Existence of positive weak solutions for sublinear Kirchhoff parabolic systems with multiple parameters

Mohammed Said Touati Brahim, Salah Boulaaras, Rafik Guefaifia and Tarek Alloush

Abstract. In this paper, we study of the existence of weak positive solutions for a sublinear Kirchhoff parabolic systems in bounded domains via sub-super solutions method combined with comparison principle

M.S.C. 2010: 35J60; 35B30, 35B40.
Key words: Kirchhoff parabolic systems; existence; positive solutions; sub-supersolution; multiple parameters.

1 Introduction

In this paper, we study of the existence of weak positive solutions for the following sublinear Kirchhoff parabolic systems

$$(1.1)\qquad \begin{cases} -A\left(\int\limits_{\Omega} |\nabla u|^2\,dx\right)\triangle u + u_t = \lambda_1 u^a + \mu_1 v^b \text{ in } Q_T = \Omega\times[0,T]\,, \\ -B\left(\int\limits_{\Omega} |\nabla u|^2\,dx\right)\triangle v + v_t = \lambda_2 u^c + \mu_2 v^d \text{ in } Q_T = \Omega\times[0,T]\,, \\ u = v = 0 \text{ on } \partial Q_T, \\ u(x,0) = \varphi(x)\,, \end{cases}$$

where $\Omega \subset \mathbb{R}^N \quad (N \geq 3)$ is a bounded smooth domain with C^2 boundary $\partial\Omega$, and A, B :$\mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions, $\lambda_1, \lambda_2, \mu_1$, and μ_2 are positive parameters, where $a + c < 1$ and $b + d < 1$. The peculiarity of this type of problem, and by far the most important, is that it is not local. This is based on the presence of the operator $-A\left(\int\limits_{\Omega} |\nabla u|^2\,dx\right)\triangle u$ (resp. $-B\left(\int\limits_{\Omega} |\nabla u|^2\,dx\right)\triangle u$), which contains an integral on all the field, implies that the equation is not a specific identity. It is clear that these problems contribute to the transition from academia to application. Indeed, very popular for its physical motivations, the problem (1.1) is none other than

Applied Sciences, Vol. 22, 2020, pp. 52-65.

a stationary version of the model which regulates the behavior of elastic whose ends are fixed and which is subjected to non-linear vibrations

$$\begin{cases} u_{tt} - M\left(\int\limits_{\Omega} |\nabla u|^2 \, dx\right) \triangle u = h\left(x, u\right), \text{ in } \Omega \times (0, T), \\ u = 0, \text{in } \ \partial\Omega \times (0, T), \\ u\left(x, 0\right) = u_0\left(x\right), \ u_t\left(x, 0\right) = u_1\left(x\right), \end{cases}$$

where T is a positive constant, u_0, u_1 are given functions. In such problems, u expresses the displacement, $h(x, u)$ the extreme force, $M(r) = a_1 r + b_1$, b_1 the initial tension, and a_1 relates to the intrinsic properties of the wire material (such as the Young's modulus). For more details, see [21], as well as their references. Basically, this is a generalization to larger dimensions of the model originally proposed in one dimension by Kirchhoff [16] in (1883)

$$\frac{\partial^2 u}{\partial t^2} - \left(\rho_0 + \rho_1 \int\limits_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right) \frac{\partial u}{\partial x} = 0, \tag{1.2}$$

where ρ_0 is the initial tension, ρ_1 represents the Young's modulus of the material of the wire and L its length. The latter is known to be an extension of the equation of D'Alembert waves.

By using Euler time scheme on (1), we obtain the following problems

$$\begin{cases} u_k - \tau' A\left(\int\limits_{\Omega} |\nabla u_k|^2 \, dx\right) \triangle u_k = \tau' \left[\lambda_1 u_k^a + \mu_1 v_k^b\right] + u_{k-1} \text{ in } \Omega, \\ v_k - \tau' B\left(\int\limits_{\Omega} |\nabla u|^2 \, dx\right) \triangle v = \tau' \left[\lambda_2 u_k^c + \mu_2 v_k^d \ \right] + v_{k-1} \text{in } \Omega, \\ u_k = v_k = 0 \text{ on } \partial\Omega, \\ u_0 = \varsigma, \end{cases}$$

where $N\tau' = T$, $0 < \tau' < 1$, and for $1 \leq k \leq N$.

Indeed, Kirchhoff took into account the changes caused by transverse oscillations along the length of the wire. With their implications in other disciplines, and given the breadth of their fields of application, non-local problems will be used to model several physical phenomena, they also intervene in biological systems or describe a process dependent on its average, such as particle density. population. Moreover, With this significant impact strengthening the field of applications, this type of problem has caught the interest of mathematicians and a lot of work on the existence of solutions has emerged. Particularly after the coup de force provided by the famous Lions article [21], where the latter has adopted an approach based on functional analysis. Nevertheless, in most of these articles, the benefit method is purely topological. It is only in the last decades that this approach has been removed from variational methods when Alves and his colleagues ([1]) obtained for the first time the results of their existence through these methods. Since then, a very fruitful development has given rise to many works based on this advantageous axis, see ([21]).

Motivated by the ideas of [15], which the authors considered a system (1.1) in the case $A(t) = B(t) = 1$. More precisely, under suitable conditions on f, g, we shall show that system (1.1) has a positive solution for $\lambda > \lambda^*$ large enough. In current paper, motivated by previous works in ([5] and [15]), we discuss the existence of weak positive solution for sublinear Kirchhoff elliptic systems in bounded domains by using subs-upersolutions method combined with comparison principle see (Lemma 2.1 in [1]).

The outline of the paper is as follows. In the second section, we give some assumptions and definitions related to problem (1.1). In section 3, we prove our main result.

2 Assumptions and definitions

Let us assume the following assumption:

$(H1)$ Assume that $A, B : \mathbb{R}^+ \to \mathbb{R}^+$ are two continuous and increasing functions and there exists $a_i, b_i > 0, i = 1, 2$, such that

$$a_1 \leq A(t) \leq a_2,\ b_1 \leq B(t) \leq b_2 \text{ for all } t \in \mathbb{R}^+,$$

$(H2)$ Suppose that $a, d \geq 0$, b, $c > 0$, $a + c < 1$ and $b + d < 1$.

Now, in order to discuss our main result of problem (1.1), we need the following two definitions:

Definition 2.1. Let $(u_k, v_k) \in \left(H_0^1(\Omega) \times H_0^1(\Omega)\right)$, (u_k, v_k) is said a weak solution of (1.1) if it satisfies

$$A\left(\|u_k\|^2\right) \int_\Omega \nabla u_k \nabla \phi dx = \int_\Omega \left[\lambda_1 u_k^a + \mu_1 v_k^b - \tfrac{u_k - u_{k-1}}{\tau'}\right] \phi dx \text{ in } \Omega,$$

$$B\left(\|v_k\|^2\right) \int_\Omega \nabla v_k \nabla \psi dx = \int_\Omega \left[\lambda_2 u_k^c + \mu_2 v_k^d - \tfrac{v_k - v_{k-1}}{\tau'}\right] \psi dx \text{ in } \Omega$$

for all $(\phi, \psi) \in \left(H_0^1(\Omega) \times H_0^1(\Omega)\right)$.

Definition 2.2. A pair of nonnegative functions $(\underline{u}_k, \underline{v}_k), (\overline{u}_k, \overline{v}_k)$ in $\left(H_0^1(\Omega) \times H_0^1(\Omega)\right)$ are called a weak subsolution and supersolution of (1.1) if they satisfy $(\underline{u}_k, \underline{v}_k) = (\overline{u}_k, \overline{v}_k) = (0, 0)$ on $\partial\Omega$

$$A\left(\|u_k\|^2\right) \int_\Omega \nabla u_k \nabla \phi dx \leq \int_\Omega \left[\lambda_1 u_k^a + \mu_1 v_k^b - \tfrac{u_k - u_{k-1}}{\tau'}\right] \phi dx \text{ in } \Omega,$$

$$B\left(\|v_k\|^2\right) \int_\Omega \nabla v_k \nabla \psi dx \leq \int_\Omega \left[\lambda_2 u_k^c + \mu_2 v_k^d - \tfrac{v_k - v_{k-1}}{\tau'}\right] \psi dx \text{ in } \Omega$$

and

$$A\left(\|u_k\|^2\right)\int_{\Omega}\nabla u_k\nabla\phi dx\geq\int_{\Omega}\left[\lambda_1 u_k^a+\mu_1 v_k^b-\tfrac{u_k-u_{k-1}}{\tau'}\right]\phi dx \text{ in } \Omega,$$

$$B\left(\|v_k\|^2\right)\int_{\Omega}\nabla v_k\nabla\psi dx\geq\int_{\Omega}\left[\lambda_2 u_k^c+\mu_2 v_k^d-\tfrac{v_k-v_{k-1}}{\tau'}\right]\psi dx \text{ in } \Omega$$

for all $(\phi,\psi)\in\left(H_0^1(\Omega)\times H_0^1(\Omega)\right)$.

Lemma 2.1. *([1])Assume that $M:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous and nonincreasing function satisfying*

$$M(s)>m_0,\ \text{for all } s\geq s_0, \tag{2.1}$$

where m_0 is a positive constant and assume that u,v are two non-negative functions such that

$$\begin{cases}-M\left(\|u\|^2\right)\triangle u\geq -M\left(\|v\|^2\right)\triangle v \ \text{in } \Omega,\\ \\ u=v=0 \ \text{on } \partial\Omega,\end{cases} \tag{2.2}$$

then $u\geq v$ a.e. in Ω.

3 Main Result

In this section, we shall state and prove the main result of this paper.

Theorem 3.1. *Suppose that $(H1)-(H2)$hold, and M is a nonincreasing function satisfying (2.1). Then problem (1.1) has a large positive weak solution for each positive parameters $\lambda_1,\lambda_2,\mu_1,$ and μ_2.*

Proof of Theorem 1. Let σ be the first eigenvalue of $-\triangle$ with Dirichlet boundary conditions and ϕ_1 the corresponding eigenfunction with $\|\phi_1\|=1$.satisfying

$\phi_1>0$ in Ω and $|\nabla\phi_1|>0$ on $\partial\Omega$.

Since $bc<(1-a)(1-d)$, we can take k such that

$$\frac{c}{1-d}<\rho<\frac{b}{1-a}. \tag{2.4}$$

We shall verify that $(\underline{u}_k,\underline{v}_k)=\left(\varepsilon\phi_1^2,\varepsilon^{\rho}\phi_1^2\right)$ is a subsolution of problem (1.1), where $\varepsilon>0$ is small and specified later.

A simple calculation

$$\begin{aligned}
A\left(\|\underline{u}_k\|^2\right)\int_\Omega \nabla\underline{u}_k.\nabla\phi dx &= 2\varepsilon A\left(\|\underline{u}_k\|^2\right)\int_\Omega \phi_1\nabla\phi_1.\nabla\phi dx \\
&= 2\varepsilon A\left(\|\underline{u}_k\|^2\right)\times \\
&\quad \left\{\int_\Omega \nabla\phi_1\nabla\left(\phi_1.\phi\right)dx - \int_\Omega \left|\nabla\phi_1\right|^2\phi dx\right\} \\
&= 2\varepsilon A\left(\|\underline{u}_k\|^2\right)\int_\Omega \left(\sigma\phi_1^2 - \left|\nabla\phi_1\right|^2\right)\phi dx \\
&\leq 2a_2\varepsilon\int_\Omega \left(\sigma\phi_1^2 - \left|\nabla\phi_1\right|^2\right)\phi dx.
\end{aligned}$$

Similarly,

$$\begin{aligned}
B\left(\|\underline{v}_k\|^2\right)\int_\Omega \nabla\underline{v}_k.\nabla\psi dx &= 2\varepsilon^\rho B\left(\|\underline{v}_k\|^2\right)\int_\Omega \left(\sigma\phi_1^2 - \left|\nabla\phi_1\right|^2\right)\phi dx \\
&\leq 2b_2\varepsilon^\rho\int_\Omega \left(\sigma\phi_1^2 - \left|\nabla\phi_1\right|^2\right)\phi dx.
\end{aligned}$$

Let $\eta > 0$, $\mu > 0$ be such that

$$\sigma\phi_1^2 - \left|\nabla\phi_1\right|^2 \leq 0,\ x\in\overline{\Omega}_\eta, \tag{2.5}$$

and $\mu \leq \phi_1 \leq 1$ on $\Omega\backslash\overline{\Omega}_\eta$ where $\overline{\Omega}_\eta = \{x\in\Omega : d\left(x,\partial\Omega\right)\leq\eta\}$.
We have from (2.5) that

$$A\left(\int_{\overline{\Omega}_\eta}\left|\nabla\underline{u}_k\right|^2 dx\right)\int_{\overline{\Omega}_\eta}\nabla\underline{u}_k.\nabla\phi dx \leq 0 \leq \int_\Omega\left[\lambda_1 u_k^a + \mu_1 v_k^b - \frac{u_k - u_{k-1}}{\tau'}\right]\phi dx, \tag{2.6}$$

and

$$B\left(\int_{\overline{\Omega}_\eta}\left|\nabla\underline{v}_k\right|^2 dx\right)\int_{\overline{\Omega}_\eta}\nabla\underline{v}_k.\nabla\psi dx \leq 0 \leq \int_\Omega\left[\lambda_2 u_k^c + \mu_2 v_k^d - \frac{v_k - v_{k-1}}{\tau'}\right]\psi dx. \tag{2.7}$$

On the other hand, in $\Omega\backslash\overline{\Omega}_\eta$, let

$$\begin{aligned}
r_1 &= \frac{1-a}{c},\ r_2 = \frac{1-a}{1-a-c}, \\
s_1 &= \frac{1-d}{b},\ \ s_2 = \frac{1-d}{1-d-b}.
\end{aligned}$$

Note that

$$\frac{1}{r_1}+\frac{1}{r_2}=1,\ \frac{1}{s_1}+\frac{1}{s_2}=1.$$

We have from (2.4) that

$$\begin{aligned} 1-\frac{a}{r_1}-\frac{kb}{r_2} &\geq 1-a-kb>0, \\ k\left(1-\frac{d}{s_2}\right)-\frac{c}{s_1} &\geq k(1-d)-c>0. \end{aligned}$$

Thus we choose $\varepsilon>0$ such that

$$\begin{aligned} 2a_2\varepsilon^{1-\frac{a}{r_1}-\frac{kb}{r_2}}\sigma\phi_1^2 &\leq \lambda_1^{\frac{1}{r_1}}\mu_1^{\frac{1}{r_2}}\mu^{2+a\delta},\ x\in\Omega\backslash\overline{\Omega}_\eta, \\ 2b_2\varepsilon^{\rho\left(1-\frac{d}{s_2}\right)-\frac{c}{s_1}}\sigma\phi_1^2 &\leq \lambda_2^{\frac{1}{s_1}}\mu_2^{\frac{1}{s_2}}\mu^{2+\gamma d},\ x\in\Omega\backslash\overline{\Omega}_\eta, \end{aligned}$$

where $\delta=\frac{2}{1-a}$, $\gamma=\frac{2}{1-d}$. Furthermore

$$\begin{aligned} a\delta r_1 &= \frac{2a}{1-a-c}\geq 2a, \\ \gamma d s_2 &= \frac{2d}{1-d-b}\geq 2d \end{aligned}$$

and

$$\begin{aligned} 2s_1 &= 2\left(\frac{1-d}{b}\right)>2\left(\frac{c}{1-a}\right)\geq 2c, \\ 2r_2 &= 2\left(\frac{1-a}{c}\right)>2\left(\frac{b}{1-d}\right)\geq 2b. \end{aligned}$$

These relations and Young inequality show that

$$\begin{aligned} &2a_2\varepsilon\int_{\Omega\backslash\overline{\Omega}_\eta}\left(\sigma\phi_1^2-|\nabla\phi_1|^2\right)\phi dx\leq 2a_2\varepsilon\int_{\Omega\backslash\overline{\Omega}_\eta}\sigma\phi_1^2.\phi dx \\ &\leq\int_{\Omega\backslash\overline{\Omega}_\eta}\left(\lambda_1^{\frac{1}{r_1}}\varepsilon^{\frac{a}{r_1}}\mu^{a\delta}\right)\left(\mu_1^{\frac{1}{r_2}}\varepsilon^{\frac{\rho b}{r_2}}\mu^2\right)\phi dx \\ &\leq\int_{\Omega\backslash\overline{\Omega}_\eta}\left[\frac{\left(\lambda_1^{\frac{1}{r_1}}\varepsilon^{\frac{a}{r_1}}\mu^{a\delta}\right)^{r_1}}{r_1}+\frac{\left(\mu_1^{\frac{1}{r_2}}\varepsilon^{\frac{\rho b}{r_2}}\mu^2\right)^{r_2}}{r_2}\right]\phi dx \end{aligned} \tag{2.8}$$

$$
\begin{aligned}
&\leq \int_{\Omega\setminus\overline{\Omega}_\eta} \left[\left(\lambda_1^{\frac{1}{r_1}}\varepsilon^{\frac{a}{r_1}}\mu^{a\delta}\right)^{r_1} + \left(\mu_1^{\frac{1}{r_2}}\varepsilon^{\frac{\rho b}{r_2}}\mu^{2}\right)^{r_2}\right]\phi dx \\
&= \int_{\Omega\setminus\overline{\Omega}_\eta} \left(\lambda_1\varepsilon^{a}\mu^{a\delta r_1} + \mu_1\varepsilon^{\rho b}\mu^{2r_2}\right)\phi dx \\
&\leq \int_{\Omega\setminus\overline{\Omega}_\eta} \left(\lambda_1\varepsilon^{a}\phi_1^{2a} + \mu_1\varepsilon^{\rho b}\phi_1^{2b}\right)\phi dx \\
&= \int_{\Omega\setminus\overline{\Omega}_\eta} \left(\lambda_1\underline{u}_k^{a} + \mu_1\underline{v}_k^{b}\right)\phi\ dx \\
&\leq \int_{\Omega} \left[\lambda_1 u_k^{a} + \mu_1 v_k^{b} - \frac{u_k - u_{k-1}}{\tau'}\right]\phi dx \qquad (3.1)
\end{aligned}
$$

and

$$
\begin{aligned}
&2b_2\varepsilon^{\rho}\int_{\Omega\setminus\overline{\Omega}_\eta}\left(\sigma\phi_1^2 - |\nabla\phi_1|^2\right)\psi\ dx \leq 2b_2\varepsilon^{\rho}\int_{\Omega\setminus\overline{\Omega}_\eta}\sigma\phi_1^2.\psi\ dx \\
&\leq \int_{\Omega\setminus\overline{\Omega}_\eta}\left(\lambda_2^{\frac{1}{s_1}}\varepsilon^{\frac{c}{s_1}}\mu^{2}\right)\left(\mu_2^{\frac{1}{s_2}}\varepsilon^{\frac{\rho d}{s_2}}\mu^{\gamma d}\right)\psi\ dx \qquad (2.9) \\
&\leq \int_{\Omega\setminus\overline{\Omega}_\eta}\left[\frac{\left(\lambda_2^{\frac{1}{s_1}}\varepsilon^{\frac{c}{s_1}}\mu^{2}\right)^{s_1}}{s_1} + \frac{\left(\mu_2^{\frac{1}{s_2}}\varepsilon^{\frac{\rho d}{s_2}}\mu^{\gamma d}\right)^{s_2}}{s_2}\right]\psi\ dx
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_{\Omega\setminus\overline{\Omega}_\eta} \left[\left(\lambda_2^{\frac{1}{s_1}}\varepsilon^{\frac{c}{s_1}}\mu^{2}\right)^{s_1} + \left(\mu_2^{\frac{1}{s_2}}\varepsilon^{\frac{\rho d}{s_2}}\mu^{\gamma d}\right)^{s_2}\right]\psi\ dx \\
&= \int_{\Omega\setminus\overline{\Omega}_\eta} \left(\lambda_2\varepsilon^{c}\mu^{2s_1} + \mu_2\varepsilon^{\rho d}\mu^{\gamma d s_2}\right)\psi\ dx \\
&\leq \int_{\Omega\setminus\overline{\Omega}_\eta} \left(\lambda_2\varepsilon^{c}\mu^{2c} + \mu_2\varepsilon^{\rho d}\mu^{2d}\right)\psi\ dx \\
&\leq \int_{\Omega\setminus\overline{\Omega}_\eta} \left(\lambda_2\varepsilon^{c}\phi_1^{2c} + \mu_2\varepsilon^{\rho d}\phi_1^{2d}\right)\psi\ dx \\
&= \int_{\Omega\setminus\overline{\Omega}_\eta} \left(\lambda_2\underline{u}_k^{c} + \ \mu_2\underline{v}_k^{d}\right)\psi\ dx \\
&\leq \int_{\Omega} \left[\lambda_2 u_k^{c} + \mu_2 v_k^{d} - \frac{v_k - v_{k-1}}{\tau'}\right]\psi dx. \qquad (3.2)
\end{aligned}
$$

Hence from $(2.6), (2.7), (2.8)$ and (2.9), it follows that

$$
\begin{aligned}
&A\left(\int_\Omega |\nabla \underline{u}_k|^2\, dx\right)\left[\int_{\Omega_\eta} \nabla \underline{u}_k \nabla\phi dx + \int_{\Omega\setminus\overline{\Omega}_\eta} \nabla \underline{u}_k \nabla\phi dx\right] \\
&= A\left(\int_\Omega |\nabla \underline{u}_k|^2\, dx\right)\int_\Omega \nabla \underline{u}_k \nabla\phi dx \le \int_\Omega \left[\lambda_1 u_k^a + \mu_1 v_k^b - \tfrac{u_k - u_{k-1}}{\tau'}\right]\phi dx,
\end{aligned}
\tag{2.10}
$$

and

$$
\begin{aligned}
&B\left(\int_\Omega |\nabla \underline{v}_k|^2\, dx\right)\left[\int_{\Omega_\eta} \nabla \underline{v}_k \nabla\psi dx + \int_{\Omega\setminus\overline{\Omega}_\eta} \nabla \underline{v}_k \nabla\psi dx\right] \\
&= B\left(\int_\Omega |\nabla \underline{v}_k|^2\, dx\right)\int_\Omega \nabla \underline{v}_k \nabla\psi dx \le \int_\Omega \left[\lambda_2 u_k^c + \mu_2 v_k^d - \tfrac{v_k - v_{k-1}}{\tau'}\right]\psi dx..
\end{aligned}
\tag{2.11}
$$

Then, by (2.10) and (2.11), $(\underline{u}, \underline{v})$ is a subsolution of (1.1).

Next We shall construct a supersolution of problem (1.1). Let ω be the solution of the following problem

$$
\begin{cases}
-\triangle e = 1 \text{ in } \Omega, \\
e = 0 \text{ on } \partial\Omega.
\end{cases}
\tag{2.12}
$$

Let

$$
\overline{u}_k = C_1 e,\ \overline{v}_k = C_2 e,
$$

where e is given by (2.12) and $C_1, C_2 > 0$ are a large positive real number to be chosen later. We shall verify that $(\overline{u}_k, \overline{v}_k)$ is a supersolution of problem (1.1). Let $\phi \in H_0^1(\Omega)$ with $\phi \ge 0$ in Ω. Then we obtain from (2.12) and the condition $(H1)$ that

$$
\begin{aligned}
A\left(\int_\Omega |\nabla \overline{u}_k|^2\, dx\right)\int_\Omega \nabla \overline{u}_k.\nabla\phi dx &= A\left(\int_\Omega |\nabla \overline{u}_k|^2\, dx\right) C_1 \int_\Omega \nabla e.\nabla\phi dx \\
&= A\left(\int_\Omega |\nabla \overline{u}_k|^2\, dx\right) C_1 \int_\Omega \phi dx \\
&\ge a_1 C_1 \int_\Omega \phi dx
\end{aligned}
$$

and

$$\begin{aligned} B\left(\int_\Omega |\nabla \overline{v}_k|^2 dx\right)\int_\Omega \nabla \overline{v}_k.\nabla \psi dx &= B\left(\int_\Omega |\nabla \overline{v}_k|^2 dx\right) C_2 \int_\Omega \nabla e.\nabla \psi dx \\ &= B\left(\int_\Omega |\nabla \overline{v}_k|^2 dx\right) C_2 \int_\Omega \psi dx \\ &\geq b_1 C_2 \int_\Omega \psi dx. \end{aligned}$$

Let $l = \|e\|_\infty$. Since $a < 1$, $d < 1$, these imply that there exist positive large constants $\alpha = a_1 C_1, \beta = b_1 C_2$ such that

$$\begin{aligned} \alpha &\geq \lambda_1 (\alpha l)^a + \mu_1 (\beta l)^b, \\ \beta &\geq \lambda_2 (\alpha l)^c + \mu_2 (\beta l)^d. \end{aligned}$$

Thus

$$\begin{aligned} a_1 C_1 \int_\Omega \phi dx \geq \int_\Omega \left(\lambda_1 \overline{u}_k^a + \mu_1 \overline{v}_k^b\right) \phi dx \\ \geq \int_\Omega \left(\lambda_1 \overline{u}_k^a + \mu_1 \overline{v}_k^b\right) \phi dx - \int_\Omega \frac{u_k - u_{k-1}}{\tau'} \phi dx \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} b_1 C_2 \int_\Omega \psi dx &\geq \int_\Omega \left(\lambda_2 \overline{u}_k^c + \mu_2 \overline{v_k}^d\right) \phi dx \\ &\geq \int_\Omega \left(\lambda_2 \overline{u}_k^c + \mu_2 \overline{v_k}^d\right) \phi dx - \int_\Omega \frac{v_k - v_{k-1}}{\tau'} \phi dx \end{aligned}$$

From (2.12) and (2.13) we have $(\overline{u}, \overline{v})$ is a subsolution of problem (1.1) with $\underline{u}_k \leq \overline{u}_k$ and $\underline{v}_k \leq \overline{v}_k$ for C_1, C_2 large.

In order to obtain a weak solution of problem (1.1) we shall use the arguments by Azzouz and Bensedik [5]. For this purpose, we define a sequence $\{(u_n, v_n)\} \subset \left(H_0^1(\Omega) \times H_0^1(\Omega)\right)$ as follows: $u_0 := \overline{u}, v_0 = \overline{v}$ and (u_n, v_n) is the unique solution of the system

$$\begin{cases} -A\left(\int_\Omega |\nabla u_n|^2 dx\right) \triangle u_n = \lambda_1 u_{n-1}^a + \mu_1 v_{n-1}^b - \frac{u_k - u_{k-1}}{\tau'} \text{in } \Omega, \\ -B\left(\int_\Omega |\nabla v_n|^2 dx\right) \triangle v_n = \lambda_2 u_{n-1}^c + \mu_2 v_{n-1}^d - \frac{v_k - v_{k-1}}{\tau'} \text{ in } \Omega, \\ u_n = v_n = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.14}$$

Problem (2.14) is (A,B) −linear in the sense that, if $(u_{n-1},v_{n-1})\in\left(H_0^1(\Omega)\times H_0^1(\Omega)\right)$ is a given, the right hand sides of (2.14) is independent of u_n,v_n.

Set $A(t)=tA\left(t^2\right),\ B(t)=tB\left(t^2\right)$. Then since $A(\mathbb{R})=\mathbb{R},\quad B(\mathbb{R})=\mathbb{R}$, $f(u_{n-1})=u_{n-1}^a$, $h(v_{n-1})=v_{n-1}^b, g(u_{n-1})=u_{n-1}^c$, and $\tau(v_{n-1})=v_{n-1}^d\in L^2(\Omega)$

we deduce from a result in [1] that system (2.14) has a unique solution $(u_n,v_n)\in\left(H_0^1(\Omega)\times H_0^1(\Omega)\right)$.

By using (2.14) and the fact that (u_0,v_0) is a supersolution of (1.1), we have

$$\begin{cases} -A\left(\int\limits_\Omega|\nabla u_0|^2dx\right)\triangle u_0\geq\lambda_1u_0^a+\ \mu_1v_0^b-\frac{u_k-u_{k-1}}{\tau'}=-A\left(\int\limits_\Omega|\nabla u_1|^2dx\right)\triangle u_1,\\ -B\left(\int\limits_\Omega|\nabla v_0|^2dx\right)\triangle v_0\geq\lambda_2u_0^c+\mu_2v_0^d\ -\frac{v_k-v_{k-1}}{\tau'}\ =-B\left(\int\limits_\Omega|\nabla v_1|dx\right)\triangle v_1\end{cases}$$

and by Lemma 1, $u_0\geq u_1$ and $v_0\geq v_1$. Also, since $u_0\geq\underline{u}$, $v_0\geq\underline{v}$ and the monotonicity of f, h, g, and τ one has

$$\begin{aligned}-A\left(\int\limits_\Omega|\nabla u_1|^2dx\right)\triangle u_1&=\lambda_1u_0^a+\ \mu_1v_0^b-\frac{u_k-u_{k-1}}{\tau'}\\&\geq\lambda_1\underline{u}^a+\ \mu_1\underline{v}^b-\frac{u_k-u_{k-1}}{\tau'}\geq-A\left(\int\limits_\Omega|\nabla\underline{u}|^2dx\right)\triangle\underline{u},\\-B\left(\int\limits_\Omega|\nabla v_1|^2dx\right)\triangle v_1&=\lambda_2u_0^c+\mu_2v_0^d\ -\frac{v_k-v_{k-1}}{\tau'}\\&\geq\lambda_2\underline{u}^c+\mu_2\underline{v}^d\ -\frac{v_k-v_{k-1}}{\tau'}\ \geq-B\left(\int\limits_\Omega|\nabla\underline{v}|^2dx\right)\triangle\underline{v}\end{aligned}$$

from which, according to Lemma 1, $u_1\geq\underline{u}$, $v_1\geq\underline{v}$. for u_2,v_2 we write

$$\begin{aligned}-A\left(\int\limits_\Omega|\nabla u_1|^2dx\right)\triangle u_1&=\lambda_1u_0^a+\ \mu_1v_0^b-\frac{u_k-u_{k-1}}{\tau'}\\&\geq\lambda_1u_1^a+\ \mu_1v_1^b-\frac{u_k-u_{k-1}}{\tau'}=-A\left(\int\limits_\Omega|\nabla u_2|^2dx\right)\triangle u_2,\\-B\left(\int\limits_\Omega|\nabla v_1|dx\right)\triangle v_1&=\lambda_2u_0^c+\mu_2v_0^d-\frac{v_k-v_{k-1}}{\tau'}\\&\geq\lambda_2u_1^c+\mu_2v_1^d-\frac{v_k-v_{k-1}}{\tau'}\ =-B\left(\int\limits_\Omega|\nabla v_2|^2dx\right)\triangle v_2\end{aligned}$$

and then $u_1 \geq u_2$, $v_1 \geq v_2$. Similarly, $u_2 \geq \underline{u}$ and $v_2 \geq \underline{v}$ because

$$\begin{aligned}
-A\left(\int_\Omega |\nabla u_2|^2\,dx\right)\triangle u_2 &= \lambda_1 u_0^a + \mu_1 v_0^b - \frac{u_k - u_{k-1}}{\tau'} \\
&\geq \lambda_1 \underline{u}^a + \mu_1 \underline{v}^b - \frac{u_k - u_{k-1}}{\tau'} \geq -A\left(\int_\Omega |\nabla \underline{u}|^2\,dx\right)\triangle \underline{u}, \\
-B\left(\int_\Omega |\nabla v_2|^2\,dx\right)\triangle v_2 &= \lambda_2 u_1^c + \mu_2 v_1^d - \frac{v_k - v_{k-1}}{\tau'} \\
&\geq \lambda_2 \underline{u}^c + \mu_2 \underline{v}^d - \frac{v_k - v_{k-1}}{\tau'} \geq -B\left(\int_\Omega |\nabla \underline{v}|^2\,dx\right)\triangle \underline{v}.
\end{aligned}$$

Repeating this argument we get a bounded monotone sequence $\{(u_n, v_n)\} \subset \left(H_0^1(\Omega) \times H_0^1(\Omega)\right)$ satisfying

$$\overline{u} = u_0 \geq u_1 \geq u_2 \geq ... \geq u_n \geq ... \geq \underline{u} > 0, \tag{2.15}$$

$$\overline{v} = v_0 \geq v_1 \geq v_2 \geq ... \geq v_n \geq ... \geq \underline{v} > 0. \tag{2.16}$$

Using the continuity of the functions f, h, g, and τ and the definition of the sequences $\{u_n\}, \{v_n\}$,there exist constants $C_i > 0$, $i = 1, ..., 4$ independent of n such that

$$|f(v_{n-1})| \leq C_1, \quad |h(u_{n-1})| \leq C_2, \; |g(u_{n-1})| \leq C_3 \tag{2.17}$$

and

$$|\tau(u_{n-1})| \leq C_4 \text{ for all } n.$$

From (2.17), multiplying the first equation of (2.14) by u_n, integrating, using the Holder inequality and Sobolev embedding we can show that

$$\begin{aligned}
a_1 \int_\Omega |\nabla u_n|^2\,dx &\leq A\left(\int_\Omega |\nabla u_n|^2\,dx\right)\int_\Omega |\nabla u_n|^2\,dx \\
&= \lambda_1 f(v_{n-1})\,u_n dx + \mu_1 \int_\Omega h(u_{n-1})\,u_n dx - \int_\Omega \frac{u_k - u_{k-1}}{\tau'} u_n dx \\
&\leq \lambda_1 \int_\Omega |f(v_{n-1})|\;|u_n|\,dx + \mu_1 \int_\Omega |h(u_{n-1})|\;|u_n|\,dx - \int_\Omega \frac{u_k - u_{k-1}}{\tau'}|u_n|\,dx \\
&\leq C_1\lambda_1 \int_\Omega |u_n|\,dx + C_2\mu_1 \int_\Omega |u_n|\,dx - \int_\Omega \frac{u_k - u_{k-1}}{\tau'}|u_n|\,dx \\
&\leq C_5 \|u_n\|_{H_0^1(\Omega)}
\end{aligned}$$

or

$$\|u_n\|_{H_0^1(\Omega)} \leq C_5,\ \forall n, \tag{2.18}$$

where $C_5 > 0$ is a constant independent of n. Similarly, there exist $C_6 > 0$ independent of n such that

$$\|v_n\|_{H_0^1(\Omega)} \leq C_6, \quad \forall n. \tag{2.19}$$

From (2.18) and (2.19), we infer that $\{(u_n, v_n)\}$ has a subsequence which weakly converges in $H_0^1\left(\Omega, \mathbb{R}^2\right)$ to a limit (u, v) with the properties $u \geq \underline{u} > 0$ and $v \geq \underline{v} > 0$. Being monotone and also using a standard regularity argument, $\{(u_n, v_n)\}$ converges itself to (u, v). Now, letting $n \to +\infty$ in (2.14), we deduce that (u, v) is a positive solution of system (1.1). The proof of theorem is now completed. □

References

[1] C. O. Alves, F. J. S. A. Correa, *On existence of solutions for a class of problem involving a nonlinear operator*, Communications on Applied Nonlinear Analysis, 8 (2001), 43-56.

[2] Y. Bouizem, S. Boulaaras and B. Djebbar, *Some existence results for an elliptic equation of Kirchhoff-type with changing sign data and a logarithmic nonlinearity*, Mathematical Methods in the Applied Sciences, 2019, Vol. 42, Issue 7, 2465-2474.

[3] S. Boulaaras, *Existence of positive solutions of nonlocal $p(x)$-Kirchhoff hyperbolic systems via sub-super solutions concept*, Journal of Intelligent & Fuzzy Systems, 38, 4 (2020), 4301-4313.

[4] S. Boulaaras, Y. Bouizem, R. Guefaifia, *Existence of positive solutions of $(p(x), q(x))$-Laplacian parabolic systems with right hand side defined as a multiplication of two separate functions*, Mathematical Methods in the Applied Sciences, 43 (2020), 2615-2625.

[5] S. Boulaaras, R.Guefaifia, *Existence of positive weak solutions for a class of Kirrchoff elliptic systems with multiple parameters*, Mathematical Methods in the Applied Sciences, 41, 13 (2018), 5203-5210.

[6] S. Boulaaras, R. Guefaifia, A. Alharbi, B. Cherif, *Existence of 3-weak solutions for a new class of an overdetermined system of fractional partial integro-differential equations*, Fractals, Vol. 2020, doi: 10.1142/S0218348X20400368.

[7] S. Boulaaras, R.Guefaifia and Tahar Bouali, *Existence of positive solutions for a new class of quasilinear singular elliptic systems involving Caffarelli–Kohn–Nirenberg exponent with sign-changing weight functions*, Indian Journal of Pure and Applied Mathematics, 49, 4 (2018), 705-715.

[8] S. Carl, V. K. Le, and D. Motreanu, *The sub-supersolution method and extremal solutions for quasilinear hemivariational inequalities*, Differential and Integral Equations, 17 (2004), 165-178.

[9] M. Chipot, B. Lovat, *Some remarks on nonlocal elliptic and parabolic problems*, Nonlinear Anal. 30 (1997), 4619-4627.

[10] F. J. S. A. Correa, G. M. Figueiredo, *On an elliptic equation of $p-$Kirchhoff type via variational methods*, Bull. Austral. Math. Soc. 74 (2006), 263-277.

[11] F. J. S. A. Correa, G. M. Figueiredo, *On a $p-$Kirchhoff equation type via Krasnoselkii's genus*, Appl. Math. Lett. 22 (2009), 819-822.

[12] E. N. Dancer and G. Sweers, *On the existence of a maximal weak solution for a semilinear elliptic equation*, Differential and Integral Equations, 2 (1989), 533-540.

[13] K. Fares , R. Guefaifia, S. Boulaaras, A. Alharbi, *Existence of weak solutions for a new class of fractional p-Laplacian boundary value systems*, Mathematics 2020, 8(4), 475; https://doi.org/10.3390/math8040475.

[14] R Guefaifia, S. M Boulaaras, S. Alodhaibi, S. Alkhalaf, *Existence of positive weak solutions for a new class of (p,q)-Laplacian nonlinear elliptic system with sign-changing weights*, Hindawi Complexity, Vol. 2020, Article ID 5302513, 6 pages, doi.org/10.1155/2020/5302513.

[15] D. D. Hai, R. Shivaji, *An existence result on positive solutions for a class of $p-$Laplacian systems*, Nonlinear Anal. 56 (2004), 1007-1010.

[16] G. Kirchhoff, *Mechanik*, Teubner, Leipzig, Germany, 1883.

[17] H. Knobloch and K. Schmitt, *Nonlinear boundary value problems for systems of second order differential equations*, Proc. Roy. Soc. Edinburgh 78A (1977), 139-159.

[18] V. K. Le, *On some equivalent properties of sub- and supersolutions in second order quasilinear elliptic equations*, Hiroshima Math. J. 28 (1998), 373-380.

[19] V. K. Le and K. Schmitt, *Some general concepts of sub- and supersolutions for nonlinear elliptic problems*, Topological Methods in Nonlinear Analysis, 28 (2006), 87-103.

[20] V. K. Le and K. Schmitt, *On boundary value problems for degenerate quasilinear elliptic equations and inequalities*, J. Differential Equations 144 (1998), 170-218.

[21] J. L. Lions, *On some questions in boundary value problems of mathematical physics*, In: "Contemporary Developments in Continuum Mechanics and Partial Differential Equations", North-Holland Math. Stud. 30, North-Holland, Amsterdam, 1978, 284-346.

[22] T. F. Ma, *Remarks on an elliptic equation of Kirchhoff type*, Nonlinear Anal. 63 (2005), 1967-1977.

[23] A. Menaceur, S. M. Boulaaras, R. Guefaifia, A.Alharbi, *Existence of positive weak solutions for quasi-linear Kirchhoff elliptic systems via sub-supersolutions concept*, Hindawi Mathematical Problems in Engineering, Vol. 2020, Article ID 6527672, 6 pages, doi.org/10.1155/2020/6527672.

[24] K. Perera, Z. Zhang, *Nontrivial solutions of Kirchhoff -type problems via the Yang index*, J. Diff. Eq. 221 (2006), 246-255.

[25] B. Ricceri, *On an elliptic Kirchhoff -type problem depending on two parameters*, J. Global Optim. 46 (2010), 543-549.

[26] K. Schmitt, *Periodic solutions of second order equations a variational approach*, The first 60 years of nonlinear analysis of Jean Mawhin, 213-220, World Sci. Publ., River Edge, NJ, 2004.

[27] S. Zediri, R. Guefaifia, S. Boulaaras, *Existence of positive solutions of a new class of nonlocal p(x)-Kirchhoff parabolic systems via sub-super-solutions concept*, J. Appl. Anal. 2020; aop.doi.org/10.1515/jaa-2020-2002.

Authors' addresses:

Mohammed Said Touati Brahim
Laboratory of mathematics, Informatics and Systems (LAMIS),
Larbi Tebessi University, 12002 Tebessa, Algeria.
E-mail:Mohammed.Said.Touati.Brahim@univ-tebessa.dz

Salah Boulaaras
Department Of Mathematics, College Of Sciences and Arts,
Al-Rass, Qassim University , Kingdom Of Saudi Arabia.
E-mail:s.boularas@qu.edu.sa

Rafik Guefaifia
Department of Mathematics and Computer Science,
Larbi Tebessi University, 12002 Tebessa, Algeria.
E-mail:rafikik982@gmail.com

Tarek Alloush
Department Of Mathematics, College Of Sciences and Arts,
Al-Rass, Qassim University , Kingdom Of Saudi Arabia.
E-mail: sap777@hotmail.com