Existence and exponential stability of solutions for laminated viscoelastic Timoshenko beams

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Abstract. In this paper, we consider a laminated Timoshenko beams with a viscoelastic damping. We prove well-posedness by using Faedo-Galerkin method and establish an exponential decay result by introducing a suitable Lyaponov functional.

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1 Introduction

Hansen and Spies [5] introduced a mathematical model for 2-layered beams with structural damping due to the interfacial slip. The model is as follows

$$(1.1)\qquad \begin{cases} \rho w_{tt}(x,t) + G(\psi - w_x)_x = 0, \\ \\ I_\rho(3S_{tt} - \psi_{tt}) - G(\psi - w_x) - D(3S_{xx} - \psi_{xx}) = 0, \\ \\ I_\rho S_{tt}(x,t) + G(\psi - w_x) + \frac{4}{3}\gamma S + \frac{4}{3}\beta S_t - DS_{xx} = 0. \end{cases}$$

where $w(x,t)$ is the transversal displacement, $\psi(x,t)$ denotes the rotational displacement and $S(x,t)$ is proportional to the amount of slip along the interface at time t and longitudinal spatial variable x. The coefficients $\rho, G, I_\rho, D, \gamma, \beta$ represent density, shear stiffness, mass moment of inertia, flexural rigidity, adhesive stiffness, and adhesive damping parameter, respectively.

In this paper, we consider a laminated viscoelastic Timoshenko beams. We use the notion of effective rotation angle $\xi = 3S - \psi$ in (1.1) with $\frac{4}{3}\gamma = 0$ and $\frac{4}{3}\beta = \mu$. Then adding the viscoelastic terms, we have, for $(x,t) \in (0,1) \times (0,+\infty)$

$$(1.2)\qquad \begin{cases} \rho w_{tt}(x,t) + G(3S - \xi - w_x)_x = 0, \\ \\ I_\rho \xi_{tt}(x,t) - G(3S - \xi - w_x) - D\xi_{xx} + \int_0^t \varpi_1(r)\xi_{xx}(t-r)dr = 0, \\ \\ I_\rho S_{tt}(x,t) + G(3S - \xi - w_x) - DS_{xx} + \mu S_t + \int_0^t \varpi_2(r)S_{xx}(t-r)dr = 0. \end{cases}$$

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Under the boundary conditions

$$\begin{cases} w(0,t) = \xi(0,t) = S(0,t) = 0, \quad t > 0, \\ \xi_x(1,t) = S_x(1,t) = 0, \quad t > 0, \\ 3S(1,t) - \xi(1,t) - w_x(1,t) = 0, \quad t > 0, \end{cases} \tag{1.3}$$

and the initial conditions

$$(w, \xi, S)|_{t=0} = (w_0, \xi_0, S_0), \quad (w_t, \xi_t, S_t)|_{t=0} = (w_1, \xi_1, S_1). \tag{1.4}$$

In the absence of the viscoelastic term (that is, if $\varpi_1 = \varpi_2 = 0$), with constant delay and boundary feedbacks B. Feng [2] considered the following Timoshenko system

$$\begin{cases} \rho w_{tt}(x,t) + G(3S - \xi - w_x)_x + a_1 w_t(x, t-\tau) = 0, \\ I_\rho \xi_{tt}(x,t) - G(3S - \xi - w_x) - D\xi_{xx} + a_2 \xi_t(x, t-\tau) = 0, \\ I_\rho S_{tt}(x,t) + G(3S - \xi - w_x) - DS_{xx} + a_3 S_t(x, t-\tau) = 0, \end{cases} \tag{1.5}$$

where $(x,t) \in (0, L) \times (0, \infty)$ and under the following boundary conditions

$$\begin{aligned} & G(3s(L,t) - \xi(L,t) - w_x(L,t)) = \alpha w_t(L,t), \quad t > 0, \\ & D\xi_x(L,t) = -\mu \xi_t(L,t), \quad 3DS_x(L,t) = -\nu S_t(L,t) \quad t > 0, \\ & w(0,t) = \xi(0,t) = S(0,t) = 0, \quad t > 0. \end{aligned} \tag{1.6}$$

The author proved the global well-posedness of solutions and exponential decay of energy to the system. (See [4, 3])
In [9] M. I. Mustafa considered the following system

$$\begin{cases} \rho \varphi_{tt}(x,t) + G(\psi - \varphi_x)_x = 0, \\ I_\rho (3\omega - \psi)_{tt}(x,t) - G(\psi - \varphi_x) \\ \qquad -D(3\omega - \psi)_{xx} + \int_0^t g(t-s)(3\omega - \psi)_{xx}(s)ds = 0, \\ I_\rho \omega_{tt}(x,t) + G(\psi - \varphi_x) + \dfrac{4}{3}\gamma\omega + \dfrac{4}{3}\beta\omega_t - DS_{xx} = 0, \end{cases} \tag{1.7}$$

in $(0, 1) \times (0, +\infty)$, the author proved the well-posedness and for a wider class of relaxation functions, a generalized stability result for this system is establish. (See [6, 8, 10, 11])
In the present paper, the well-posedness of the problem is analyzed in Section 3 using the Faedo-Galerkin method. In Section 4, we prove the exponential decay of the energy when time goes to infinity.

2 Preliminaries and statement of main results

In this section, we present some materials that will be used to prove our main results. We denote $V = \{v \in H^1(0,1) : v(0) = 0\}$. For the relaxation functions ϖ_1, ϖ_2, we assume

(A1) $\varpi_i \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ $(for\, i = 1,2)$ satisfying

$$\varpi_1(0) > 0, \quad 0 < \beta_1(t) := D - \int_0^t \varpi_1(r)\mathrm{d}r \quad and \quad 0 < \beta_1^0 := D - \int_0^\infty \varpi_1(r)\mathrm{d}r,$$

$$\varpi_2(0) > 0, \quad 0 < \beta_2(t) := D - \int_0^t \varpi_2(r)\mathrm{d}r \quad and \quad 0 < \beta_2^0 := D - \int_0^\infty \varpi_2(r)\mathrm{d}r.$$

(A2) There exist non-increasing functions $\chi_i(t) \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ such that

$$\varpi_i'(t) \leq -\chi_i(t)\varpi_i(t), \quad \forall t \geq 0 \quad \text{and} \quad \int_0^\infty \chi_i(t)\mathrm{d}t = +\infty, \quad for \quad i = 1,2.$$

(A3) The wave speeds are equal that is $\frac{\rho}{G} = \frac{I_\rho}{D}$

Remark 2.1. The hypotheses (A1)-(A3) imply that

$$\begin{cases} \beta_1^0 \leq \beta_1(t) \leq D, \\ \beta_2^0 \leq \beta_2(t) \leq D. \end{cases} \tag{2.1}$$

Let us introduce the following notations

$$(\varpi_i * h)(t) := \int_0^t \varpi_i(t-r)h(r)dr,$$

$$(\varpi_i \circ h)(t) := \int_0^t \varpi_i(t-r)|h(t) - h(r)|^2 dr,$$

Lemma 2.1. *For any $\varpi, h \in C^1(\mathbb{R})$, the following equation holds*

$$2[\varpi * h]h' = \varpi' \circ h - \varpi(t)|h|^2 - \frac{d}{dt}\Big(\varpi \circ h - \int_0^t \varpi(s)ds|h|^2\Big).$$

The existence and uniqueness result is stated as follows

Theorem 2.2. *Assume that $(A1-A3)$ hold. Then given $(w_0, \xi_0, S_0) \in H^2(0,1) \cap V$, $(w_1, \xi_1, S_1) \in V$, there exists a unique regular solution w, ξ, S of problem (1.2) such that*

$$(w, \xi, S) \in C([0, +\infty[, H^2(0,1) \cap V) \cap C^1([0, +\infty[, V).$$

For any regular solution of (1.2), we define the energy as

$$\begin{aligned} \mathcal{E}(t) &= \frac{1}{2}\int_0^1 \Big(\rho w_t^2(x,t) + I_\rho \xi_t^2(x,t)dx \\ &+ 3I_\rho S_t^2(x,t) + G|3S - \xi - w_x|^2 + \beta_1(t)\xi_x^2 + \beta_2(t)S_x^2\Big)dx \\ &+ \frac{1}{2}\int_0^1 (\varpi_1 \circ \xi_x)dx + \frac{3}{2}\int_0^1 (\varpi_2 \circ S_x)dx. \end{aligned} \tag{2.2}$$

Our decay result reads as follows

Theorem 2.3. *Let (w, ξ, S) be the solution of (1.2). Assume that $(A1)-(A3)$ hold. Then there exist two positive constants C and d, such that*

$$\mathcal{E}(t) \leq Ce^{-d\int_0^t \chi(r)dr}, \quad \forall t \geq 0. \tag{2.3}$$

3 Well-posedness of the problem

In this section, we will prove the existence and uniqueness of problem (1.2) by using Faedo-Galerkin method.

Proof. We divide the proof of Theorem 2.2 into two steps: the Faedo-Galerkin approximation and the energy estimates. (See [1])

Step 1 : **Faedo-Galerkin approximation.**
We construct approximations of the solution (w, ξ, S) by the Faedo-Galerkin method as follows. For $n \geq 1$, let $V_n = \text{span}\,\{v_1, \dots, v_i\}$ be a Hilbert basis of the space $H^2(0,1) \cap V$.
We choose sequences

$$(w_0^n), (w_1^n), (\xi_0^n), (\xi_1^n), (S_0^n), (S_1^n)$$

in V_n such that

$$(w_0^n, \xi_0^n, S_0^n, w_1^n, \xi_1^n, S_1^n) \to (w_0, \xi_0, S_0, w_1, \xi_1, S_1)$$

strongly in $H^2(0,1) \cap V$ as $n \to \infty$.
We search the approximate solutions

$$w^n(x,t) = \sum_{i=1}^n f_i^n(t) v_i(x),$$

$$\xi^n(x,t) = \sum_{i=1}^n h_i^n(t) v_i(x)$$

$$S^n(x,t) = \sum_{i=1}^n k_i^n(t) v_i(x)$$

to the finite dimensional Cauchy problem

$$\begin{cases} \int_0^1 \big(\rho w_{tt}^n v_i - G(3S^n - \xi^n - w_x^n) v_{i,x}\big) dx = 0, \\ \int_0^1 \big(I_\rho \xi_{tt}^n v_i - G(3S^n - \xi^n - w_x^n) v_i + D\xi_x^n v_{i,x}\big) dx \\ \quad - \int_0^1 (\varpi_1(r) * \xi_x^n) v_{i,x} dx = 0, \\ \int_0^1 \big(I_\rho S_{tt}^n v_i + G(3S^n - \xi^n - w_x^n) v_i + DS_x^n v_{i,x} - \mu S_t^n v_i\big) dx \\ \quad - \int_0^1 (\varpi_2(r) * S_x^n) v_{i,x} dx = 0, \\ (w^n(0), \xi^n(0), S^n(0)) = (w_0^n, \xi_0^n, S_0^n), \\ \quad (w_t^n(0), \xi_t^n(0), S_t^n(0)) = (w_1^n, \xi_1^n, S_1^n). \end{cases} \tag{3.1}$$

According to the standard theory of ordinary differential equations, the finite dimensional problem (3.1) has solution $f_i^n(t), h_i^n(t), k_i^n(t)$ defined on $[0,t)$. The a priori estimates that follow imply that in fact $t_n = T$.

Step 2: **Energy estimates.**
Multiplying the first, the second and the third equation of (3.1) by $\big(f_i^n(t)\big)'$, $\big(h_i^n(t)\big)'$ and $3\big(k_i^n(t)\big)'$ respectively, we obtain

$$(3.2)\qquad \begin{cases} \int_0^1 \big(w_{tt}^n w_t^n - G(3S^n - \xi^n - w_x^n) w_{xt}^n\big) dx = 0, \\ \int_0^1 \big(\xi_{tt}^n \xi_t^n - G(3S^n - \xi^n - w_x^n)\xi_t^n + D\xi_x^n \xi_{xt}^n\big) dx \\ \quad - \int_0^1 (\varpi_1(r) * \xi_x^n)\xi_{xt}^n dx = 0, \\ \int_0^1 3\big(S_{tt}^n S_t^n + 3G(3S^n - \xi^n - w_x^n)S_t^n + DS_x^n S_{xt}^n + 3\mu|S_t^n|^2\big) dx \\ \quad - \int_0^1 3(\varpi_2(r) * S_x^n) S_{xt}^n dx = 0. \end{cases}$$

Integrating (3.2) over $(0,t)$, then summing all the equations and using Lemma (2.1), we obtain

$$(3.3)\qquad \begin{aligned} &\mathcal{E}_n(t) + 3\mu \int_0^t \int_0^1 |S_t^n|^2 dx dr - \frac{1}{2}\int_0^t \int_0^1 (\varpi_1' \circ \xi_x^n) dx dr \\ &+ \frac{1}{2}\int_0^t \int_0^1 \varpi_1(t)|\xi_x^n|^2 dx ds - \frac{3}{2}\int_0^t \int_0^1 (\varpi_2' \circ S_x^n) dx dr \\ &+ \frac{3}{2}\int_0^t \int_0^1 \varpi_2(t) |S_x^n|^2 dx ds \\ &= \mathcal{E}_n(0), \end{aligned}$$

where

$$(3.4)\qquad \begin{aligned} \mathcal{E}_n(t) &= \frac{1}{2}\int_0^1 \big(\rho(w_t^n)^2(x,t) + I_\rho(\xi_t^n)^2(x,t) + dx \\ &+ 3I_\rho(S_t^n)^2(x,t) + G|3S^n - \xi^n - w_x^n|^2\big) dx \\ &+ \frac{1}{2}\int_0^1 \big(\beta_1(t)(\xi_x^n)^2 + \beta_2(t)(S_x^n)^2\big) dx \\ &+ \frac{1}{2}\int_0^1 (\varpi_1 \circ \xi_x^n) dx + \frac{3}{2}\int_0^1 (\varpi_2 \circ S_x^n) dx. \end{aligned}$$

Consequently, we have the following estimate

$$(3.5)\qquad \begin{aligned} \mathcal{E}_n(t) &- \frac{1}{2}\int_0^t \int_0^1 (\varpi_1' \circ \xi_x^n) dx dr + \frac{1}{2}\int_0^t \int_0^1 \varpi_1(t)|\xi_x^n|^2 dx ds \\ &- \frac{3}{2}\int_0^t \int_0^1 (\varpi_2' \circ S_x^n) dx dr + \frac{3}{2}\int_0^t \int_0^1 \varpi_2(t)|S_x^n|^2 dx dr \\ &\le \mathcal{E}_n(0). \end{aligned}$$

Now, since the sequences $\left(w_0^n\right)_{n\in\mathbb{N}}$, $\left(w_1^n\right)_{n\in\mathbb{N}}$, $\left(\xi_0^n\right)_{n\in\mathbb{N}}$, $\left(\xi_1^n\right)_{n\in\mathbb{N}}$, $\left(S_0^n\right)_{n\in\mathbb{N}}$, $\left(S_1^n\right)_{n\in\mathbb{N}}$ converge and using (A2), in both cases, we can find a positive constant c independent of n such that

$$\mathcal{E}_n(t) \le c. \tag{3.6}$$

Therefore, using the fact that $\beta_i(t) \ge \beta_i(0)$, the estimate (3.6) together with (3.5) give us, for all $n \in \mathbb{N}$, $t_n = T$,

$$\begin{array}{lll} \left(w^n\right)_{n\in\mathbb{N}} & \text{is bounded in} & L^\infty(0,T;V), \\ \left(\xi^n\right)_{n\in\mathbb{N}} & \text{is bounded in} & L^\infty(0,T;V), \\ \left(S^n\right)_{n\in\mathbb{N}} & \text{is bounded in} & L^\infty(0,T;V), \\ \left(w_t^n\right)_{n\in\mathbb{N}} & \text{is bounded in} & L^\infty(0,T;V), \\ \left(\xi_t^n\right)_{n\in\mathbb{N}} & \text{is bounded in} & L^\infty(0,T;V), \\ \left(S_t^n\right)_{n\in\mathbb{N}} & \text{is bounded in} & L^\infty(0,T;V). \end{array} \tag{3.7}$$

Consequently, we conclude that

$$\begin{array}{lll} w^n \rightharpoonup^* u & \text{in} & L^\infty(0,T;V), \\ \xi^n \rightharpoonup^* u & \text{in} & L^\infty(0,T;V), \\ S^n \rightharpoonup^* v & \text{in} & L^\infty(0,T;V), \\ w_t^n \rightharpoonup^* u_t & \text{in} & L^\infty(0,T;V), \\ \xi_t^n \rightharpoonup^* v_t & \text{in} & L^\infty(0,T;V), \\ S_t^n \rightharpoonup^* u_t & \text{in} & L^\infty(0,T;V). \end{array} \tag{3.8}$$

From (3.7), we have $\left(w^n\right)_{n\in\mathbb{N}}$, $\left(\xi^n\right)_{n\in\mathbb{N}}$, $\left(S^n\right)_{n\in\mathbb{N}}$ are bounded in $L^\infty(0,T)$. Then $\left(w^n\right)_{n\in\mathbb{N}}$, $\left(\xi^n\right)_{n\in\mathbb{N}}$, $\left(S^n\right)_{n\in\mathbb{N}}$ are bounded in $L^2(0,T;V)$. Consequently, $\left(w^n\right)_{n\in\mathbb{N}}$, $\left(w^n\right)_{n\in\mathbb{N}}$, $\left(S^n\right)_{n\in\mathbb{N}}$ are bounded in $H^1(0,T;V)$. Since the embedding

$$H^1(0,T;H^1(0,1)) \hookrightarrow L^2(0,T;L^2(0,1)),$$

is compact, using Aubin Lion's theorem [7], we can extract subsequences $\left(w^k\right)_{k\in\mathbb{N}}$ of $\left(w^n\right)_{n\in\mathbb{N}}$, $\left(\xi^k\right)_{k\in\mathbb{N}}$ of $\left(\xi^n\right)_{n\in\mathbb{N}}$ and $\left(S^k\right)_{k\in\mathbb{N}}$ of $\left(S^n\right)_{n\in\mathbb{N}}$ such that

$$w^k \to w \quad \text{strongly in} \quad L^2(0,T;L^2(0,1)),$$

$$\xi^k \to \xi \quad \text{strongly in} \quad L^2(0,T;L^2(0,1)),$$

and

$$S^k \to S \quad \text{strongly in} \quad L^2(0,T;L^2(0,1)).$$

Therefore,

$$w^k \to w \quad \text{strongly and a.e} \quad (0,T)\times(0,1),$$

$$\xi^k \to \xi \quad \text{strongly and a.e} \quad (0,T)\times(0,1),$$

and

$$S^k \to S \quad \text{strongly and a.e} \quad (0,T)\times(0,1),$$

The proof now can be completed arguing as in Theorem 3.1 of [7].

□

4 Exponential stability

In this section we study the asymptotic behavior of the system (1.2). For the proof of Theorem 2.3 we need a several Lemmas.

Lemma 4.1. *Let (w, ξ, S) be the solution of (1.2), then we have the inequality*

$$\begin{aligned} \frac{d\mathcal{E}(t)}{dt} &\leq -3\mu \int_0^1 |S_t(x,t)|^2\,dx - \tfrac{1}{2}\varpi_1(t)\int_\Omega |\xi_x(x,t)|^2\,dx \\ &+\tfrac{1}{2}\int_\Omega(\varpi_1' \circ \xi_x)dx - \tfrac{3}{2}\varpi_2(t)\int_\Omega |S_x(x,t)|^2\,dx + \tfrac{3}{2}\int_\Omega(\varpi_2' \circ S_x)dx \\ &\leq 0. \end{aligned} \tag{4.1}$$

Proof. Multiplying the first, the second and the third equation of (1.2) by w_t, ξ_t and $3S_t$ respectively, then summing and integrating it over $(0, t)$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \big(\rho w_t^2(x,t) + I_\rho \xi_t^2(x,t) + dx + 3I_\rho S_t^2(x,t) \\ &+G|3S-\xi-w_x|^2 + D\xi_x^2 + 3DS_x^2\big)dx \\ &= -3\mu\int_0^1 |S_t|^2 dx + \int_0^1\int_0^t \varpi_1(r)\xi_x(r)\xi_x(t)drdx \\ &+3\int_0^1\int_0^t \varpi_2(r)S_x(r)S_x(t)drdx. \end{aligned} \tag{4.2}$$

Owing to Lemma 2.1, the last term in the RHS of (4.2) can be rewritten as

$$\begin{aligned} &\int_0^t \varpi_1(r)\int_0^1 \xi_x(r)\xi_x(t)drdx + \frac{1}{2}\varpi_1(t)\int_0^1 |\xi_x|^2(x,t)dx \\ &= \frac{1}{2}\frac{d}{dt}\Big(\int_0^t \varpi_1(r)dr\int_0^1 |\xi_x|^2(x,t)dx - \int_0^1(\varpi_1\circ\xi_x)(t)dx\Big) \\ &+\frac{1}{2}\int_0^1(\varpi_1'\circ\xi_x)(t)dx, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &\int_0^t \varpi_2(r)\int_0^1 S_x(r)\xi_x(t)drdx + \frac{1}{2}\varpi_2(t)\int_0^1 |S_x|^2(x,t)dx \\ &= \frac{1}{2}\frac{d}{dt}\Big(\int_0^t \varpi_2(r)dr\int_0^1 |S_x|^2(x,t)dx - \int_0^1(\varpi_2\circ S_x)(t)dx\Big) \\ &+\frac{1}{2}\int_0^1(\varpi_2'\circ S_x)(t)dx. \end{aligned} \tag{4.4}$$

So, from (2.2), $\frac{dE}{dt}$ becomes

$$\begin{aligned} \frac{d\mathcal{E}}{dt} &= -3\mu\int_0^1 |S_t|^2dx - \tfrac{1}{2}\varpi_1(t)\int_0^1 |\xi_x|^2(x,t)dx \\ &+\tfrac{1}{2}\int_0^1(\varpi_1'\circ\xi_x)(t)dx - \tfrac{3}{2}\varpi_2(t)\int_0^1 |S_x|^2(x,t)dx + \tfrac{3}{2}\int_0^1(\varpi_2'\circ S_x)(t)dx. \end{aligned} \tag{4.5}$$

This completes the proof. □

Now, we define a functional F_1 as follows

$$F_1(t) = -\int_0^1 \left(\rho w w_t + I_\rho \xi \xi_t + 3I_\rho S S_t + \frac{3}{2}\mu S^2\right) dx. \tag{4.6}$$

Then, we have the following estimate

Lemma 4.2.

$$\begin{aligned} F_1'(t) \leq & -\int_0^1 (\rho w_t^2 + I_\rho \xi_t^2 + 3I_\rho S_t^2) dx + G\int_0^1 |3S - \xi - w_x|^2 dx \\ & + c\int_0^1 \xi_x^2 dx + c\int_0^1 S_x^2 dx + c\int_0^1 (\varpi_1 \circ \xi_x) dx + c\int_0^1 (\varpi_2 \circ S_x) dx. \end{aligned} \tag{4.7}$$

Proof. Taking the derivative of $F_1(t)$ with respect to t and using (1.2), we find that

$$\begin{aligned} F_1'(t) &= -\int_0^1 (\rho w_t^2 + I_\rho \xi_t^2 + 3I_\rho S_t^2) dx + G\int_0^1 w(3S - \xi - w_x)_x dx \\ &- \int_0^1 \xi\Big(G(3S - \xi - w_x) + D\xi_{xx} - \int_0^t \varpi_1(r)\xi_{xx}(t-r)dr\Big) dx \\ &- \int_0^1 3S\Big(DS_{xx} - G(3S - \xi - w_x) - \mu S_t - \int_0^t \varpi_2(r) S_{xx}(t-r)dr\Big) dx \\ &- 3\mu \int_0^1 S_t S dx \\ &= -\int_0^1 (\rho w_t^2 + I_\rho \xi_t^2 + 3I_\rho S_t^2) dx + G\int_0^1 |3S - \xi - w_x|^2 dx \\ &+ D\int_0^1 \xi_x^2 dx + 3D\int_0^1 S_x^2 dx \\ &- \int_0^1 \xi_x(x,t)\Big(\int_0^t \varpi_1(r)\xi_x(t-r)dr\Big) dx \\ &- 3\int_0^1 S_x(x,t)\Big(\int_0^t \varpi_2(r) S_x(t-r)dr\Big) dx. \end{aligned} \tag{4.8}$$

Using the fact that

$$\begin{aligned} & \int_0^1 \int_0^t \varpi_1(r)|\xi_x(s) - \xi_x(t)|\xi_x(x,t) dr dx \\ & \leq \delta \int_0^1 |\xi_x|^2(x,t) dx + \frac{1}{4\delta}\int_0^1 \Big(\int_0^t \varpi_1(r)|\xi_x(s) - \xi_x(t)| dr\Big)^2 dx \\ & \leq \delta \int_0^1 |\xi_x|^2(x,t) dx + \frac{D - \beta_1(t)}{4\delta}\int_0^1 (\varpi_1 \circ \xi_x)(t) d. \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} & \int_0^1 \int_0^t \varpi_2(r)|S_x(s) - S_x(t)|S_x(x,t) dr dx \\ & \leq \delta \int_0^1 |S_x|^2(x,t) dx + \frac{D - \beta_2(t)}{4\delta}\int_0^1 (\varpi_2 \circ S_x)(t) dx. \end{aligned} \tag{4.10}$$

Inserting the estimate (4.9), (4.10) into (4.8) and using Young's, Poincaré's inequalities lead to the desired estimate.
This completes the proof. $\square$

Lemma 4.3. *Let (w, ξ, S) be the solution of (1.2). Assume that*

$$\frac{\rho}{G} = \frac{I_\rho}{D}. \tag{4.11}$$

Then the functional F_2 defined by

$$\begin{aligned} F_2(t) &= -I_\rho \int_0^1 (w_t\xi_x + 3w_tS_x)dx + I_\rho \int_0^1 \xi_t(3S - \xi - w_x)\,dx \\ &+ 3I_\rho \int_0^1 S_t(3S - \xi - w_x)\,dx + \frac{\rho}{G}\int_0^1 w_t \int_0^t \varpi_1(t-r)\xi_x(r)\,dr\,dx \\ &+ \frac{3\rho}{G}\int_0^1 w_t \int_0^t \varpi_2(t-r)S_x(r)\,dr\,dx, \end{aligned} \tag{4.12}$$

satisfies the following estimate

$$\begin{aligned} F_2'(t) &\leq -\Big[(D\xi_x - \int_0^t \varpi_1(t-r)\xi_x(r)\,dr)w_x\Big]_{x=0}^{x=1} \\ &- 3\Big[(DS_x - \int_0^t \varpi_2(t-r)S_x(r)\,dr)w_x\Big]_{x=0}^{x=1} \\ &+ c\int_0^1 w_t^2dx - I_\rho\int_0^1 \xi_t^2dx + c\int_0^1 S_t^2dxdx + c\int_0^1 \xi_x^2dx + c\int_0^1 S_x^2dx \\ &- (2G - \frac{3}{2}\mu)\int_0^1 |3S - \xi - w_x|^2dx + c\int_0^1 (\varpi_1' \circ \xi_x)dx + c\int_0^1 (\varpi_2' \circ S_x)dx. \end{aligned} \tag{4.13}$$

Proof. Taking the derivative of $F_2(t)$ with respect to t, using (1.2) and some integrations by parts, we obtain

$$\begin{aligned} F_2'(t) &= -\Big[(D\xi_x - \int_0^t \varpi_1(t-r)\xi_x(r)\,dr)w_x\Big]_{x=0}^{x=1} \\ &- 3\Big[(DS_x - \int_0^t \varpi_2(t-r)S_x(r)\,dr)w_x\Big]_{x=0}^{x=1} \\ &- 2G\int_0^1 |3S - \xi - w_x|^2dx - I_\rho\int_0^1 \xi_t^2dx + 9I_\rho\int_0^1 S_t^2dx \\ &- 3\mu\int_0^1 (3S - \xi - w_x)S_tdx + \frac{\rho}{G}\varpi_1(t)\int_0^1 w_t\xi_xdx \\ &+ \frac{\rho}{G}\int_0^1 w_t\Big(\int_0^t \varpi_1'(t-r)\xi_x(t-r)dr\Big)dx + \frac{3\rho}{G}\varpi_1(t)\int_0^1 w_tS_xdx \\ &+ \frac{3\rho}{G}\int_0^1 w_t\Big(\int_0^t \varpi_2'(t-r)S_x(t-r)dr\Big)dx. \end{aligned} \tag{4.14}$$

By (4.9), (4.10) and using Young, Poincaré's inequalities lead to the desired estimate.
This completes the proof. $\square$

Lemma 4.4. *Let $m \in C^1([0,1])$ be a function satisfying $m(0) = -m(1) = 2$. Then there exists $c > 0$ such that, for any $0 < \varepsilon < 1$, the functional F_3 defined by*

$$\begin{aligned} F_3(t) &= \tfrac{1}{4\varepsilon}\int_0^1 I_\rho m(x)\xi_t\bigg(D\xi_x - \int_0^t \varpi_1(t-r)\xi_x(r)\,dr\bigg)dx + \tfrac{4\varepsilon}{G}\int_0^1 \rho m(x) w_t w_x\,dx \\ &\quad + \tfrac{3}{4\varepsilon}\int_0^1 I_\rho m(x) S_t\bigg(DS_x - \int_0^t \varpi_2(t-r)S_x(r)\,dr\bigg)dx, \end{aligned} \tag{4.15}$$

satisfies the following estimate

$$\begin{aligned} F_3'(t) \leq\ & -\frac{1}{4\varepsilon}\bigg[\bigg(D\xi_x(1,t) - \int_0^t \varpi_1(t-r)\xi_x(1,t-r)\,dr\bigg)^2 \\ & + \bigg(D\xi_x(0,t) - \int_0^t \varpi_1(t-r)\xi_x(0,t-r)\,dr\bigg)^2\bigg] \\ & - \frac{3}{4\varepsilon}\bigg[\bigg(DS_x(1,t) - \int_0^t \varpi_2(t-r)S_x(1,t-r)\,dr\bigg)^2 \\ & + \big(DS_x(0,t) - \int_0^t \varpi_2(t-r)S_x(0,t-r)\,dr\big)^2\bigg] \\ & - 4\varepsilon\big(w_x^2(1,t) + w_x^2(0,t)\big) \\ & + \frac{2\rho\varepsilon c}{G}\int_0^1 w_t^2 dx + c\int_0^1 \xi_t^2 dx + c\int_0^1 S_t^2 dx dx \\ & + c\int_0^1 \xi_x^2 dx + c\int_0^1 S_x^2 dx + Gc\int_0^1 |3S - \xi - w_x|^2 dx \\ & + c\int_0^1 (\varpi_1 \circ \xi_x)dx + c\int_0^1 (\varpi_2 \circ S_x)dx \\ & + c\int_0^1 (\varpi_1' \circ \xi_x)dx + c\int_0^1 (\varpi_2' \circ S_x)dx. \end{aligned} \tag{4.16}$$

Proof. Taking the derivative of $F_3(t)$ with respect to t, using (1.2) and some integrations by parts, we obtain

$$
\begin{aligned}
F_3'(t) \quad = \quad & \frac{1}{4\varepsilon}\left[\frac{1}{2}m(x)\left(D\xi_x(x,t)-\int_0^t \varpi_1(t-r)\xi_x(x,t-r)\,dr\right)^2\right]_{x=0}^{x=1} \\
& -\frac{1}{4\varepsilon}\int_0^1 \frac{1}{2}m'(x)\left(D\xi_x-\int_0^t \varpi_1(t-r)\xi_x(r)\,dr\right)^2 dx \\
& +\frac{G}{4\varepsilon}\int_0^1 m(x)(3S-\xi-w_x)\left(D\xi_x-\int_0^t \varpi_1(t-r)\xi_x(r)\,dr\right)dx \\
& -\frac{I_\rho D}{4\varepsilon}\int_0^1 \frac{1}{2}m'(x)\xi_t^2 dx \\
& -\frac{I_\rho}{4\varepsilon}\varpi_1(t)\int_0^1 \xi_t\xi_x dx-\frac{I_\rho}{4\varepsilon}\int_0^1 m(x)\xi_t\left(\int_0^t \varpi_1'(t-r)\xi_x(t-r)dr\right)dx \\
& +\frac{3}{4\varepsilon}\left[\frac{1}{2}m(x)\left(DS_x(x,t)-\int_0^t \varpi_2(t-r)S_x(x,t-r)\,dr\right)^2\right]_{x=0}^{x=1} \\
& -\frac{3}{4\varepsilon}\int_0^1 \frac{1}{2}m'(x)\left(DS_x-\int_0^t \varpi_2(t-r)S_x(r)\,dr\right)^2 dx \\
& -\frac{3G}{4\varepsilon}\int_0^1 m(x)(3S-\xi-w_x)\left(DS_x-\int_0^t \varpi_2(t-r)S_x(r)\,dr\right)dx \\
& -\frac{3\mu}{4\varepsilon}\int_0^1 m(x)S_t\left(DS_x-\int_0^t \varpi_2(t-r)S_x(r)\,dr\right)dx \\
& -\frac{3I_\rho D}{4\varepsilon}\int_0^1 \frac{1}{2}m'(x)S_t^2 dx-\frac{3I_\rho}{4\varepsilon}\varpi_2(t)\int_0^1 S_tS_x dx \\
& -\frac{3I_\rho}{4\varepsilon}\int_0^1 m(x)S_t\left(\int_0^t \varpi_2'(t-r)S_x(t-r)dr\right)dx \\
& -\frac{2\varepsilon\rho D}{G}\int_0^1 \frac{1}{2}m'(x)w_t^2 dx-12\varepsilon\int_0^1 \frac{1}{2}m(x)w_xS_x dx \\
& +4\varepsilon\int_0^1 \frac{1}{2}m(x)w_x\xi_x dx-4\varepsilon\int_0^1 \frac{1}{2}m'(x)w_x^2 dx+4\varepsilon\Big[-(w_x^2(1,x)+w_x^2(0,x))\Big].
\end{aligned}
\tag{4.17}
$$

Then by Young's, Poincaré's inequalities and using (4.9), (4.10) and the fact that

$$w_x^2 \le 2(3S-\xi-w_x)^2+36cS_x+c\xi_x,$$

leads to the desired estimate. □

Lemma 4.5. *Assume that (A1) hold. Then, the functional $F(t)$ defined by*

$$F(t)=F_1(t)+F_2(t)+F_3(t),$$

satisfies, along the solution, the estimate

$$
\begin{aligned}
F'(t) \leq\ & -(\rho-2c)\int_0^1 w_t^2dx-(2I_\rho-c)\int_0^1\xi_t^2dx \\
& +c-(3I_\rho-2c)\int_0^1 S_t^2dx \\
& -\{G(1-c)-\frac{3}{2}\mu\}\int_0^1|3S-\xi-w_x|^2dx+c\int_0^1\xi_x^2dx \\
& +c\int_0^1 S_x^2dx+c\int_0^1(\varpi_1\circ\xi_x)dx+c\int_0^1(\varpi_2\circ S_x)dx \\
& +c\int_0^1(\varpi_1'\circ\xi_x)dx+c\int_0^1(\varpi_2'\circ S_x)dx.
\end{aligned}
\tag{4.18}
$$

Proof. Using (4.7), (4.13), (4.16) and the fact that

$$
\begin{aligned}
& \Big[(D\xi_x-\int_0^t\varpi_1(t-r)\xi_x(r)\,dr)w_x\Big]_{x=0}^{x=1} \\
& -3\Big[(DS_x-\int_0^t\varpi_2(t-r)S_x(r)\,dr)w_x\Big]_{x=0}^{x=1} \\
& \leq\frac{1}{4\varepsilon}\Big[(D\xi_x(1,t)-\int_0^t\varpi_1(t-r)\xi_x(1,r)\,dr)^2\Big] \\
& +\frac{1}{4\varepsilon}\Big[(D\xi_x(0,t)-\int_0^t\varpi_1(t-r)\xi_x(0,r)\,dr)^2\Big] \\
& +\frac{3}{4\varepsilon}\Big[(DS_x(0,t)-\int_0^t\varpi_2(t-r)S_x(0,r)\,dr)^2\Big] \\
& +\frac{3}{4\varepsilon}\Big[(DS_x(1,t)-\int_0^t\varpi_2(t-r)S_x(1,r)\,dr)^2\Big] \\
& +4\varepsilon\Big[w_x^2(1)+w_x^2(0)\Big],
\end{aligned}
\tag{4.19}
$$

for any $0<\varepsilon<1$, we obtain (4.18).
This completes the proof. □

Next, we introduce the following functional

$$
D(t)=\int_0^1(I_\rho\xi\xi_t\psi+\rho w_t\sigma+I_\rho SS_t-\rho w_t\theta+\frac{\mu}{2}S^2)dx,
\tag{4.20}
$$

where σ,θ are the solutions of

$$
\begin{aligned}
&-\sigma_{xx}=\xi_x, \qquad \sigma(0)=\sigma(1)=0, \\
&-\theta_{xx}=S_x, \qquad \theta(0)=\theta(1)=0.
\end{aligned}
\tag{4.21}
$$

Then we have the following estimate

Lemma 4.6.

$$
\begin{aligned}
D(t) &\le c(\delta)\int_0^1 \xi_t^2 dx + \rho\delta\int_0^1 w_t^2 dx + c(\delta)\int_0^1 S_t^2 dx\\
&- \big(D - \int_0^\infty \varpi_1(r)dr - \delta\big)\int_0^1 \xi_x^2\,dx\\
&- \big(D - \int_0^\infty \varpi_2(r)dr - \delta\big)\int_0^1 S_x^2, dx\\
&+ c\int_0^1 (\varpi_1\circ\xi_x)dx + c\int_0^1(\varpi_2\circ S_x)dx.
\end{aligned} \tag{4.22}
$$

Proof. Using Equation (1.2), we find that

$$
\begin{aligned}
D'(t) &= I_\rho\int_0^1 \xi_t^2dx + \rho\int_0^1 w_t^2\sigma_t dx - \big(D - \int_0^\infty \varpi_1(r)dr\big)\int_0^1 \xi_x^2dx - G\int_0^1 \xi^2dx\\
&-G\int_0^1 \xi w_x dx + 3G\int_0^1 \xi S dx - 3G\int_0^1 \sigma S_x dx - G\int_0^1 \xi_x\sigma dx + G\int_0^1 \sigma w_{xx}dx\\
&-G\int_0^1 \sigma_t w_t dx + \int_0^1\bigg(\int_0^t \varpi_1(t-r)(\xi_x(s)-\xi_x(t))\,ds\bigg)\xi_x\,dx + I_\rho\int_0^1 S_t^2dx\\
&-\rho\int_0^1 w_t^2\theta_t dx - \big(D - \int_0^\infty \varpi_2(r)dr\big)\int_0^1 S_x^2dx - 3G\int_0^1 S^2dx + G\int_0^1 S\xi dx\\
&+G\int_0^1 S w_x dx + 3G\int_0^1 \theta S_x dx - G\int_0^1 \theta\xi_x dx - G\int_0^1 \theta w_{xx}dx\\
&+\int_0^1\bigg(\int_0^t \varpi_2(t-r)(S_x(s)-S_x(t))\,ds\bigg)S_x\,dx.
\end{aligned} \tag{4.23}
$$

Integrating by parts and (4.21), we obtain

$$
\begin{aligned}
D'(t) &= I_\rho\int_0^1 \xi_t^2dx + \rho\int_0^1 w_t^2\sigma_t dx - \big(D - \int_0^\infty \varpi_1(r)dr\big)\int_0^1 \xi_x^2dx\\
&+\int_0^1\bigg(\int_0^t \varpi_1(t-r)(\xi_x(s)-\xi_x(t))\,ds\bigg)\xi_x\,dx + I_\rho\int_0^1 S_t^2dx - \rho\int_0^1 w_t^2\theta_t dx\\
&-\big(D - \int_0^\infty \varpi_2(r)dr\big)\int_0^1 S_x^2dx + \int_0^1\bigg(\int_0^t \varpi_2(t-r)(S_x(s)-S_x(t))\,ds\bigg)S_x\,dx.
\end{aligned} \tag{4.24}
$$

Observing that, for $\delta > 0$

$$
\int_0^1\bigg(\int_0^t \varpi_1(t-r)(\xi_x(s)-\xi_x(t))\,ds\bigg)\xi_x\,dx \le c(\delta)\int_0^1(\varpi_1\circ\xi_x)dx + \delta\int_0^1 \xi_x^2dx, \tag{4.25}
$$

and

$$
\int_0^1 \sigma_t^2 dx \le \int_0^1 \sigma_{tx}^2 dx \le \int_0^1 \xi_t^2 dx. \tag{4.26}
$$

Young's inequality and (4.25), (4.26) yield then the desired result. □

Proof. (Of Theorem 2.3) We define the Lyapunov functional

$$\mathcal{L}(t) = N_1\mathcal{E}(t) + N_2F(t) + D(t), \tag{4.27}$$

where N_1 and N_2 are positive constants that will be fixed later.
Taking the derivative of (4.27) with respect to t and making use of (4.1), (4.18) and (4.23), we obtain

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) \leq\ & -\{N_2(\rho - 2c) - \rho\delta\}\int_0^1 w_t^2 dx - \{N_2(2I_\rho - c) - c(\delta)\}\int_0^1 \xi_t^2 dx \\
& -\{3N_1\mu + N_2(3I_\rho - 2c) - c(\delta)\}\int_0^1 S_t^2 dx \\
& -\{\frac{1}{2}\varpi_1(t)N_1 + (\beta_1^0 - \delta) - cN_2\}\int_0^1 \xi_x^2 dx \\
& -\{\frac{3}{2}\varpi_2(t)N_1 + (\beta_2^0 - \delta) - cN_2\}\int_0^1 S_x^2\, dx \\
& -N_2\{G(1-c) - \frac{3}{2}\mu\}\int_0^1 |3s - \xi - w_x|^2 dx \\
& +c\int_0^1 (\varpi_1 \circ \xi_x)dx + c\int_0^1 (\varpi_2 \circ S_x)dx + c\int_0^1 (\varpi_1' \circ \xi_x)dx + c\int_0^1 (\varpi_2' \circ S_x)dx.
\end{aligned} \tag{4.28}$$

At this point, we choose our constants in (4.28), carefully, such that all the coefficients in (4.28) will be negative. We first choose N_2 satisfying

$$N_2(\rho - 2c) - \rho\delta > 0,$$

and

$$N_2(2I_\rho - c) - c(\delta) > 0.$$

Then, we pick the constant $N1$ sufficiently large such that

$$3N_1\mu + N_2(3I_\rho - 2c) - c(\delta) > 0,$$

and

$$\frac{1}{2}\varpi_1(t)N_1 + (\beta_1^0 - \delta) - cN_2 > 0,$$

and

$$\frac{3}{2}\varpi_2(t)N_1 + (\beta_2^0 - \delta) - cN_2 > 0.$$

Consequently, from the above, we deduce that there exist positive constants η_1, η_2 and η_3 such that (4.28) becomes

$$\frac{d\mathcal{L}(t)}{dt} \leq -\eta_1\mathcal{E}(t) + \eta_2\int_\Omega (\varpi_1 \circ \xi_x)dx + \eta_3\int_\Omega (\varpi_2 \circ \nabla v)dx. \tag{4.29}$$

Therefore, if $\chi(t) = min\{\chi_1(t), \chi_2(t)\}, \quad \forall t \geq 0$, then using (A2) and (4.1), we get

$$\begin{aligned}
\xi(t)\mathcal{L}'(t) &\leq -\eta_1\chi(t)\mathcal{E}(t) + \eta_2\chi(t)\int_\Omega(\varpi_1 \circ \xi_x)dx + \eta_3\xi(t)\int_\Omega(\varpi_2 \circ S_x)dx \\
&\leq -\eta_1\chi(t)\mathcal{E}(t) + \eta_2\chi_1(t)\int_\Omega(\varpi_1 \circ \xi_x)dx + \eta_3\chi_2(t)\int_\Omega(\varpi_2 \circ S_x)dx \\
&\leq -\eta_1\chi(t)\mathcal{E}(t) + \eta_2\int_\Omega\int_0^t \chi_1(t-r)\varpi_1(t-r)|\xi_x(t) - \xi_x(r)|^2 drdx \\
&+ \eta_3\int_\Omega\int_0^t \chi_2(t-r)\varpi_2(t-r)|S_x(t) - S_x(r)|^2 drdx \\
&\leq -\eta_1\chi(t)\mathcal{E}(t) - \eta_2\int_\Omega\int_0^t \varpi_1'(t-r)|\xi_x(t) - \xi_x(r)|^2 drdx \\
&- \eta_3\int_\Omega\int_0^t \varpi_2'(t-r)|S_x(t) - S_x(s)|^2 drdx \\
&\leq -\eta_1\xi(t)\mathcal{E}(t) - c\mathcal{E}'(t), \quad \forall t \geq 0.
\end{aligned} \tag{4.30}$$

Which gives

$$\big(\chi(t)\mathcal{L}(t) + c\mathcal{E}(t)\big)' - \chi'(t)\mathcal{L}(t) \leq -\eta_1\chi(t)\mathcal{E}(t).$$

Using the fact that $\chi'(t) \leq 0, \quad \forall t \geq 0$ and letting

$$J(t) = \chi(t)\mathcal{L}(t) + c\mathcal{E}(t) \sim \mathcal{E}(t), \tag{4.31}$$

we obtain

$$J'(t) \leq -\eta_1\chi(t)\mathcal{E}(t) \leq -\eta_3\chi(t)J(t). \tag{4.32}$$

A simple integration of (4.32) over $(0, t)$ leads to

$$J(t) \leq J(0)e^{-\eta_3\int_0^t \chi(r)dr} \quad \forall t \geq 0. \tag{4.33}$$

A combination of (4.31) and (4.33) leads to (2.3).
This completes the proof. □

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