi}^{3\Gamma_\gamma^\alpha}(G,x)$ is also a linear space.*

Proof. Omitted. □

Definition 3.1. The triple sequence $\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ is called triple lacunary if there exist three increasing sequences of integers such that

$$m_0 = 0, h_i = m_i - m_{i-1} \to \infty \text{ as } i \to \infty$$

and

$$n_0 = 0, \overline{h_\ell} = n_\ell - n_{\ell-1} \to \infty \text{ as } \ell \to \infty,$$
$$k_0 = 0, \overline{h_j} = k_j - k_{j-1} \to \infty \text{ as } j \to \infty.$$

Let $m_{i,\ell,j} = m_i n_\ell k_j, h_{i,\ell,j} = h_i \overline{h_\ell h_j}$, and $\theta_{i,\ell,j}$ is determine by

$$I_{i,\ell,j} = \{(m,n,k) : m_{i-1} < m < m_i \,\text{and}\, n_{\ell-1} < n \leq n_\ell \,\text{and}\, k_{j-1} < k \leq k_j\},$$
$$q_i = \frac{m_i}{m_{i-1}}, \overline{q_\ell} = \frac{n_\ell}{n_{\ell-1}}, \overline{q_j} = \frac{k_j}{k_{j-1}}.$$

Definition 3.2. A triple sequence $A = (A_{mnk})$ is said to be rough six dimensional triple geometric difference sequence of interval numbers of Wijsman $r-$ convergent to A denoted by $A_{mnk} \to^r A$, provided that

$$\forall \epsilon > 0\ \exists (m_\epsilon, n_\epsilon, k_\epsilon) \in \mathbb{N}^3 : m \geq m_\epsilon, n \geq n_\epsilon, k \geq k_\epsilon \Rightarrow_G \lim_{rst} \tfrac{1}{rst} \left| \{ m \leq r, n \leq s, \right.$$
$$\left. k \leq t : \psi_{mnk} a^{ij\ell}_{mnk} \left| d(A, A_{mnk}) - d(A, A) \right|^G < r + \epsilon \right\}, \psi \in \mathbb{M} \Big| = 0$$

The set

$$_G LIM^r A = \left\{ L \in \mathbb{R}^3 : A_{mnk} \to^r A \right\}$$

is called the Wijsman $r-$ limit set of the triple sequences of interval numbers.

Definition 3.3. A rough six dimensional triple geometric difference sequence of interval numbers $A = (A_{mnk})$ is said to be Wijsman $r-$ convergent if $_G LIM^r A \neq \phi$. In this case, r is called the Wijsman convergence degree of the triple sequence $A = (A_{mnk})$. For $r = 0$, we get the ordinary convergence.

Definition 3.4. A rough six dimensional triple geometric sequence of interval numbers (A_{mnk}) is said to be Wijsman $r-$ statistically convergent to A, denoted by $A_{mnk} \to^{rst} A$, provided that the set

$$_G \lim_{rst} \frac{1}{rst} \left| \left\{ (m,n,k) \in \mathbb{N}^3 : \psi_{mnk} a^{ij\ell}_{mnk} \left| d(A, A_{mnk}) - d(A, A) \right|^G \geq r + \epsilon \right\}, \psi \in \mathbb{M} \right| = 0$$

has natural density zero for every $\epsilon > 0$, or equivalently, if the condition

$$st -_G \limsup\ \psi_{mnk} a^{ij\ell}_{mnk} \left| d(A, A_{mnk}) - d(A, A) \right|^G \leq r$$

is satisfied.

In addition, we can write $A_{mnk} \to^{rst} A$ if and only if the inequality

$$_G \lim_{rst} \frac{1}{rst} \left| \left\{ m \leq r, n \leq s, k \leq t : \psi_{mnk} a^{ij\ell}_{mnk} \left| d(A, A_{mnk}) - \right. \right. \right.$$
$$\left. \left. d(A, A) \right|^G < r + \epsilon \right\} \Big| = 0$$

holds for every $\epsilon > 0$ and almost all (m,n,k). Here r is called the Wijsman roughness of degree. If we take $r = 0$, then we obtain the ordinary Wijsman statistical convergence of triple sequence.

In a similar fashion to the idea of classic Wijsman rough convergence, the idea of Wijsman rough statistical convergence of a six dimensional triple geometric sequence spaces can be interpreted as follows:

Assume that a rough six dimensional triple geometric difference sequences of interval numbers of $B = (B_{mnk})$ is Wijsman statistically convergent and cannot be measured or calculated exactly; one has to do with an approximated (or Wijsman statistically approximated) triple sequence $A = (A_{mnk})$ satisfying

$$\psi_{mnk} a^{ij\ell}_{mnk} \left| d\left(A - B, A_{mnk}\right) - d\left(A - B, A\right)\right|^G \leq r \text{ for all } m, n, k$$

(or for almost all (m,n,k)), i.e.,

$$\delta\left({}_G\lim_{rst} \frac{1}{rst}\left|\left\{m \leq r, n \leq s, k \leq t : \psi_{mnk} a^{ij\ell}_{mnk} \left| d\left(A - B, A_{mnk}\right) - \right.\right.\right.\right.$$
$$\left.\left.\left.\left. d\left(A - B, A\right)\right|^G > r\right\}\right|\right) = 0.$$

Then the rough six dimensional triple geometric difference sequences of interval numbers of x is not statistically convergent any more, but as the inclusion

$${}_G\lim_{rst} \frac{1}{rst}\left\{\psi_{mnk} a^{ij\ell}_{mnk} \left| d\left(B, A_{mnk}\right) - d\left(B, A\right)\right| \geq \epsilon\right\} \supseteq$$
$${}_G\lim_{rst} \frac{1}{rst}\left\{\psi_{mnk} a^{ij\ell}_{mnk} \left| d\left(A, A_{mnk}\right) - d\left(A, A\right)\right|^G \geq r + \epsilon\right\}$$

holds and we have

$$\delta\left({}_G\lim_{rst} \frac{1}{rst}\left|\left\{(m,n,k) \in \mathbb{N}^3 : \left|B_{mnk} - l\right|^G \geq r + \epsilon\right\}\right|\right) = 0,$$

i.e., we get

$$\delta\left({}_G\lim_{rst} \frac{1}{rst}\left|\left\{m \leq r, n \leq s, k \leq t : \psi_{mnk} a^{ij\ell}_{mnk} \left| d\left(A, A_{mnk}\right) - \right.\right.\right.\right.$$
$$\left.\left.\left.\left. d\left(A, A\right)\right|^G \geq r + \epsilon\right\}\right|\right) = 0,$$

i.e., rough six dimensional triple geometric difference sequences of interval numbers of x is Wijsman $r-$ statistically convergent in the sense of definition 3.4.

In general, the Wijsman rough statistical limit of a triple geometric difference sequences of interval numbers may not unique for the Wijsman roughness degree $r > 0$. So we have to consider the so called Wijsman $r-$ statistical limit set of a rough six dimensional triple geometric difference sequence of interval numbers of $A = (A_{mnk})$, which is defined by

$$st -_G LIM^r A_{mnk} = \left\{L \in \mathbb{R} : A_{mnk} \rightarrow^{rst} A\right\}.$$

Definition 3.5. A rough six dimensional triple geometric difference sequence of interval numbers (A_{mnk}) is said to be Wijsman $r-I$ convergent to A, if for every $\epsilon > 0$ and for each $A \in X$,

$$A(x,\epsilon) = \left\{(m,n,k) \in \mathbb{N}^3 : \psi_{mnk} a^{ij\ell}_{mnk} \left|d(A, A_{mnk}) - d(A,A)\right|^G \geq r + \epsilon\right\} \in I$$

Definition 3.6. A rough six dimensional triple geometric difference sequence of interval numbers (A_{mnk}) is said to be Wijsman $r-I$ statistical convergent to A, if for every $\epsilon > 0$, $\delta > 0$ and for each $A \in X$,

$$\left\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{rst}\left|\left\{(r,s,t) \leq (m,n,k) : \psi_{mnk} a^{ij\ell}_{mnk} \left|d(A, A_{mnk}) - \right.\right.\right.\right.$$
$$\left.\left.\left.\left. d(A,A)\right|^G \geq r + \epsilon\right\}\right| \geq \delta\right\} \in I.$$

In this case, we write $A_{mnk} \to^{s(I_W)} A$.

Definition 3.7. Let θ be a lacunary sequence. A rough six dimensional triple geometric difference sequence of interval numbers (A_{mnk}) is said to be Wijsman strongly $r-I$ convergent to A, if for every $\epsilon > 0$ and for each $A \in X$,

$$\left\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{h_{rst}} \sum_{(m,n,k) \in I_{rst}} \psi_{mnk} a^{ij\ell}_{mnk} \left|d(A, A_{mnk}) - d(A,A)\right|^G \geq r + \epsilon\right\} \in I.$$

In this case, we write $A_{mnk} \to^{N_\theta(I_W)} A$.

Definition 3.8. Let α be a proper fraction,β be nonnegative real number, f be an Musielak-Orlicz function and $\theta = \{m_r n_s k_t\}_{(rst) \in \mathbb{N} \bigcup 0}$ be the six dimensional triple geometric difference sequence of interval numbers of Wijsman rough lacunary sequence spaces of $\left(\Delta^\alpha_\gamma d(A, A_{mnk})\right)$ is said to be $\Delta^\alpha_\gamma-$ Wijsman rough lacunary statistically convergent to a number $\bar{0}$ if for any $\epsilon > 0$,

$$_G \lim_{rst \to \infty} \frac{1}{h_{rst}}\left|\left\{(m,n,k) \in I_{rst} : \psi_{mnk} a^{ij\ell}_{mnk} \left[f_{mnk} \left|\Delta^\alpha_\gamma \left(d(A, A_{mnk}) - \right.\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left. d(A,A)\right), \bar{0}\right|^G\right] \geq \beta + \epsilon\right\}\right| = 0,$$

where

$$I_{r,s,t} = \left\{(m,n,k) : m_{r-1} < m < m_r \,\text{and}\, n_{s-1} < n \leq n_s \,\text{and}\, k_{t-1} < k \leq k_t\right\},$$
$$q_r = \frac{m_r}{m_{r-1}}, \overline{q_s} = \frac{n_s}{n_{s-1}}, \overline{q_t} = \frac{k_t}{k_{t-1}}.$$

In this case write $\Delta^\alpha_\gamma X \to^{S_\theta} \Delta^\alpha_\gamma x$.

Definition 3.9. If α be a proper fraction, β be nonnegative real number,f be an Musielak-Orlicz function and $\theta = \{m_r n_s k_t\}_{(rst) \in \mathbb{N} \bigcup 0}$ be the six dimensional triple sequence of interval numbers of geometric difference sequence spaces of Wijsman rough lacunary. A number X is said to be $\Delta_\gamma^\alpha - N_\theta -$ convergent to a real number $\bar{0}$ if for every $\epsilon > 0$,

$$_G \lim_{rst\to\infty} \frac{1}{h_{rst}} \sum_{m\in I_r} \sum_{n\in I_s} \sum_{k\in I_t} \psi_{mnk} a_{mnk}^{ij\ell} \left[f_{mnk} \left| \Delta_\gamma^\alpha \left(d\left(A, A_{mnk}\right) - d\left(A, A\right)\right), \bar{0} \right|^G \right] = 0.$$

In this case we write $\Delta_\gamma^\alpha \left(A_{mnk} - A\right) \to^{N_\theta} \bar{0}$.

Definition 3.10. Let α be a proper fraction, β be nonnegative real number, f be an Musielak-Orlicz function and arbitary sequence $p = (p_{rst})$ of strictly positive real numbers. A six dimensional triple geometric difference sequence of interval numbers of random variables is said to be $\Delta_\gamma^\alpha -$ Wijsman rough lacunary statistically convergent in probability to $\Delta_\gamma^\alpha X : W^3 \to \mathbb{R}^3$ with respect to the roughness of degree β if for any $\epsilon, \delta > 0$,

$$_G \lim_{rst\to\infty} \frac{1}{h_{rst}} \left| \left\{ (m,n,k) \in I_{rst} : P\left(\left[\psi_{mnk} a_{mnk}^{ij\ell} \left(f_{mnk} \left| \Delta_\gamma^\alpha \left(d\left(A, A_{mnk}\right) - \right.\right.\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left.\left. d\left(x, A\right)\right) \right|^G \right) \right]^{p_{rst}} \geq \beta + \epsilon \right) \geq \delta \right\} \right| = 0$$

and we write $\Delta_\gamma^\alpha \left(A_{mnk} - A\right) \to_\beta^{S^P} \bar{0}$. It will be denoted by βS_θ^P.

Definition 3.11. Let α be a proper fraction, β be nonnegative real number,f be an Musielak-Orlicz function and arbitary sequence $p = (p_{rst})$ of strictly positive real numbers. A six dimensional triple geometric difference sequence spaces of interval numbers of random variables is said to be $\Delta_\gamma^\alpha -$ Wijsman rough $N_\theta -$ convergent in probability to $\Delta_\gamma^\alpha X : W^3 \to \mathbb{R}^3$ with respect to the roughness of degree β if for any $\epsilon > 0$,

$$_G \lim_{rst\to\infty} \frac{1}{h_{rst}} \sum_{m\in I_r} \sum_{n\in I_s} \sum_{k\in I_t} \left| \left\{ P\left(\left[\psi_{mnk} a_{mnk}^{ij\ell} \left(f_{mnk} \left| \Delta_\gamma^\alpha \left(d\left(A, A_{mnk}\right) - \right.\right.\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left.\left. d\left(A, A\right)\right) \right|^G \right) \right]^{p_{rst}} \geq \beta + \epsilon \right) \right\} \right| = 0,$$

and we write $\Delta_\gamma^\alpha A_{mnk} \to_\beta^{N_\theta^P} \Delta_\gamma^\alpha A$. The class of all $\beta - N_\theta -$ convergent six dimensional triple geometric difference sequence spaces of interval numbers of Wijsman rough random variables in probability will be denoted by βN_θ^P.

Definition 3.12. Let $\theta = \{m_r n_s k_t\}_{(rst)\in\mathbb{N}\bigcup 0}$ be Wijsman rough lacunary six dimensional triple geometric difference sequence spaces of Wijsman rough lacunary refinement of θ is a six dimensional triple geometric difference sequence of interval numbers of Wijsman rough lacunary sequence spaces of $\theta' = \left\{ m_r' n_s' k_t' \right\}_{(rst)\in\mathbb{N}\bigcup 0}$ satisfying $\theta = \{m_r n_s k_t\}_{(rst)\in\mathbb{N}\bigcup 0} \subset \left\{ m_r' n_s' k_t' \right\}_{(rst)\in\mathbb{N}\bigcup 0}$.

Note 3.13. *Let f be an Musielak-Orlicz function, $a_{mnk}^{ij\ell}$ be a six dimensional matrix and ψ_{mnk} be a rough triple geometric difference sequence spaces of interval numbers* *such that*

$$\left\|\chi^{3G}_{f\psi}, \left(d\left(x_1\right), d\left(x_2\right), \ldots, d\left(x_{n-1}\right)\right)\right\|_p = \psi_{mnk} a^{ij\ell}_{mnk} \left[f_{mnk}\left(\left\|\mu_{mnk}\left(X\right), \left(d\left(x_1\right), d\left(x_2\right), \ldots, d\left(x_{n-1}\right)\right)\right\|_p\right)\right],$$

where $\mu_{mnk}\left(G, X\right) = \left(\left(\left(m+n+k\right)! \left|\Gamma^{\alpha}_{\gamma}\left(d\left(A, A_{mnk}\right) - d\left(A, A\right)\right)\right|^{G}\right)^{1/m+n+k}, \bar{0}\right)$.

4 Main Results

In this section by using the operator Γ^{α}_{γ}, we introduce some new six dimensional triple difference sequence spaces involving Wijsman Weierstrass gamma function of rough lacunary statistical and arbitrary sequence $p = (p_{rst})$ of strictly positive real numbers, α be a proper fraction, β be nonnegative real number, f be an Musielak-Orlicz function, $a^{ij\ell}_{mnk}$ be six dimensional matrix and ψ_{mnk} be a rough triple geometric sequence spaces involving Wijsman Weierstrass gamma function of interval numbers, the following theorems are obtained:

Theorem 4.1. *Let $\theta = \{m_r n_s k_t\}_{(rst) \in \mathbb{N} \bigcup 0}$ be a rough six dimensional triple geometric difference lacunary statistical sequence. Then the followings are equivalent:*

(i) $\left\|\chi^{3G}_{f\psi}, \left(d\left(x_1\right), d\left(x_2\right), \ldots, d\left(x_{n-1}\right)\right)\right\|_p$ *is $\beta-$ six dimensional triple geometric difference of Wijsman Weierstrass gamma function of rough lacunary statistically convergent in probability to $\bar{0}$.*

(ii) $\left\|\left(\chi^{3G}_{f\psi}, d\left(x_2\right), \ldots, d\left(x_{n-1}\right)\right)\right\|_p$ *is $\beta - N_{\theta}$ convergent in probability to $\bar{0}$.*

Proof. (i)$\Longrightarrow$ (ii) First suppose that $\left\|\chi^{3G}_{f\psi}, \left(d\left(x_1\right), d\left(x_2\right), \ldots, d\left(x_{n-1}\right)\right)\right\|_p \rightarrow^{S^P_{\theta}}_{\beta} \bar{0}$. Then we can write

$$\frac{1}{h_{rst}} \sum_{m \in I_r} \sum_{n \in I_s} \sum_{k \in I_t} \left|\left\{P\left(\left\|\psi_{mnk} a^{ij\ell}_{mnk} \left[f_{mnk}\left(\left(\mu_{mnk} - \mu\right)\left(\Gamma^{\alpha}_{\gamma}\left(G, X\right)\right), \left(d\left(x_1\right), d\left(x_2\right), \ldots, d\left(x_{n-1}\right)\right)\right\|_p\right] \geq \beta + \epsilon\right)\right\}\right|$$

$$= \frac{1}{h_{rst}} \sum_{m \in I_r} \sum_{n \in I_s} \sum_{k \in I_t, P\left(\left\|\psi_{mnk} a^{ij\ell}_{mnk}\left[f_{mnk}\left(\left(\mu_{mnk}-\mu\right)\left(\Gamma^{\alpha}_{\gamma}(G,X)\right), \left(d(x_1), d(x_2), \ldots, d\left(x_{n-1}\right)\right)\right\|_p\right)\right] \geq \beta+\epsilon\right) \geq \frac{\delta}{2}} \left|\left\{P\left(\left\|\psi_{mnk} a^{ij\ell}_{mnk} \left[f_{mnk}\left(\left(\mu_{mnk} - \mu\right)\left(\Gamma^{\alpha}_{\gamma}\left(G, X\right)\right), \left(d\left(x_1\right), d\left(x_2\right), \ldots, d\left(x_{n-1}\right)\right)\right\|_p\right)\right] \geq \beta + \epsilon\right)\right\}\right|$$

$$+\frac{1}{h_{rst}}\sum_{m\in I_r}\sum_{n\in I_s}\sum_{k\in I_t, P\left(\left\|\psi_{mnk}a^{ij\ell}_{mnk}\left[f_{mnk}\left((\mu_{mnk}-\mu)\left(\Gamma^{\alpha}_{\gamma}(G,X)\right),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\right\|_p\right)\right]\geq\beta+\epsilon\right)<\frac{\delta}{2}}$$

$$\left|\left\{P\left(\left\|\psi_{mnk}a^{ij\ell}_{mnk}\left[f_{mnk}\left((\mu_{mnk}-\mu)\left(\Gamma^{\alpha}_{\gamma}(G,X)\right),\right)\right.\right.\right.\right.$$
$$\left.\left.\left.\left.(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|_p\right)\right]\geq\beta+\epsilon\right)\right\}\right|$$

$$\leq\frac{1}{h_{rst}}\left|\left\{P\left(\left\|\psi_{mnk}a^{ij\ell}_{mnk}\left[f_{mnk}\left((\mu_{mnk}-\mu)\left(\Delta^{\alpha}_{\gamma}(G,X)\right),\right.\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left.(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|_p\right)\right]\geq\beta+\epsilon\right)\geq\frac{\delta}{2}\right\}\right|+\frac{\delta}{2}.$$

(ii)$\Longrightarrow$ (i) Next suppose that condition (ii) holds. Then

$$\sum_{m\in I_r}\sum_{n\in I_s}\sum_{k\in I_t}\left|\left\{P\left(\left\|\psi_{mnk}a^{ij\ell}_{mnk}\left[f_{mnk}\left(((\mu_{mnk}-\mu)\left(\Gamma^{\alpha}_{\gamma}(G,X)\right),\right.\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left.(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|_p\right)\right]\geq\beta+\epsilon\right)\right\}\right|$$

$$\geq\sum_{m\in I_r}\sum_{n\in I_s}\sum_{k\in I_t}\left|\left\{P\left(\left\|\psi_{mnk}a^{ij\ell}_{mnk}\left[f_{mnk}\left(\mu_{mnk}\left(\Gamma^{\alpha}_{\gamma}X\right)-\mu\left(\Gamma^{\alpha}_{\gamma}(G,X)\right),\right.\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left.(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|_p\right)\right]\geq\beta+\epsilon\right)\geq\delta\right\}\right|$$

$$\geq\delta\left|\left\{P\left(\left\|\psi_{mnk}a^{ij\ell}_{mnk}\left[f_{mnk}\left((\mu_{mnk}-\mu)\left(\Gamma^{\alpha}_{\gamma}(G,X)\right),\right.\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left.(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|_p\right)\right]\geq\beta+\epsilon\right)\geq\delta\right\}\right|.$$

Therefore

$$\frac{1}{\delta}\frac{1}{h_{rst}}\sum_{m\in I_r}\sum_{n\in I_s}\sum_{k\in I_t}\left|\left\{P\left(\left\|\psi_{mnk}a^{ij\ell}_{mnk}\left[f_{mnk}\left(((\mu_{mnk}-\mu)\Gamma^{\alpha}_{\gamma}(G,X)\right),\right.\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left.(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|_p\right)\right]\geq\beta+\epsilon\right)\right\}\right|$$

$$\geq\frac{1}{h_{rst}}\left|\left\{P\left(\left\|\psi_{mnk}a^{ij\ell}_{mnk}\left[f_{mnk}\left(((\mu_{mnk}-\mu)\Gamma^{\alpha}_{\gamma}(G,X)\right),\right.\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left.(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|_p\right)\right]\geq\beta+\epsilon\right)\geq\delta\right\}\right|.$$

Hence $\left\|\chi^{3G}_{f\psi},(d(x_1),d(x_2),\ldots,d(x_{n-1}))\right\|_p\rightarrow^{S^P_\theta}_{\beta}\bar{0}$. □

Theorem 4.2. *If* $\left\|\chi^{3(G,X)}_{f\psi},(d(x_1),d(x_2),\ldots,d(x_{n-1}))\right\|_p\rightarrow^{S^P_\theta}_{\beta}\bar{0}$ *and* $\left\|\chi^{3(G,Y)}_{f\psi},(d(x_1),d(x_2),\ldots,d(x_{n-1}))\right\|_p\rightarrow^{S^P_\theta}_{\beta}\bar{0}$ *then* $P\left(\left|\left\{\left\|\chi^{3(G,X-Y)}_{f\psi},\bar{0}\right),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\right\|_p\right|\geq\beta+\epsilon\right)\right\}=\bar{0}.$

Proof. Consider $\left\|\chi^{3(G,X)}_{f\psi}, (d(x_1), d(x_2), \dots, d(x_{n-1}))\right\|_p \to^{S^P_\theta}_{\beta} \bar{0}$ and $\left\|\chi^{3(G,Y)}_{f\psi}, (d(x_1), d(x_2), \dots, d(x_{n-1}))\right\|_p \to^{S^P_\theta}_{\beta} \bar{0}$. Then we can write

$$\frac{1}{h_{rst}} \sum_{m\in I_r} \sum_{n\in I_s} \sum_{k\in I_t} \left|\left\{P\left(\left\|\psi_{mnk} a^{ij\ell}_{mnk}\left[f_{mnk}\left(\mu_{mnk}\left(\Gamma^\alpha_\gamma(G,X) - \mu\left(\Gamma^\alpha_\gamma(G,Y)\right)\right),\right.\right.\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left.\left.(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p\right)\right] \geq \beta+\epsilon\right)\right\}\right|$$

$$= \frac{1}{h_{rst}} \sum_{m\in I_r} \sum_{n\in I_s} \sum_{k\in I_t, P\left(\left\|\psi_{mnk}a^{ij\ell}_{mnk}\left[f_{mnk}\left(\mu_{mnk}\left(\Gamma^\alpha_\gamma(G,X)-\mu\left(\Gamma^\alpha_\gamma(G,Y)\right)\right), (d(x_1),d(x_2),\dots,d(x_{n-1}))\right\|_p\right)\right]\geq\beta+\epsilon\right)\geq\frac{\delta}{2}}$$
$$\left|\left\{P\left(\left\|\psi_{mnk} a^{ij\ell}_{mnk}\left[f_{mnk}\left(\mu_{mnk}\left(\Gamma^\alpha_\gamma(G,X) - \mu\left(\Gamma^\alpha_\gamma(G,Y)\right)\right),\right.\right.\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left.\left.(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p\right)\right] \geq \beta+\epsilon\right)\right\}\right|$$

$$+ \frac{1}{h_{rst}} \sum_{m\in I_r} \sum_{n\in I_s} \sum_{k\in I_t, P\left(\left\|\psi_{mnk}a^{ij\ell}_{mnk}\left[f_{mnk}\left(\mu_{mnk}\left(\Gamma^\alpha_\gamma(G,X)-\mu\left(\Gamma^\alpha_\gamma(G,Y)\right)\right), (d(x_1),d(x_2),\dots,d(x_{n-1}))\right\|_p\right)\right]\geq\beta+\epsilon\right)<\frac{\delta}{2}}$$
$$\left|\left\{P\left(\left\|\psi_{mnk} a^{ij\ell}_{mnk}\left[f_{mnk}\left(\mu_{mnk}\left(\Gamma^\alpha_\gamma(G,X) - \mu\left(\Delta^\alpha_\gamma X\right)\right),\ \right)\right.\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left.(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p\right)\right] \geq \beta+\epsilon\right)\right\}\right|$$

$$\leq \frac{1}{h_{rst}} \left|\left\{P\left(\left\|\psi_{mnk} a^{ij\ell}_{mnk}\left[f_{mnk}\left(\mu_{mnk}\left(\Gamma^\alpha_\gamma(G,X) - \mu\left(\Gamma^\alpha_\gamma(G,Y)\right)\right),\right.\right.\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left.\left.(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p\right)\right] \geq \beta+\epsilon\right) \geq \frac{\delta}{2}\right\}\right| + \frac{\delta}{2}.$$

Therefore

$$\frac{1}{\delta}\frac{1}{h_{rst}} \sum_{m\in I_r} \sum_{n\in I_s} \sum_{k\in I_t} \left|\left\{P\left(\left\|\psi_{mnk} a^{ij\ell}_{mnk}\left[f_{mnk}\left(\mu_{mnk}\left(\Gamma^\alpha_\gamma(G,X) - \mu\left(\Gamma^\alpha_\gamma(G,Y)\right)\right),\right.\right.\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left.\left.(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p\right)\right] \geq \beta+\epsilon\right)\right\}\right|$$

$$\geq \frac{1}{h_{rst}} \left|\left\{P\left(\left\|\psi_{mnk} a^{ij\ell}_{mnk}\left[f_{mnk}\left(\mu_{mnk}\left(\Gamma^\alpha_\gamma(G,X) - \mu\left(\Gamma^\alpha_\gamma(G,Y)\right)\right),\right.\right.\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left.\left.(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p\right)\right] \geq \beta+\epsilon\right) \geq \delta\right\}\right|.$$

Hence

$$\left\|\chi^{3(G,X-Y)}_{f\psi}\left(\Gamma^\alpha_\gamma(G,X_{mnk}) - \Gamma^\alpha_\gamma(G,Y_{mnk})\right), (d(x_1), d(x_2), \dots, d(x_{n-1}))\right\|_p \to^{S^P_\theta}_{\beta} \bar{0}.$$

□

Theorem 4.3. *Let* $\theta' = \left\{m'_r n'_s k'_t\right\}_{(rst)\in\mathbb{N}\bigcup 0}$ *be a triple Wijsman Weierstrass gamma function of rough lacunary refinement of the six dimensional triple geometric difference sequence of interval numbers of* $\theta = \{m_r n_s k_t\}_{(rst)\in\mathbb{N}\bigcup 0}$. *Let* $h_r = (m_{r-1}, m_r]$, $h_s = (n_{s-1}, n_s]$, $h_t = (k_{t-1}, h_r]$, $r, s, t = 1, 2, 3, \ldots$. *If there exists a* $\eta > 0$ *such that* $\frac{|h_{rst}|}{|I_{rst}|} > \eta$ *for every* $h_{rst} \subseteq I_{rst}$. *Then*

$$\left\|\chi^{3(G,X)}_{f\psi}, (d(x_1), d(x_2), \ldots, d(x_{n-1}))\right\|_p \rightarrow^{S^P_\theta}_{\beta} \bar{0} \Longrightarrow \left\|\chi^{3(G,X)}_{f\psi}, (d(x_1), d(x_2), \ldots, d(x_{n-1}))\right\|_p \rightarrow^{S^P_{\theta'}}_{\beta} \bar{0}.$$

Proof. Let $\left\|\chi^{3(G,X)}_{f\psi}, (d(x_1), d(x_2), \ldots, d(x_{n-1}))\right\|_p \rightarrow^{S^P_\theta}_{\beta} \bar{0}$ and $\epsilon, \delta > 0$. Therefore

$$\lim_{rst\to\infty} \frac{1}{|I_{rst}|} \left|\left\{P\left(\left\|\psi_{mnk} a^{ij\ell}_{mnk}\left[f_{mnk}\left(\mu_{mnk}\left(\Gamma^\alpha_\gamma(G,X) - \mu\left(\Gamma^\alpha_\gamma(G,X)\right)\right)\right),\right.\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left.(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p\right)\right] \geq \beta + \epsilon\right) \geq \delta\right\}\right| = 0.$$

For every h_{rst} we can find I_{rst} such that $h_{rst} \subseteq I_{rst}$. We obtain

$$\frac{1}{|h_{rst}|} \left|\left\{P\left(\left\|\psi_{mnk} a^{ij\ell}_{mnk}\left[f_{mnk}\left(\mu_{mnk}\left(\Gamma^\alpha_\gamma(G,X) - \mu\left(\Gamma^\alpha_\gamma(G,X)\right)\right)\right),\right.\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left.(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p\right)\right] \geq \beta + \epsilon\right) \geq \delta\right\}\right|$$

$$= \frac{|I_{rst}|}{|h_{rst}|}\frac{1}{|I_{rst}|} \left|\left\{P\left(\left\|\psi_{mnk} a^{ij\ell}_{mnk}\left[f_{mnk}\left(\mu_{mnk}\left(\Gamma^\alpha_\gamma(G,X) - \mu\left(\Gamma^\alpha_\gamma(G,X)\right)\right)\right),\right.\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left.(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p\right)\right] \geq \beta + \epsilon\right) \geq \delta\right\}\right|$$

$$\leq \frac{1}{\eta}\frac{1}{|I_{rst}|} \left|\left\{P\left(\left\|\psi_{mnk} a^{ij\ell}_{mnk}\left[f_{mnk}\left(\mu_{mnk}\left(\Gamma^\alpha_\gamma(G,X) - \mu\left(\Gamma^\alpha_\gamma(G,X)\right)\right)\right),\right.\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left.(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p\right)\right] \geq \beta + \epsilon\right) \geq \delta\right\}\right|.$$

□

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