Decomposition of groups and the wreath product of permutation groups

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Abstract. In this work, we study decomposition of groups. Let G be a group, and let N be a normal subgroup of G; we shall show how multiplication in G can be viewed as a two-step process consisting of multiplication in the quotient G/N followed by multiplication in N. The object of wreath product of permutation groups is defined by the actions on Cartesian product of two sets. In this paper we consider $S(\Gamma)$ and $S(\Delta)$ - the permutation groups on Γ and Δ respectively, and $S(\Gamma)^{\Delta}$ - the set of all maps of Δ into the permutations group $S(\Gamma)$, to provide the wreath product W of $S(\Gamma)$ by $S(\Delta)$, and the action of W on $\Gamma \times \Delta$.

1 Introduction

In Mathematics, the wreath product in group theory is a specialized product of two groups. The wreath product is an important tool in the classification of permutation groups, and also provides a way of constructing interesting examples of groups. The wreath product and its generalizations play an important role in algebraic theory. For example, it can be used to prove the theorem on the decomposition of every finite semi-group automation into a step wise combination of flip-flop and simple group automata.

The remainder of this paper is organized as follows. In Section 2 we introduce the concept of decomposition of groups. In Section 3, we provide some mathematical preliminaries. In Section 4, we give a proposition on wreath product of groups. In Section 5, we introduce the wreath product of permutation groups and the notion of group actions on a set, and adjacent concepts like the orbit and the stabilizer. Finally, we draw our conclusions in Section 6.

2   Preliminaries

Let $S(X)$ be the set of one to one and onto functions on the n-element set X, with multiplication to composition of functions. The elements of $S(X)$ are called permutations and $S(X)$ is called the symmetric group on X.

A group homomorphism is a well-defined map $\varphi : G \longrightarrow H$ between two groups G and H, which preserves the multiplicative structure. In other words, $\varphi(xy) = \varphi(x)\varphi(y)$, for all $x, y \in G$. A bijective homomorphism is called an isomorphism. When there is an isomorphism between two groups G and H, we say G and H are isomorphic and we write $G \cong H$.

Let G and H be group and $\varphi : G \longrightarrow H$ be a homomorphism. Then $N = \ker \varphi$ is a normal subgroup of G and the induced map $\bar{\varphi} : G/N \longrightarrow \text{Im}(\varphi) \leq H, Ng \longmapsto \varphi(g)$ is an isomorphism between the quotient group G/N and the image $\text{Im}(\varphi)$.

Let G be a group and X be a non empty set. We say that G acts on the set X if to each g in G and each x in X, there corresponds a unique point $g \cdot x$ in X such that, for all x in X and g_1, g_2 in G, we have that

$$(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x) \text{ and } 1_G \cdot x = x.$$

To be explicit, we assume that G acts on the set X on the left. The stabilizer of an element $x \in X$ under the action of G is defined by:

$$G_x = \{g \in G : g \cdot x = x\}.$$

The kernel of an action $G \times X \longrightarrow X, (g, x) \longmapsto g \cdot x$ is given by:

$$Ker = \{g \in G : g \cdot x = x \text{ for all } x \in X\}.$$

We define the orbit containing $x \in X$ to be $G \cdot x = \{g \cdot x, g \in G\}$.

Let G be a group acting on a set X. Then, for all $x \in X$, we have $|G_x| \, |G \cdot x| = |G|$.

Let G and K be two groups. We say that G acts on K as a group if to each k in K there corresponds a unique element k^g in K such that for g_1, g_2, g in G and k_1, k_2, k in K we have that

$$(k^{g_1})^{g_2} = k^{g_1 g_2}, k^{1_G} = k \text{ and } (k_1 k_2)^g = k_1^g k_2^g.$$

Given any groups G and H and a morphism $\theta : G \longrightarrow Aut(H)$, we denote the automorphism $\theta(g)$ by θ_g, and then $G \times H$ is a group with the multiplication $(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 \theta_{g_1}(h_2))$, where $g_1, g_2 \in G$ and $h_1, h_2 \in H$. The group $(G \times H, \cdot)$ is called the semidirect product of G and H with respect to θ.

3   Decomposition of groups

In this section, we introduce the concept of decomposition of groups.

Proposition 3.1. *Let $(G, \cdot)$ be a group, and let N be a normal subgroup of G. Let $S = \{g_i : i \in I\}$ be a complete set of coset representatives of N. If $g \in G$, then we denote by $[g]$ the chosen representative of Ng. Thus,*

1. for all $n_1, n_2 \in N, g_i, g_j \in S : (n_1 g_i)(n_2 g_j) = ng$, where $g = [g_i g_j]$ and $n = n_1 \left(g_i n_2 g_i^{-1}\right)$.

2. consider the set $N \times G/N$ the Cartesian product of N and G/N, and define a multiplication " $*$ " on $N \times G/N$ as follows:

$$(n_1, Ng_i) * (n_2, Ng_j) = \left(n_1 g_i n_2 g_i^{-1}, Ng_i g_j\right)$$

The result is a group.

3. $(N \times G/N, *)$ and $(G, \cdot)$ are isomorphic.

Proof. 1) For all $n_1, n_2 \in N, g_i, g_j \in S$, we have $n_1 g_i \in Ng_i, n_2 g_j \in Ng_j$ and $Ng_i Ng_j = Ng_i g_j$, then $(n_1 g_i)(n_2 g_j) \in Ng_i g_j$. This implies that there exists $n \in N$ such that
$(n_1 g_i)(n_2 g_j) = ng_i g_j$ and consequently $g = [g_i g_j]$ and $n = n_1 \left(g_i n_2 g_i^{-1}\right)$.
2) We will prove that the set $N \times G/N$ forms a group such that for any $(n_1, Ng_i), (n_2, Ng_j) \in N \times G/N, (n_1, Ng_i) * (n_2, Ng_j) = \left(n_1 g_i n_2 g_i^{-1}, Ng_i g_j\right)$.
a) $N \times G/N$ is non-empty and is closed with respect to multiplication.
b) We will prove that " $*$ " is associative on $N \times G/N$:
let $(n_1, Ng_i), (n_2, Ng_j), (n_3, Ng_k) \in N \times G/N$, we have
$((n_1, Ng_i) * (n_2, Ng_j)) * (n_3, Ng_k) = \left(n_1 g_i n_2 g_i^{-1}, Ng_i g_j\right) * (n_3, Ng_k)$
$= \left(n_1 g_i n_2 g_j n_3 g_j^{-1} g_i^{-1}, N(g_i g_j) g_k\right)$. Also we have
$(n_1, Ng_i) * ((n_2, Ng_j) * (n_3, Ng_k)) = \left(n_1 g_i n_2 g_j n_3 g_j^{-1} g_i^{-1}, Ng_i(g_j g_k)\right)$.
Then " $*$ " is associative on $N \times G/N$.
c) For $(n, Ng_i) \in N \times G/N$, we have
$(n, Ng_i) * (1_G, N) = (1_G, N) * (n, Ng_i) = (n, Ng_i)$.
The identity element in $(N \times G/N, *)$ is $(1_G, N)$.
d) We show that every element of $(N \times G/N, *)$ is invertible.
Let $(n, Ng_i) \in N \times G/N$. We have
$(n, Ng_i) * \left(g_i^{-1} ng_i, Ng_i^{-1}\right) = \left(g_i^{-1} ng_i, Ng_i^{-1}\right) * (n, Ng_i) = (1_G, N)$.
3) We define the mapping $\varphi : N \times G/N \longrightarrow G, \varphi(n, Ng_i) = ng_i$; the mapping φ is morphism because for all $(n_1, Ng_i), (n_2, Ng_j) \in N \times G/N$,
$\varphi((n_1, Ng_i) * (n_2, Ng_j)) = \varphi\left(n_1 g_i n_2 g_i^{-1}, Ng_i g_j\right) = n_1 g_i n_2 g_i^{-1} g_i g_j = n_1 g_i n_2 g_j$. It is clear that φ is onto. The mapping φ is one-to-one, because we have for all $(n_1, Ng_i), (n_2, Ng_j) \in N \times G/N$,
$\varphi(n_1, Ng_i) = \varphi(n_2, Ng_j) \Longrightarrow n_1 g_i = n_2 g_j \Longrightarrow n_2^{-1} n_1 = g_j g_i^{-1} \Longrightarrow g_j g_i^{-1} \in N$
$\Longrightarrow g_j g_i^{-1} = 1_G \Longrightarrow (n_1 = n_2$ and $Ng_i = Ng_j)$. $\qquad \square$

Example 3.1. Let $G = (S_3, \circ)$ be the symmetric group on $\{1, 2, 3\}$, where
$S_3 = \{e = (1), \tau_1 = (23), \tau_2 = (13), \tau_3 = (12), \sigma_1 = (123), \sigma_2 = (132)\}$.
The Cayley table of $(S_3, \circ)$ is defined as follows (see Table 1):

$\circ$	e	τ_1	τ_2	τ_3	σ_1	σ_2
e	e	τ_1	τ_2	τ_3	σ_1	σ_2
τ_1	τ_1	e	σ_1	σ_2	τ_2	τ_3
τ_2	τ_2	σ_2	e	σ_1	τ_3	τ_1
τ_3	τ_3	σ_1	σ_2	e	τ_1	τ_2
σ_1	σ_1	τ_3	τ_1	τ_2	σ_2	e
σ_2	σ_2	τ_2	τ_3	τ_1	e	σ_1

Let $N = \langle (123) \rangle = \langle \sigma_1 \rangle$ the subgroup of $(S_3, \circ)$ generated by σ_1.
We have $N = \{e, \sigma_1, \sigma_2\}$. N is a normal subgroup of $(S_3, \circ)$ because the index of N in S_3 $[S_3 : N] = 2$.
The quotient $S_3/N = \{N \circ \sigma : \sigma \in S_3\} = \{\{\tau_1, \tau_2, \tau_3\}, \{e, \sigma_1, \sigma_2\}\} = \{[e], [\tau_1]\}$.
The Cayley table of $(S_3/N, \cdot)$ is defined as follows (see Table 2):

$\cdot$	$[e]$	$[\tau_1]$
$[e]$	$[e]$	$[\tau_1]$
$[\tau_1]$	$[\tau_1]$	$[e]$

$N \times S_3/N = \{(e, [e]), (e, [\tau_1]), (\sigma_1, [e]), (\sigma_1, [\tau_1]), (\sigma_2, [e]), (\sigma_2, [\tau_1])\}$.
The Cayley table of $(N \times S_3/N, *)$ is defined as follows (see Table 3):

$*$	$(e, [e])$	$(e, [\tau_1])$	$(\sigma_1, [e])$	$(\sigma_1, [\tau_1])$	$(\sigma_2, [e])$	$(\sigma_2, [\tau_1])$
$(e, [e])$	$(e, [e])$	$(e, [\tau_1])$	$(\sigma_1, [e])$	$(\sigma_1, [\tau_1])$	$(\sigma_2, [e])$	$(\sigma_2, [\tau_1])$
$(e, [\tau_1])$	$(e, [\tau_1])$	$(e, [e])$	$(\sigma_2, [\tau_1])$	$(\sigma_2, [e])$	$(\sigma_1, [\tau_1])$	$(\sigma_1, [e])$
$(\sigma_1, [e])$	$(\sigma_1, [e])$	$(\sigma_1, [\tau_1])$	$(\sigma_2, [e])$	$(\sigma_2, [\tau_1])$	$(e, [e])$	$(e, [\tau_1])$
$(\sigma_1, [\tau_1])$	$(\sigma_1, [\tau_1])$	$(\sigma_1, [e])$	$(e, [\tau_1])$	$(e, [e])$	$(\sigma_2, [\tau_1])$	$(\sigma_2, [e])$
$(\sigma_2, [e])$	$(\sigma_2, [e])$	$(\sigma_2, [\tau_1])$	$(e, [e])$	$(e, [\tau_1])$	$(\sigma_1, [e])$	$(\sigma_1, [\tau_1])$
$(\sigma_2, [\tau_1])$	$(\sigma_2, [\tau_1])$	$(\sigma_2, [e])$	$(\sigma_1, [\tau_1])$	$(\sigma_1, [e])$	$(e, [\tau_1])$	$(e, [e])$

Finally $(N \times G/N, *)$ and $(S_3, \circ)$ are isomorphic.

4   The wreath product of groups

In this section, we introduce the concept of wreath product of groups.

Theorem 4.1. *Let G and H be two groups. Let H^G be the set of all functions defined on G with values in H.*

1. The set H^G forms a group such that for any $\varphi, \psi \in H^G$, let $\varphi\psi \in H^G$ in H^G be defined for all $x \in G$ by:

$$(\varphi\psi)(x) = \varphi(x)\psi(x).$$

2. The group G acts on H^G as a group, in the following way:

$$\text{if } a \in G, \varphi \in H^G, \text{ then } (a \cdot \varphi)(x) = \varphi^a(x) = \varphi(xa^{-1}) \text{ for } x \in G.$$

3. The set of all pairs (a, φ) with $a \in G, \varphi \in H^G$, with the multiplication operation given by:

$$(a, \varphi)(b, \psi) = (ab, \varphi^b \psi) \text{ where } a, b \in G \text{ and } \varphi, \psi \in H^G$$

provides as resulting group W, called the wreath product of G and H, and denoted by GW_rH.

Proof. (1) First we will prove that the set H^G forms a group; for any $\varphi, \psi \in H^G$, let
$\varphi\psi \in H^G$ in H^G,
be defined for all $x \in G$ by $(\varphi\psi)(x) = \varphi(x)\psi(x)$.

(i) H^G is non-empty and is closed with respect to multiplication. If $\varphi, \psi \in H^G$,
then $\varphi(x), \psi(x) \in H$, for all $x \in G$. Hence $\varphi(x)\psi(x) \in H$. This implies that
$(\varphi\psi)(x) \in H$ and so $\varphi\psi \in H^G$.

(ii) Since multiplication in H is associative, so is the multiplication in H^G, as well.

(iii) The identity element in H^G is the map $e : G \longrightarrow H$ given by
$e(x) = 1_H$, for all $x \in G$, where 1_H is the identity element of H.

(iv) For every element $\varphi \in H^G$, is defined for all $x \in G$ by
$\varphi^{-1}(x) = (\varphi(x))^{-1}$.

Thus H^G is a group with respect to the multiplication defined above.

(2) We shall further prove that G acts on H^G as group; assume that G acts
on H^G as follows $G \times H^G \longrightarrow H^G; (a, \varphi) \longrightarrow \varphi^a$, such that for $x \in G$ we have
$\varphi^a(x) = \varphi(xa^{-1})$, $a \in G, \varphi \in H^G$. Take $\varphi, \psi \in H^G$ and $a, b \in G$; then

(i) $(\varphi^a)^b(x) = \varphi^a(xb^{-1}) = \varphi((xb^{-1})a^{-1}) = \varphi(x(ab)^{-1}) = \varphi^{ab}(x)$.

(ii) $\varphi^{1_G}(x) = \varphi(x1_G^{-1}) = \varphi(x)$.

(iii) $(\varphi\psi)^a(x) = \varphi\psi(xa^{-1}) = \varphi(xa^{-1})\psi(xa^{-1}) = \varphi^a(x)\psi^a(x)$.

(3) Now we can construct the wreath product W of G and H, that is, the semidirect
product of G and H^G; then we shall prove that $G \times H^G$ is a group with multiplication
$(a, \varphi)(b, \psi) = (ab, \varphi^b\psi)$. Then we have the following:

(i) The closure property follows from the definition of multiplication.

(ii) Take $\varphi, \psi, \eta \in H^G$ and $a, b, c \in G$; then

$$((a, \varphi)(b, \psi))(d, \eta) = (ab, \varphi^b\psi)(d, \eta) = ((ab)d, (\varphi^b\psi)^d \eta).$$

Also we have $(a, \varphi)((b, \psi)(d, \eta)) = (a, \varphi)(bd, \psi^d\eta) = (a(bd), \varphi^{bd}\psi^d\eta)$
$= ((ab)d, \varphi^{bd}\psi^d\eta)$.

Now if $x \in G$, then $(\varphi^b\psi)^d \eta(x) = (\varphi^b\psi)^d(x)\eta(x) = (\varphi^b)^d(x)\psi^d(x)\eta(x)$
$= \varphi^b(xd^{-1})\psi(xd^{-1})\eta(x) = \varphi(xd^{-1}b^{-1})\psi(xd^{-1})\eta(x) = \varphi(x(bd)^{-1})\psi(xd^{-1})\eta(x)$
$= \varphi^{bd}(x)\psi^d(x)\eta(x)$. As well, $\varphi^{bd}\psi^d\eta(x) = \varphi^{bd}(x)\psi^d(x)\eta(x)$. Thus we have estab-
lished the associativity of the multiplication on the set $G \times H^G$.

(iii) We know that for every $\varphi \in H^G, \varphi^{1_G} = \varphi$, for every $g \in G$, the map $\varphi \longrightarrow \varphi^g$
is an automorphism of H^G. Also, if e is the identity element in H^G, then $e^g = e$. We have $(a, \varphi)(1_G, e) = (a1_G, \varphi^{1_G}e) = (a, \varphi e) = (a, \varphi)$. Also $(1_G, e)(a, \varphi) = (1_G a, e^a\varphi) = (a, e\varphi) = (a, \varphi)$. Hence the identity element exists.

(iv) We have $(a, \varphi)\left(a^{-1}, (\varphi^{-1})^{(a)^{-1}}\right) = \left(a^{-1}, (\varphi^{-1})^{(a)^{-1}}\right)(a, \varphi) = (1_G, e)$. Thus
the inverse element of (a, φ) is $\left(a^{-1}, (\varphi^{-1})^{(a)^{-1}}\right)$. Hence $G \times H^G$ is a group with
respect to the multiplication defined above. $\qquad\square$

In the following proposition, we show that the group H^G is a normal subgroup of
W and G is a subgroup of W.

Proposition 4.2.   *1. If G and H^G are finite groups, then the wreath product W
is a finite group of order $|W| = |G|.|H|^{|G|}$.*

2. *The group H^G is a normal subgroup of W and G is a subgroup of W.*

3. $G \cap H^G = (1_G, e)$.

4. $GW_r H^G = G \times H^G$.

Proof. (1) It is clear.

(2) We have the injective maps $\Phi : H^G \longrightarrow G \times H^G$ given by $f \longmapsto (1_G, f)$, and $\Psi : G \longrightarrow G \times H^G$ given by $a \longmapsto (a, e)$. Both of them are homomorphisms, since $\Phi(f_1 f_2) = (1_G, f_1 f_2) = \left(1_G 1_G, f_1^{(1_G)} f_2\right) = (1_G, f_1) W_r (1_G, f_2) = \Phi(f_1) W_r \Phi(f_2)$.

As well, $\Psi(ab) = (ab, e) = \left(ab, e^b e\right) = (a, e) W_r (b, e) = \Psi(a) W_r \Psi(b)$.

Then $H^G \cong \mathrm{Im}(\Phi) \leq G \times H^G$, and $G \cong \mathrm{Im}(\Psi) \leq G \times H^G$. These injective homomorphisms let us think of H^G and G as subgroups of $G \times H^G$. Finally, we must show that H^G is normal in $G \times H^G$; from the calculations, there follows:

$$(a, e)(1_G, f)(a, e)^{-1} = (a, e)(1_G, f)\left(a^{-1}, \left(e^{-1}\right)^{a^{-1}}\right) = (a, e)(1_G, f)\left(a^{-1}, e\right)$$
$$= \left(a1_G, e^{1_G} f\right)\left(a^{-1}, e\right) = \left(aa^{-1}, f\right) = (1_G, f).$$

(3) It is clear that $G \cap H^G = (1_G, e)$.

(4) We have $GW_r H^G = G \times H^G$ since $(a, e) W_r (1_G, f) = \left(a1_G, e^{1_G} f\right)$ $= (a, f)$ for all $(a, f) \in G \times H^G$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5   Wreath product of permutation groups

This section is essentially an upgrade of the results of Ibrahim and Audu (see [2]) on the wreath product of permutation groups. We introduce the notion of group actions on a set and its adjacent concepts, like the orbit and the stabilizer.

Theorem 5.1. *Let $S(\Gamma)$ and $S(\Delta)$ be permutation groups on Γ and Δ respectively. Let $S(\Gamma)^\Delta$ be the set of all maps of Δ into the permutations group $S(\Gamma)$. That is, $S(\Gamma)^\Delta = \{f : \Delta \longrightarrow S(\Gamma)\}$. For any f_1, f_2 in $S(\Gamma)^\Delta$, let $f_1 f_2$ in $S(\Gamma)^\Delta$ be defined for all δ in Δ by $(f_1 f_2)(\delta) = f_1(\delta) f_2(\delta)$. With respect to this operation of multiplication, $S(\Gamma)^\Delta$ acquires the structure of a group.*

Proof. (i) $S(\Gamma)^\Delta$ is non-empty and is closed with respect to multiplication. If $f_1, f_2 \in S(\Gamma)^\Delta$, then $f_1(\delta), f_2(\delta) \in S(\Gamma)$. Hence $f_1(\delta) f_2(\delta) \in S(\Gamma)$. This implies that $(f_1 f_2)(\delta) \in S(\Gamma)$ and so $f_1 f_2 \in S(\Gamma)^\Delta$.

(ii) Since multiplication is associative, so is the multiplication in $S(\Gamma)^\Delta$.

(iii) The identity element in $S(\Gamma)^\Delta$ is the map $e : \Delta \longrightarrow S(\Gamma)$ given by $e(\delta) = id_\Gamma$ for all $\delta \in \Delta$ where id_Γ is the identity element of $S(\Gamma)$.

(iv) Every element $f \in S(\Gamma)^\Delta$ is defined for all $\delta \in \Delta$ by $f^{-1}(\delta) = (f(\delta))^{-1}$.

Thus $S(\Gamma)^\Delta$ is a group with respect to the multiplication defined above. We denote this group by P. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 5.2. *Assume that $S(\Delta)$ acts on P as follows: $S(\Delta) \times S(\Gamma)^\Delta \longrightarrow S(\Gamma)^\Delta, (s, f) \longmapsto s \cdot f = f^s$, where $f^s(\delta) = \left(f \circ s^{-1}\right)(\delta)$ $= \left(f s^{-1}\right)(\delta)$ for all $\delta \in \Delta$. Then $S(\Delta)$ acts on P as a group.*

Proof. Take, $f, f_1, f_2 \in S(\Gamma)^\Delta$ and $s, s_1, s_2 \in S(\Delta)$, then

(i) $((s_1 s_2) \cdot f)(\delta) = f^{(s_1 s_2)}(\delta) = \left(f \left(s_1 s_2 \right)^{-1} \right)(\delta) = \left(f \left(s_2^{-1} s_1^{-1} \right) \right)(\delta)$

$= \left(f s_2^{-1} \right) \left(s_1^{-1}(\delta) \right) = (s_1 \cdot (s_2 \cdot f))(\delta).$

(ii) $f^{id_\Delta}(\delta) = \left(f id_\Delta^{-1} \right)(\delta) = (f id_\Delta)(\delta) = (f)(\delta).$

(iii) $(f_1 f_2)^s (\delta) = \left(f_1 f_2 \circ s^{-1} \right)(\delta) = f_1 f_2 \left(s^{-1}(\delta) \right)$

$= f_1 \left(s^{-1}(\delta) \right) f_2 \left(s^{-1}(\delta) \right) = f_1^s(\delta) f_2^s(\delta).$ $\qquad \square$

Proposition 5.3. *The set of all ordered (f, s) with $f \in S(\Gamma)^\Delta$ and $s \in S(\Delta)$ acquires the structure of a group, when we define for all $f_1, f_2 \in S(\Gamma)^\Delta$ and $s_1, s_2 \in S(\Delta)$.*

$$(f_1, s_1)(f_2, s_2) = \left(f_1 f_2^{s_1^{-1}}, s_1 s_2 \right).$$

Thus $S(\Gamma)^\Delta \times S(\Delta)$ is a group with respect to the multiplication defined above. We denote this group by W. The resulting group W is called the wreath product of $S(\Gamma)$ by $S(\Delta)$, and is denoted by $W = S(\Gamma) W_r S(\Delta)$.

Proof. (i) The closure property follows from the definition of multiplication.

(ii) Take $f_1, f_2, f_3 \in S(\Gamma)^\Delta$ and $s_1, s_2, s_3 \in S(\Delta)$. Then, $[(f_1, s_1)(f_2, s_2)](f_3, s_3)$

$= \left(f_1 f_2^{s_1^{-1}}, s_1 s_2 \right)(f_3, s_3) = \left(f_1 f_2^{s_1^{-1}} f_3^{(s_1 s_2)^{-1}}, s_1 s_2 s_3 \right) = \left(f_1 f_2^{s_1^{-1}} f_3^{s_2^{-1} s_1^{-1}}, s_1 s_2 s_3 \right).$

Also, we get in the same manner that, $(f_1, s_1)[(f_2, s_2)(f_3, s_3)] = (f_1, s_1) \left(f_2 f_3^{s_2^{-1}}, s_2 s_3 \right)$

$= \left(f_1 \left(f_2 f_3^{s_2^{-1}} \right)^{s_1^{-1}}, s_1 s_2 s_3 \right) = \left(f_1 f_2^{s_1^{-1}} f_3^{s_2^{-1} s_1^{-1}}, s_1 s_2 s_3 \right).$

Hence, multiplication is associative.

(iii) We know that for every $f \in S(\Gamma)^\Delta$, $f^{id_\Delta} = f$. Now for every $s \in S(\Delta)$, the map $f \longmapsto f^s$ is an automorphism of $S(\Gamma)^\Delta$. Also, if e is the identity element in $S(\Gamma)^\Delta$, then $e^s = e$. Also, $\left(f^{-1} \right)^s = (f^s)^{-1}$. Now, $(f, s)(e, id_\Delta) = \left(f e^{s^{-1}}, s \circ id_\Delta \right) = (f, s).$

Also, using the reverse order, we have that, $(e, id_\Delta)(f, s) = \left(e f^{(id_\Delta)^{-1}}, id_\Delta \circ s \right)$

$= (f, s)$. Thus the identity element exists.

(iv) $(f, s) \left(\left(f^{-1} \right)^s, s^{-1} \right) = \left(\left(f^{-1} \right)^s, s^{-1} \right)(f, s) = (e, id_\Delta).$ $\qquad \square$

In the following proposition, we show that the group $S(\Gamma)^\Delta$ is a normal subgroup of W, and $S(\Delta)$ is a subgroup of W.

Proposition 5.4. *1. If $S(\Delta)$ and $S(\Gamma)$ are finite groups, then the wreath product W is a finite group of order $|W| = |S(\Gamma)|^{|\Delta|} \cdot |S(\Delta)|$.*

2. *The group $S(\Gamma)^\Delta$ is a normal subgroup of W and $S(\Delta)$ is a subgroup of W.*

3. *$S(\Gamma)^\Delta \cap S(\Delta) = (e, id_\Delta)$.*

4. *$S(\Gamma)^\Delta W_r S(\Delta) = S(\Gamma)^\Delta \times S(\Delta)$.*

5. *The action of W on $\Gamma \times \Delta$ is given by: $(f, s)(\gamma, \delta) = (f(\delta)(\gamma), s(\delta))$ for all $(f, s) \in S(\Gamma)^\Delta \times S(\Delta)$ and $(\gamma, \delta) \in \Gamma \times \Delta$.*

Proof. (1) Obvious.

(2) We have the injective maps $\Phi : S(\Gamma)^\Delta \longrightarrow S(\Gamma)^\Delta \times S(\Delta)$ given by $f \longmapsto (f, id_\Delta)$, and $\Psi : S(\Delta) \longrightarrow S(\Gamma)^\Delta \times S(\Delta)$ given by $s \longmapsto (e, s)$. Both of them are homomorphisms, since $\Phi(f_1 f_2) = (f_1 f_2, id_\Delta) = \left(f_1 f_2^{(id_\Delta)^{-1}}, id_\Delta \circ id_\Delta \right)$ $= (f_1, id_\Delta) W_r (f_2, id_\Delta) = \Phi(f_1) W_r \Phi(f_2)$.

As well, $\Psi(s_1 \circ s_2) = (e, s_1 \circ s_2) = \left(e e^{(s_1)^{-1}}, s_1 \circ s_2 \right)$ $= (e, s_1) W_r (e, s_2) = \Psi(s_1) W_r \Psi(s_2)$.

Then $S(\Gamma)^\Delta \cong \text{Im}(\Phi) \leq S(\Gamma)^\Delta \times S(\Delta)$. And $S(\Delta) \cong \text{Im}(\Psi) \leq S(\Gamma)^\Delta \times S(\Delta)$. These injective homomorphisms let us think of $S(\Gamma)^\Delta$ and $S(\Delta)$ as subgroups of $S(\Gamma)^\Delta \times S(\Delta)$. Finally, we must show that $S(\Gamma)^\Delta$ is normal in $S(\Gamma)^\Delta \times S(\Delta)$; and the calculation leads to

$(e, s)(f, id_\Delta)(e, s)^{-1} = (e, s)(f, id_\Delta)\left((e^{-1})^s, s^{-1} \right) = (e, s)(f, id_\Delta)(e, s^{-1})$
$= \left(e f^{(s)^{-1}}, s \circ id_\Delta \right)(e, s^{-1}) = \left(f^{(s)^{-1}}, id_\Delta \right)$.

(3) It is clear that $S(\Gamma)^\Delta \cap S(\Delta) = (e, id_\Delta)$.

(4) We have $S(\Gamma)^\Delta W_r S(\Delta) = S(\Gamma)^\Delta \times S(\Delta)$ since $(f, id_\Delta) W_r (e, s)$
$= \left(f e^{(id_\Delta)^{-1}}, id_\Delta \circ s \right) = (f, s)$ for all $(f, s) \in S(\Gamma)^\Delta \times S(\Delta)$

(5) Take, $(f_1, s_1), (f_2, s_2) \in S(\Gamma)^\Delta \times S(\Delta)$ and $(\gamma, \delta) \in \Gamma \times \Delta$, then

(i) $(e, id_\Delta)(\gamma, \delta) = (e(\delta)(\gamma), id_\Delta(\delta)) = (id_\Gamma(\gamma), \delta) = (\gamma, \delta)$.

(ii) $[(f_1, s_1)(f_2, s_2)](\gamma, \delta) = \left(f_1 f_2^{s_1^{-1}}, s_1 s_2 \right)(\gamma, \delta) = \left(f_1 f_2^{s_1^{-1}}(\delta)(\gamma), s_1 s_2(\delta) \right)$
$= \left(\left(f_1(\delta) f_2^{s_1^{-1}}(\delta) \right)(\gamma), s_1 s_2(\delta) \right) = ((f_1(\delta)(f_2 \circ s_1)(\delta))(\gamma), s_1 s_2(\delta))$.
Also, we have in the same manner that,

$(f_1, s_1)[(f_2, s_2)(\gamma, \delta)] = (f_1, s_1)(f_2(\delta)(\gamma), s_2(\delta)) = (f_1(s_2(\delta))(f_2(\delta)(\gamma)), s_1 s_2(\delta))$.
□

Proposition 5.5. *Under the action of W on $\Gamma \times \Delta$, the stabilizer of any point (γ, δ) in $\Gamma \times \Delta$ denoted by $W_{(\gamma, \delta)}$ is given by:$W_{(\gamma, \delta)} = S(\Gamma)^\Delta(\delta)_\gamma \times S(\Delta)_\delta$. Where $S(\Gamma)^\Delta(\delta)_\gamma$ is the set of all $f(\delta)$ that stabilize γ, and $S(\Delta)_\delta$ is the stabilizer of δ under the action of $S(\Delta)$ on Δ.*

Proof. We have $W_{(\gamma, \delta)} = \left\{ (f, s) \in S(\Gamma)^\Delta \times S(\Delta) / (f, s)(\gamma, \delta) = (\gamma, \delta) \right\}$
$= \left\{ (f, s) \in S(\Gamma)^\Delta \times S(\Delta) / (f(\delta)\gamma, s(\delta)) = (\gamma, \delta) \right\}$
$= \left\{ (f, s) \in S(\Gamma)^\Delta \times S(\Delta) / f(\delta)\gamma = \gamma, s(\delta) = \delta \right\} = S(\Gamma)^\Delta(\delta)_\gamma \times S(\Delta)_\delta$. □

Example 5.1. Consider the permutation groups $S(\Gamma) = \{(1), (12)\}$ and $S(\Delta) = \{(1), (12), (13), (23), (123), (132)\}$ on the sets $\Gamma = \{1, 2\}$ and $\Delta = \{1, 2, 3\}$ respectively. Let $S(\Gamma)^\Delta = \{f : \Delta \longrightarrow S(\Gamma)\}$, then $|S(\Gamma)|^{|\Delta|} = 2^3 = 8$. The mappings are follows:

$f_1 : 1 \longmapsto (1), 2 \longmapsto (1), 3 \longmapsto (1)$
$f_2 : 1 \longmapsto (1), 2 \longmapsto (1), 3 \longmapsto (12)$
$f_3 : 1 \longmapsto (1), 2 \longmapsto (12), 3 \longmapsto (1)$
$f_4 : 1 \longmapsto (1), 2 \longmapsto (12), 3 \longmapsto (12)$
$f_5 : 1 \longmapsto (12), 2 \longmapsto (1), 3 \longmapsto (1)$
$f_6 : 1 \longmapsto (12), 2 \longmapsto (1), 3 \longmapsto (12)$
$f_7 : 1 \longmapsto (12), 2 \longmapsto (12), 3 \longmapsto (1)$

$f_8 : 1 \longmapsto (12), 2 \longmapsto (12), 3 \longmapsto (12)$

We can easily verify that $S(\Gamma)^\Delta$ is a group with respect to the operation $(\varphi\psi)(\delta) = (\varphi)(\delta)(\psi)(\delta)$ where $\delta \in \Delta$.

We have

$S(\Gamma)^\Delta \times S(\Delta) = \{(f,s) / f \in S(\Gamma)^\Delta, s \in S(\Delta)\}$

$= \{(f_i, (1)), (f_i, (12)), (f_i, (12)), (f_i, (23)), (f_i, (123)), (f_i, (132)), 1 \leq i \leq 8\}$.

And $\left|S(\Gamma)^\Delta \times S(\Delta)\right| = \left|S(\Gamma)^\Delta\right| . |S(\Delta)| = 8.6 = 48$.

$S(\Gamma)^\Delta \times S(\Delta)$ is a group with respect to the operation $(\varphi, s_1)(\psi, s_2) = \left(\varphi\psi^{(s_1)^{-1}}, s_1 s_2\right)$.

We have $\Gamma \times \Delta = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$.

The stabilizer of $(1,1)$ denoted by

$W_{(1,1)} = S(\Gamma)^\Delta(1)_1 \times S(\Delta)_1 = \{f_1, f_2, f_3, f_4\} \times \{(1), (23)\}$

$= \{(f_1, (1)), (f_2, (1)), (f_3, (1)), (f_4, (1)), (f_1, (23)), (f_2, (23)), (f_3, (23)), (f_4, (23))\}$.

Then $W_{(1,1)}$ is a subgroup of $S(\Gamma)^\Delta \times S(\Delta)$ of order 8.

Also, we can infer in the same manner that,

$W_{(1,2)} = S(\Gamma)^\Delta(2)_1 \times S(\Delta)_2 = \{f_1, f_2, f_5, f_6\} \times \{(1), (13)\}$

$= \{(f_1, (1)), (f_2, (1)), (f_5, (1)), (f_6, (1)), (f_1, (13)), (f_2, (13)), (f_5, (13)), (f_6, (13))\}$.

Then $W_{(1,2)}$ is a subgroup of $S(\Gamma)^\Delta \times S(\Delta)$ of order 8.

$W_{(1,3)} = S(\Gamma)^\Delta(3)_1 \times S(\Delta)_3 = \{f_1, f_3, f_5, f_7\} \times \{(1), (12)\}$

$= \{(f_1, (1)), (f_3, (1)), (f_5, (1)), (f_7, (1)), (f_1, (12)), (f_3, (12)), (f_5, (12)), (f_7, (12))\}$.

Then $W_{(1,3)}$ is a subgroup of $S(\Gamma)^\Delta \times S(\Delta)$ of order 8.

$W_{(2,1)} = S(\Gamma)^\Delta(1)_2 \times S(\Delta)_1 = \{f_1, f_2, f_3, f_4\} \times \{(1), (23)\}$

$= \{(f_1, (1)), (f_2, (1)), (f_3, (1)), (f_4, (1)), (f_1, (23)), (f_2, (23)), (f_3, (23)), (f_4, (23))\}$.

Then $W_{(1,3)}$ is a subgroup of $S(\Gamma)^\Delta \times S(\Delta)$ of order 8.

$W_{(2,2)} = S(\Gamma)^\Delta(2)_2 \times S(\Delta)_2 = \{f_1, f_2, f_5, f_6\} \times \{(1), (13)\}$

$= \{(f_1, (1)), (f_2, (1)), (f_5, (1)), (f_6, (1)), (f_1, (13)), (f_2, (13)), (f_5, (13)), (f_6, (13))\}$.

Then $W_{(2,2)}$ is a subgroup of $S(\Gamma)^\Delta \times S(\Delta)$ of order 8.

$W_{(2,3)} = S(\Gamma)^\Delta(3)_2 \times S(\Delta)_3 = \{f_1, f_3, f_5, f_7\} \times \{(1), (12)\}$

$= \{(f_1, (1)), (f_3, (1)), (f_5, (1)), (f_7, (1)), (f_1, (12)), (f_3, (12)), (f_5, (12)), (f_7, (12))\}$.

Then $W_{(2,3)}$ is a subgroup of $S(\Gamma)^\Delta \times S(\Delta)$ of order 8.

Finally, we have $W_{(1,1)} = W_{(2,1)}, W_{(1,2)} = W_{(2,2)}, W_{(1,3)} = W_{(2,3)}$.

For $(\gamma, \delta) \in \Gamma \times \Delta$, we have $\left|W_{(\gamma,\delta)}\right| . |W(\gamma, \delta)| = |W|$, then

$|W(\gamma, \delta)| = \frac{|W|}{|W_{(\gamma,\delta)}|} = \frac{48}{8} = 6$.

In this example, we have

$(f_1, (1))(1,1) = (f_2, (1))(1,1) = (f_3, (1))(1,1) = (f_4, (1))(1,1) = (1,1)$

$(f_1, (12))(1,1) = (f_2, (12))(1,1) = (f_3, (12))(1,1)$

$= (f_4, (12))(1,1) = (1,2)$

$(f_1, (13))(1,1) = (f_2, (13))(1,1) = (f_3, (13))(1,1)$

$= (f_4, (13))(1,1) = (1,1)$

$(f_1, (23))(1,1) = (f_2, (23))(1,1) = (f_3, (23))(1,1)$

$= (f_4, (23))(1,1) = (1,1)$

$(f_1, (123))(1,1) = (f_2, (123))(1,1) = (f_3, (123))(1,1)$

$= (f_4, (123))(1,1) = (1,2)$

$(f_1, (132))(1,1) = (f_2, (132))(1,1) = (f_3, (132))(1,1)$

$= (f_4, (132))(1,1) = (1,3)$

$(f_5, (1))(1,1) = (f_6, (1))(1,1) = (f_7, (1))(1,1)$

$= (f_8, (1))(1,1) = (2,1)$

$(f_5, (12)) (1, 1) = (f_6, (12)) (1, 1) = (f_7, (12)) (1, 1)$
$= (f_8, (12)) (1, 1) = (2, 2)$
$(f_5, (13)) (1, 1) = (f_6, (13)) (1, 1) = (f_7, (13)) (1, 1)$
$= (f_8, (13)) (1, 1) = (2, 3)$
$(f_5, (23)) (1, 1) = (f_6, (23)) (1, 1) = (f_7, (23)) (1, 1)$
$= (f_8, (23)) (1, 1) = (2, 1)$
$(f_5, (123)) (1, 1) = (f_6, (123)) (1, 1) = (f_7, (123)) (1, 1)$
$= (f_8, (123)) (1, 1) = (2, 2)$
$(f_5, (132)) (1, 1) = (f_6, (132)) (1, 1) = (f_7, (132)) (1, 1)$
$= (f_8, (132)) (1, 1) = (2, 3)$
Then the orbit of $(1, 1)$ is
$\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\} = \Gamma \times \Delta.$

6   Conclusions

In this paper, study the decomposition of groups, derive results on wreath product of groups, and provide several illustrative examples.

References

[1] M. R. Adhikari, A. Adhikari, *Basic Modern Algebra with Applications*, Springer, 2014.

[2] Audu M. S, *Wreath Product of Permutation Group. A Research Oriented Course In Arithmetics of Elliptic Curves, Groups and Loops*, Lecture Notes Series, National Mathematical Centre, Abuja, 2001.

[3] B. Baumslag and B. Chandler. *Theory and Problems of Group Theory*, New York University, 1968.

[4] O. Bogopolski, *Introduction to Group Theory*, European Mathematical Society, 2008.

[5] W. Chang, *Image processing with wreath product groups*, HARVEY MUDD, 2004.

[6] L. Daniels, *Group Theory and the Rubik's Cube*, Lakehead University, Canada, 2014.

[7] D. Guin et T. Hausberger, *Algèbre Tome 1 Groupes, Corps et Théorie de Galois*, EDP Sciences, 2008.

[8] J. M. Howie, *Fundamentals of Semigroup Theory*, Oxford Science Publications, 1995.

[9] A. A. Ibrahim and M. S. Audu, *On wreath product of permutation groups*, Proyecciones, 26, 1 (2007), 73-90.

[10] D. L. Kreher, *Group Theory Notes*, 2012.

[11] W. Ledermann, *Introduction to Group Theory*, Longman Group Limited, London, 1973.

[12] J. D. P. Meldrum, *Wreath products of groups and semigroups*, Longman, 1995.

[13] A. E. Nagy and C. L. Nehaniv, *Cascade Product of Permutation Groups*, Centre for Computer Science and Informatics, U. K and Centre for Research in Mathematics, Australia, 2013.

[14] J. P. Serre, *Trees*, Springer-Velag Berlin Heidelberg New York, 1980.

[15] H. Straubing, *Finite Automata, Formal Logic, and Circuit Complexity*, Springer Science$^+$ Business Media, LLC, 1994.

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