Flag curvature of a naturally reductive homogeneous Finsler space with infinite series metric

Kirandeep Kaur and Gauree Shanker

Abstract. In this paper, first we derive an explicit formula for the flag curvature of a homogeneous Finsler space with infinite series (α, β)-metric. Next, we deduce it for naturally reductive homogeneous Finsler space with the above mentioned metric.

1 Introduction

The main purpose of this paper is to give a formula for flag curvture of a homogeneous Finsler space with infinite series (α, β)-metric. Finsler geometry is just the Riemannian geometry without the quadratic restriction, as mentioned by S. S. Chern ([4]). The notion of (α, β)-metric in Finsler geometry was first introduced by M. Matsumoto in 1972 ([20]). An (α, β)-metric is a Finsler metric of the form $F = \alpha \phi(s)$, $s = \dfrac{\beta}{\alpha}$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is induced by a Riemannian metric $\tilde{a} = a_{ij} dx^i \otimes dx^j$ on a connected smooth n-manifold M and $\beta = b_i(x)y^i$ is a 1-form on M. It is a well known fact that (α, β)-metrics are the generalizations of the Randers metric, introduced by G. Randers, in ([28]). (α, β)-metrics have various applications in physics and biology ([1]). Consider the rth series (α, β)-metric:

$$F(\alpha, \beta) = \beta \sum_{r=0}^{r=\infty} \left(\frac{\alpha}{\beta} \right)^r .$$

If $r = 1$, then it is a Randers metric.
If $r = \infty$, then

$$F = \frac{\beta^2}{\beta - \alpha}.$$

This metric is called an infinite series (α, β)-metric. Interesting fact about this metric is that, it is the difference of a Randers metric and a Matsumoto metric, and satisfies Shen's lemma (see lemma 2.1).

Some other important class of (α, β)-metrics are Randers metric, Kropina metric, Matsumoto metric and exponential metric. Various properties of (α, β)-metrics have been studied by so many authors ([21], [34], [35], [36], [37]). The study of various types of curvatures of Finsler spaces such as S-curvature, mean Berwald curvature, flag curvature always remain the central idea in the Finsler geometry.

Finsler geometry has been developing rapidly since last few decades, after its emergence in 1917 ([15]). Finsler geometry has been influenced by group theory. The celebrated Erlangen program of F. Klein, posed in 1872 ([17]), greatly influenced the development of geometry. Later, Klein proposed to categorize the geometries by their chacteristic group of transformations. Famous Myers-Steenrod theorem, published in 1939 ([23]), extended the scope of applying Lie theory to all homogeneous Riemannian manifolds.

Theorem(**Myers-Steenrod**): *Let M be a connected Riemannian manifold. Then the group of isometries $I(M)$ of M admits a differentiable structure such that $I(M)$ is a Lie transformation group of M.*

S. Deng and Z. Hou ([8]) have generalized this theorem to the Finslerian case, in 2002. This result opened the door for applying Lie theory to study Finsler geometry. The current important and interesting topics of research in Finsler geometry are homogeneous Finsler spaces, Finsler spaces with (α, β)-metrics, symmetric spaces, Rigidity problems and minimal Finsler surfaces etc. Almost all known examples of Einstein manifolds are homogeneous spaces.

To compute the geometric quantities, specially, curvatures is an interesting problem in homogeneous spaces. In 1976, Milnor ([22]) studied the curvature properties of such spaces by using the formula for the sectional curvature of a left invariant Riemannian metric on a Lie group.

In literature, the term curvature refers to the measurement of how fastly a curve is pulled away from the tangent line at some point in its domain. It should be noted that if a curve is oriented, then curvature remains invariant. The amount that a Riemannian manifold deviates from being Euclidean is defined by the term sectional (or Riemannian) curvature which is the generalization of Gaussian curvature of surfaces. Therefore, sectional curvature of Euclidean space is zero. The notion of sectional curvature was introduced by Riemann ([29]), in 1854. At that time, Riemann had not given a method to calculate the sectional curvature, it was done by Christoffel ([6]) in the year 1869.

The flag curvature in Finsler geometry is a generalization of the sectional curvature of Riemannian geometry. The notion of flag curvature was introduced by Berwald in 1926 ([3]). Flag curvature is an important geometric quantity to characterize Finsler spaces. It is very difficult to compute the flag curvature of a general Finsler space as local coordinates are invoved in computation. It can be calculated for homogeneous Finsler spaces without using local coordinates. Many authors ([22], [24], [27]) have studied homogeneous Riemannian spaces. Some authors ([9], [10], [14], [23]) have tried

to extend some of the results on homogeneous Riemannian spaces to homogeneous Finsler spaces. In the last decade, curvature properties of homogeneous Finsler spaces have been studied by some authors ([12], [13], [32], [36]). Classical Bonnet-Myers theorem, Cartan-Hadamard theorem and some other important results indicate that the sign of flag curvature governs the behaviour of geodesic flow and also it reflects the topology of the underlying manifold ([16]).

In ([9]), Deng and Hou give an algebraic description of invariant Finsler metrics on reductive homogeneous spaces and find a necessary and sufficient condition for a homogeneous space to have invariant Finsler metrics. Also, in the same year, they study invariant Randers metrics on homogeneous manifolds and obtain a formula for the flag curvature of such spaces ([10]). But in 2006, with the help of some counter examples, Esrafilian and Salimi Moghaddam ([13]) prove that the formula for flag curvature of the Randers space, given in ([10]) is incorrect and they derive an explicit formula for the flag curvature of invariant Randers metrics on naturally reductive homogeneous manifolds. In ([31]) and ([30]), Moghaddam give the formula for flag curvature of homogeneous spaces with square metric and Randers metric respectively. In 2014, Moghaddam ([33]) gives the formula for flag curvature of homogeneous space with Matsumoto metric. Recently, in 2016, Parhizkar and Latifi ([26]) have given the formula for flag curvature of invariant (α, β)-metrics on homogeneous Finsler spaces.

2   Preliminaries

In this section, we give some basic concepts of Finsler geometry that are required for next sections. For symbols and notations , we refer ([2], [5]) and ([7]).

Definition 2.1. An n-dimensional real vector space V is called a **Minkowski space** if there exists a real valued function $F : V \longrightarrow \mathbb{R}$ satisfying the following conditions:

(a) F is smooth on $V \backslash \{0\}$,

(b) $F(v) \geq 0 \ \ \forall \ v \in V$,

(c) F is positively homogeneous, i.e., $F(\lambda v) = \lambda F(v), \ \ \forall \ \lambda > 0$,

(d) For a basis $\{v_1, \ v_2, \ ..., \ v_n\}$ of V and $y = y^i v_i \in V$, the Hessian matrix $\left(g_{ij}\right) = \left(\dfrac{1}{2} F^2_{y^i y^j}\right)$ is positive-definite at every point of $V \backslash \{0\}$.

Here, F is called a Minkowski norm.

Definition 2.2. A connected smooth manifold M is called a **Finsler space** if there exists a function $F \colon TM \longrightarrow [0, \infty)$ such that F is smooth on the slit tangent bundle $TM \backslash \{0\}$ and the restriction of F to any $T_p(M), \ p \in M$, is a Minkowski norm. In this case, F is called a Finsler metric.

Let (M, F) be a Finsler space and let (x^i, y^i) be a standard coordinate system in $T_x(M)$. The induced inner product g_y on $T_x(M)$ is given by $g_y(u, v) = g_{ij}(x, y) u^i v^j$, where $u = u^i \dfrac{\partial}{\partial x^i}, \ v = v^i \dfrac{\partial}{\partial x^i} \in T_x(M)$. Also note that $F(x, y) = \sqrt{g_y(y, y)}$.

The condition for an (α, β)-metric to be a Finsler metric is given in following Shen's lemma:

Lemma 2.1. *([5]) Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, where α is a Riemannian metric and β is a 1-form whose length with respect to α is bounded above, i.e., $b := \|\beta\|_\alpha < b_0$, where b_0 is a positive real number. Then F is a Finsler metric if and only if the function $\phi = \phi(s)$ is a smooth positive function on $(-b_0, b_0)$ and satisfies the following condition:*

$$\phi(s) - s\phi'(s) + \left(b^2 - s^2\right)\phi''(s) > 0, \quad |s| \leq b < b_0.$$

Definition 2.3. Let (M, F) be a Finsler space. A diffeomorphism $\phi\colon M \longrightarrow M$ is called an **isometry** if $F\left(\phi(p), d\phi_p(X)\right) = F(p, X)$ for any $p \in M$ and $X \in T_p(M)$.

Definition 2.4. Let G be a Lie group and M a smooth manifold. If G has a smooth action on M, then G is called a Lie transformation group of M.

Definition 2.5. A connected Finsler space (M, F) is said to be homogeneous Finsler space if the action of the group of isometries of (M, F), denoted by $I(M, F)$, is transitive on M.

3 Flag Curvature

Let G be a compact Lie group and H be a closed subgroup of G with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Let $\tilde{a}_0$ be a biinvariant Riemannian metric on G. The tangent space of homogeneous space G/H is given by orthogonal complement $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to $\tilde{a}_0$. Each invariant metric $\tilde{a}$ on G/H is determined by its restriction to $\mathfrak{m}$. The $\mathrm{Ad}(H)$-invariant inner product on $\mathfrak{m}$ arising from $\tilde{a}$ can be extended to an $\mathrm{Ad}(H)$-invariant inner product on $\mathfrak{g}$ by taking $\tilde{a}_0$ for the components in $\mathfrak{h}$. In this way, $\tilde{a}$ determines a unique left invariant metric on G which we also denote by $\tilde{a}$. Also, the values of $\tilde{a}_0$ and $\tilde{a}$ at the identity e of G are inner products on $\mathfrak{g}$ and we denote them by $\langle\langle \ , \ \rangle\rangle$ and $\langle \ , \ \rangle$ respectively. The inner product $\langle \ , \ \rangle$ determines a positive definite endomorphism ψ of $\mathfrak{g}$ such that $\langle Y, Z\rangle = \langle\langle \psi(Y), \ Z\rangle\rangle \ \ \forall \ Y, \ Z \in \mathfrak{g}$.

Püttmann ([27]) has given the following formula for curvature tensor of the invariant metric $\langle \ , \ \rangle$ on the compact homogeneous space G/H:

$$\langle R(X, Y)Z, W\rangle = \frac{1}{2}\bigg\{ \langle\langle B_-(X, Y), [Z, W]\rangle\rangle + \langle\langle [X, Y], B_-(Z, W)\rangle\rangle \bigg\}$$

$$+ \frac{1}{4}\bigg\{ \langle [X, W], [Y, Z]_\mathfrak{m}\rangle - \langle [X, Z], [Y, W]_\mathfrak{m}\rangle - 2\langle [X, Y], [Z, W]_\mathfrak{m}\rangle \bigg\}$$

$$+ \left\langle\langle B_+(X, W), \psi^{-1}B_+(Y, Z)\rangle\right\rangle - \left\langle\langle B_+(X, Z), \psi^{-1}B_+(Y, W)\rangle\right\rangle,$$

where the bilinear maps B_+ (symmetric) and B_- (skew-symmetric) are defined by

$$B_+(X, Y) = \frac{1}{2}\bigg([X, \psi Y] + [Y, \psi X]\bigg),$$

$$B_-(X, Y) = \frac{1}{2}\bigg([\psi X, Y] + [X, \psi Y]\bigg),$$

and $[\ , \ ]_\mathfrak{m}$ is the projection of $[\ , \ ]$ to $\mathfrak{m}$.

Theorem 3.1. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras of the compact Lie group G and its closed subgroup H respectively. Further, let $\tilde{a}_0$ be a bi-invariant metric on G and $\tilde{a}$ be an invariant Riemannian metric on homogeneous space G/H such that $\langle Y, Z \rangle = \langle \langle \psi(Y), Z \rangle \rangle$, where $\psi \colon \mathfrak{g} \longrightarrow \mathfrak{g}$ is a positive definite endomorphism and $Y, Z \in \mathfrak{g}$. Also, suppose that $\tilde{X}$ is an invariant vector field on G/H which is parallel with respect to $\tilde{a}$ and $\sqrt{\tilde{a}(\tilde{X}, \tilde{X})} < 1$ and $\tilde{X}_H = X$. Assume that $F = \dfrac{\beta^2}{\beta - \alpha}$ be an infinite series (α, β)-metric arising from $\tilde{a}$ and $\tilde{X}$ and (P, Y) be a flag in $T_H(G/H)$ such that $\{U, Y\}$ is an orthonormal basis of P with respect to $\langle\ ,\ \rangle$. Then the flag curvature of the flag (P, Y) is given by*

(3.1)
$$K(P, Y) = \left(\frac{\langle X, Y \rangle - 1}{\langle X, Y \rangle} \right)^4 \times$$

$$\frac{\left[\langle X, R(U, Y)Y \rangle \langle X, U \rangle \left\{ \langle X, Y \rangle^2 + 2 \right\} + \langle U, R(U, Y)Y \rangle \langle X, Y \rangle^2 \left\{ \langle X, Y \rangle - 1 \right\} \right]}{\left[\langle X, Y \rangle^2 \left\{ \langle X, Y \rangle^3 + \langle X, Y \rangle^2 - 5 \langle X, Y \rangle + 3 \right\} + 2 \langle X, U \rangle^2 \left\{ \langle X, Y \rangle^2 + 4 \langle X, Y \rangle - 5 \right\} \right]},$$

where the values of $\langle X, R(U, Y)Y \rangle$ and $\langle U, R(U, Y)Y \rangle$ are given by equations (3.6) and (3.8).

Proof. Since $\tilde{X}$ is parallel with respect to $\tilde{a}$, so β is parallel with respect to $\tilde{a}$. Therefore F is a Berwald metric, i.e., the Chern connection of F coincide with the Riemannian connection of $\tilde{a}$. Thus the Finsler metric F has the same curvature tensor as that of the Riemannian metric $\tilde{a}$ and we denote it by R. Using Lemma 3.1 of ([35]), we can write

$$F(Y) = \frac{\langle X, Y \rangle^2}{\langle X, Y \rangle - \sqrt{\langle Y, Y \rangle}}.$$

Also, we know that

$$g_Y(U, V) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} F^2(Y + sU + tV) \bigg|_{s=t=0}.$$

After some calculations, we get

(3.2)
$$g_Y(U, V) = \frac{\langle X, Y \rangle^2}{\left(\langle X, Y \rangle - \sqrt{\langle Y, Y \rangle} \right)^4} \Bigg[\langle X, Y \rangle^2 \langle X, V \rangle \langle X, U \rangle - 4 \langle Y, Y \rangle^{3/2} \langle X, V \rangle \langle X, U \rangle$$

$$+ 6 \langle Y, Y \rangle \langle X, V \rangle \langle X, U \rangle + \frac{\langle X, Y \rangle^2 \langle X, V \rangle \langle U, Y \rangle}{\sqrt{\langle Y, Y \rangle}} - 4 \langle X, Y \rangle \langle X, V \rangle \langle U, Y \rangle$$

$$- \frac{\langle X, Y \rangle^3 \langle U, Y \rangle \langle V, Y \rangle}{\langle Y, Y \rangle^{3/2}} + \frac{\langle X, Y \rangle^3 \langle U, V \rangle}{\sqrt{\langle Y, Y \rangle}} + \frac{4 \langle X, Y \rangle^2 \langle U, Y \rangle \langle V, Y \rangle}{\langle Y, Y \rangle}$$

$$- \langle X, Y \rangle^2 \langle U, V \rangle + \frac{\langle X, Y \rangle^2 \langle X, U \rangle \langle V, Y \rangle}{\sqrt{\langle Y, Y \rangle}} - 4 \langle X, Y \rangle \langle X, U \rangle \langle V, Y \rangle \Bigg].$$

Since $\{U, Y\}$ is an orthonormal basis with respect to $\langle\,,\,\rangle$, therefore equation (3.2) reduces to the following equation:

(3.3)
$$g_Y(U, V) = \frac{\langle X, Y\rangle^2}{(\langle X, Y\rangle - 1)^4}\left[\langle X, Y\rangle^2 \langle X, U\rangle\left\{\langle X, V\rangle + \langle V, Y\rangle\right\} + 2\langle X, V\rangle\langle X, U\rangle\right.$$
$$\left. - 4\langle X, Y\rangle\langle X, U\rangle\langle V, Y\rangle + \langle X, Y\rangle^2\langle U, V\rangle\left\{\langle X, Y\rangle - 1\right\}\right].$$

From equation (3.3), we deduce following three equations:

$$g_Y(Y, Y) = \frac{\langle X, Y\rangle^4}{(\langle X, Y\rangle - 1)^4}\left[\langle X, Y\rangle^2 + 2\langle X, Y\rangle - 3\right],$$

$$g_Y(Y, U) = \frac{\langle X, Y\rangle^3}{(\langle X, Y\rangle - 1)^4}\left[\langle X, Y\rangle^2\langle X, U\rangle + \langle X, Y\rangle\langle X, U\rangle - 2\langle X, U\rangle\right],$$

$$g_Y(U, U) = \frac{\langle X, Y\rangle^2}{(\langle X, Y\rangle - 1)^4}\left[\langle X, U\rangle^2\langle X, Y\rangle^2 + 2\langle X, U\rangle^2 - \langle X, Y\rangle^2 + \langle X, Y\rangle^3\right].$$

Therefore,

(3.4)
$$g_Y(Y, Y)g_Y(U, U) - g_Y^2(Y, U) = \frac{\langle X, Y\rangle^6}{(\langle X, Y\rangle - 1)^8}\left[\langle X, Y\rangle^5 + \langle X, Y\rangle^4 - 5\langle X, Y\rangle^3 + 3\langle X, Y\rangle^2\right.$$
$$\left. + 2\langle X, Y\rangle^2\langle X, U\rangle^2 + 8\langle X, Y\rangle\langle X, U\rangle^2 - 10\langle X, U\rangle^2\right].$$

Also,

(3.5)
$$g_Y(U, R(U, Y)Y) = \frac{\langle X, Y\rangle^2}{(\langle X, Y\rangle - 1)^4}\left[\langle X, R(U, Y)Y\rangle\langle X, U\rangle\left\{\langle X, Y\rangle^2 + 2\right\}\right.$$
$$\langle Y, R(U, Y)Y\rangle\langle X, Y\rangle\langle X, U\rangle\left\{\langle X, Y\rangle - 4\right\}$$
$$\left. \langle U, R(U, Y)Y\rangle\langle X, Y\rangle^2\left\{\langle X, Y\rangle - 1\right\}\right].$$

Using Püttmann's formula, we have

(3.6)
$$\langle X, R(U, Y)Y\rangle = \frac{1}{4}\left(\langle\langle[\psi U, Y] + [U, \psi Y], [Y, X]\rangle\rangle + \langle\langle[U, Y], [\psi Y, X] + [Y, \psi X]\rangle\rangle\right)$$
$$+ \frac{3}{4}\langle[Y, U], [Y, X]_{\mathrm{m}}\rangle$$
$$+ \frac{1}{2}\langle\langle[U, \psi X] + [X, \psi U], \psi^{-1}[Y, \psi Y]\rangle\rangle$$
$$- \frac{1}{4}\langle\langle[U, \psi Y] + [Y, \psi U], \psi^{-1}([Y, \psi X] + [X, \psi Y])\rangle\rangle,$$

$$(3.7) \qquad\qquad\qquad \langle Y, R(U,Y)Y \rangle = 0,$$

$$
\begin{aligned}
\langle U, R(U,Y)Y \rangle = & \frac{1}{2} \Big(\langle\langle [\psi U, Y] + [U, \psi Y], [Y,U] \rangle\rangle \Big) \\
& + \frac{3}{4} \langle [Y,U], [Y,U]_{\mathfrak{m}} \rangle \\
& + \langle\langle [U, \psi U], \psi^{-1}[Y, \psi Y] \rangle\rangle \\
& - \frac{1}{4} \langle\langle [U, \psi Y] + [Y, \psi U], \psi^{-1}\big([Y, \psi U] + [U, \psi Y]\big) \rangle\rangle.
\end{aligned}
$$

(3.8)

The flag curvature is given by

$$K(P,Y) = \frac{g_{Y}(U, R(U,Y)Y)}{g_{Y}(Y,Y)g_{Y}(U,U) - g_{Y}^{2}(Y,U)}.$$

Finally, using equations (3.4) to (3.8) in the above formula, we get equation (3.1). $\square$

4 Naturally reductive homogeneous space

In the literature, there are two different definitions of naturally reductive homogeneous Finsler spaces. In 2004 , Deng and Hou ([9]) have given the first definition of naturally reductive homogeneous Finsler spaces. In 2007, Latifi ([19]) has given another definition of naturally reductive homogeneous Finsler spaces. In 2010, Deng and Hou ([11]) have shown that if a homogeneous Finsler space is naturally reductive in the sense of Latifi, then it must be naturally reductive in the sense of Deng and Hou and also is Berwaldian. Parhizkar and Moghaddam ([25]) prove that both the definitions of naturally reductive homogeneous Finsler spaces are equivalent under the consideration of a mild condition.
Next, recall([9], [18], [19]) the following definitions for later use.

Definition 4.1. A homogeneous space G/H with an invariant Riemannian metric $\tilde{a}$ is called naturally reductive if there exists an $\mathrm{Ad}(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ such that
$$\langle [X,Y]_{\mathfrak{m}}, Z \rangle + \langle Y, [X,Z]_{\mathfrak{m}} \rangle = 0, \ \forall \ X, Y, Z \in \mathfrak{m},$$
where $\langle \ , \ \rangle$ is the bilinear form on $\mathfrak{m}$ induced by $\tilde{a}$.
In particular, if we consider $H = \{e\}$, then $\tilde{a}$ is bi-invariant Riemannian metric on G and the above condition reduces to the following:
$$\langle [X,Y], Z \rangle + \langle Y, [X,Z] \rangle = 0, \ \forall \ X, Y, Z \in \mathfrak{g}.$$

Definition 4.2. A homogeneous Finsler space G/H with an invariant Finsler metric F is said to be naturally reductive if there exists an invariant Riemannian metric $\tilde{a}$ on G/H such that $(G/H, \tilde{a})$ is naturally reductive and the Chern connection of F coincides with the Levi-Civita connection of $\tilde{a}$.

Definition 4.3. A homogeneous Finsler space G/H with an invariant Finsler metric F is called naturally reductive if there exists an $\mathrm{Ad}(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ such that

$$g_Y\big([X,U]_\mathfrak{m}, V\big) \;+\; g_Y\big(U, [X,V]_\mathfrak{m}\big) \;+\; 2\,C_Y\big([X,Y]_\mathfrak{m}, U, V\big) = 0,$$

where $Y \neq 0$, $X, U, V \in \mathfrak{m}$.

Theorem 4.1. *Let $(G/H, F)$ be a homogeneous Finsler space of Berwald type with infinite series metric $F = \dfrac{\beta^2}{\beta - \alpha}$ defined by an invariant Riemannian metric $\langle\,,\,\rangle$ and an invariant vector field $\tilde{X}$ such that $\tilde{X}(H) = X$. Then $(G/H, F)$ is naturally reductive if and only if the underlying Riemannian space $(G/H, \langle\,,\,\rangle)$ is naturally reductive.*

Proof. Let $Y \neq 0$, $Z \in \mathfrak{m}$. From equation (3.2), we can write

(4.1)

$$
\begin{aligned}
g_Y(Y, [Y,Z]_\mathfrak{m}) = {} & \frac{\langle X,Y\rangle^2}{\big(\langle X,Y\rangle - \sqrt{\langle Y,Y\rangle}\big)^4}\bigg[\langle X,Y\rangle^2\,\langle X,[Y,Z]_\mathfrak{m}\rangle\,\langle X,Y\rangle \\
& - 4\langle Y,Y\rangle^{3/2}\langle X,[Y,Z]_\mathfrak{m}\rangle\langle X,Y\rangle + 6\langle Y,Y\rangle\langle X,[Y,Z]_\mathfrak{m}\rangle\langle X,Y\rangle \\
& + \frac{\langle X,Y\rangle^2\langle X,[Y,Z]_\mathfrak{m}\rangle\langle Y,Y\rangle}{\sqrt{\langle Y,Y\rangle}} - 4\langle X,Y\rangle\langle X,[Y,Z]_\mathfrak{m}\rangle\langle Y,Y\rangle \\
& - \frac{\langle X,Y\rangle^3\langle Y,Y_\mathfrak{m}\rangle\langle[Y,Z]_\mathfrak{m},Y\rangle}{\langle Y,Y\rangle^{3/2}} + \frac{\langle X,Y\rangle^3\langle Y,[Y,Z]_\mathfrak{m}\rangle}{\sqrt{\langle Y,Y\rangle}} \\
& + \frac{4\langle X,Y\rangle^2\langle Y,Y\rangle\langle[Y,Z]_\mathfrak{m},Y\rangle}{\langle Y,Y\rangle} - \langle X,Y\rangle^2\langle Y,[Y,Z]_\mathfrak{m}\rangle \\
& + \frac{\langle X,Y\rangle^2\langle X,Y\rangle\langle[Y,Z]_\mathfrak{m},Y\rangle}{\sqrt{\langle Y,Y\rangle}} - 4\langle X,Y\rangle\langle X,Y\rangle\langle[Y,Z]_\mathfrak{m},Y\rangle\bigg]. \\
= {} & \frac{\langle X,Y\rangle^3}{\big(\langle X,Y\rangle - \sqrt{\langle Y,Y\rangle}\big)^4} \times \\
& \bigg[\langle X,[Y,Z]_\mathfrak{m}\rangle\Big\{\langle X,Y\rangle^2 - 4\langle Y,Y\rangle^{3/2} + \langle X,Y\rangle\sqrt{\langle Y,Y\rangle} + 2\langle Y,Y\rangle\Big\} \\
& + \langle Y,[Y,Z]_\mathfrak{m}\rangle\bigg\{\frac{\langle X,Y\rangle^2}{\sqrt{\langle Y,Y\rangle}} - \langle X,Y\rangle\bigg\}\bigg].
\end{aligned}
$$

Since $(G/H, F)$ is of Berwald type, therefore Chern connection of $(G/H, F)$ coincides with the Riemannian connection of $(G/H, \langle\,,\,\rangle)$.
Thus from equation (4.1), we have

(4.2) $$\langle X, [Y,Z]_\mathfrak{m}\rangle = 0, \ \forall\, Z \in \mathfrak{m}.$$

Now, let $(G/H, F)$ be naturally reductive, i.e., in the sense of ([19]), we have

$$g_Y\big([Z,U]_\mathfrak{m}, V\big) + g_Y\big([Z,V]_\mathfrak{m}, U\big) + 2\,C_Y\big([Z,Y]_\mathfrak{m}, U, V\big) = 0, \ \forall\, Y \neq 0, \ U,\, V,\, Z \in \mathfrak{m}.$$

Therefore, we can write

$$g_Y\left([Y,U]_{\mathfrak{m}},V\right) \;+\; g_Y\left([Y,V]_{\mathfrak{m}},U\right) \;+\; 2\,C_Y\left([Y,Y]_{\mathfrak{m}},U,V\right)=0,\ \forall\ Y\neq 0,$$

i.e.,

(4.3) $$g_Y\left([Y,U]_{\mathfrak{m}},V\right) \;+\; g_Y\left([Y,V]_{\mathfrak{m}},U\right)=0.$$

From above equation, we have

(4.4) $$g_Y\left([Y,Z]_{\mathfrak{m}},Y\right)=0.$$

From equation (4.1), (4.2) and (4.4), we get

(4.5) $$\langle[Y,Z]_{\mathfrak{m}},Y\rangle=0.$$

From equation (3.2), we can write

$$
\begin{aligned}
g_Y([Y,U]_{\mathfrak{m}},V) =\; & \frac{\langle X,Y\rangle^2}{\left(\langle X,Y\rangle-\sqrt{\langle Y,Y\rangle}\right)^4}\Bigg[\langle X,Y\rangle^2\,\langle X,V\rangle\,\langle X,[Y,U]_{\mathfrak{m}}\rangle \\
& -4\,\langle Y,Y\rangle^{3/2}\,\langle X,V\rangle\,\langle X,[Y,U]_{\mathfrak{m}}\rangle +6\,\langle Y,Y\rangle\,\langle X,V\rangle\,\langle X,[Y,U]_{\mathfrak{m}}\rangle \\
& +\frac{\langle X,Y\rangle^2\,\langle X,V\rangle\,\langle[Y,U]_{\mathfrak{m}},Y\rangle}{\sqrt{\langle Y,Y\rangle}} -4\,\langle X,Y\rangle\,\langle X,V\rangle\,\langle[Y,U]_{\mathfrak{m}},Y\rangle \\
& -\frac{\langle X,Y\rangle^3\,\langle[Y,U]_{\mathfrak{m}},Y\rangle\,\langle V,Y\rangle}{\langle Y,Y\rangle^{3/2}} +\frac{\langle X,Y\rangle^3\,\langle[Y,U]_{\mathfrak{m}},V\rangle}{\sqrt{\langle Y,Y\rangle}} \\
& +\frac{4\,\langle X,Y\rangle^2\,\langle[Y,U]_{\mathfrak{m}},Y\rangle\,\langle V,Y\rangle}{\langle Y,Y\rangle} -\langle X,Y\rangle^2\,\langle[Y,U]_{\mathfrak{m}},V\rangle \\
& +\frac{\langle X,Y\rangle^2\,\langle X,[Y,U]_{\mathfrak{m}}\rangle\,\langle V,Y\rangle}{\sqrt{\langle Y,Y\rangle}} -4\,\langle X,Y\rangle\,\langle X,[Y,U]_{\mathfrak{m}}\rangle\,\langle V,Y\rangle\Bigg].
\end{aligned}
$$

Using equations (4.2) and (4.5) in above equation, we get

$$g_Y([Y,U]_{\mathfrak{m}},V)=\frac{\langle X,Y\rangle^4\,\langle[Y,U]_{\mathfrak{m}},V\rangle}{\left(\langle X,Y\rangle-\sqrt{\langle Y,Y\rangle}\right)^3\sqrt{\langle Y,Y\rangle}}.$$

Similarly, we get

$$g_Y([Y,V]_{\mathfrak{m}},U)=\frac{\langle X,Y\rangle^4\,\langle[Y,V]_{\mathfrak{m}},U\rangle}{\left(\langle X,Y\rangle-\sqrt{\langle Y,Y\rangle}\right)^3\sqrt{\langle Y,Y\rangle}}.$$

Adding the above two equations, we get
(4.6)
$$g_Y([Y,U]_{\mathfrak{m}},V)+g_Y([Y,V]_{\mathfrak{m}},U)=\frac{\langle X,Y\rangle^4}{\left(\langle X,Y\rangle-\sqrt{\langle Y,Y\rangle}\right)^3\sqrt{\langle Y,Y\rangle}}\left\{\langle[Y,U]_{\mathfrak{m}},V\rangle+\langle[Y,V]_{\mathfrak{m}},U\rangle\right\}.$$

From equations (4.3) and (4.6), we get

$$\langle [Y,U]_{\mathfrak{m}}, V \rangle + \langle [Y,V]_{\mathfrak{m}}, U \rangle = 0.$$

Hence, $(G/H, \langle \ , \ \rangle)$ is naturally reductive.

Conversely, let $(G/H, \langle \ , \ \rangle)$ be naturally reductive. Using equation (3.2), we can write

$$(4.7) \qquad
\begin{aligned}
g_Y\left([Z,U]_{\mathfrak{m}}, V\right) = & \frac{\langle X,Y\rangle^3}{\left(\langle X,Y\rangle - \sqrt{\langle Y,Y\rangle}\right)^4}\left[\langle [Z,U]_{\mathfrak{m}}, V\rangle \left\{ \frac{\langle X,Y\rangle^2}{\sqrt{\langle Y,Y\rangle}} - \langle X,Y\rangle \right\} \right. \\
& + \langle [Z,U]_{\mathfrak{m}}, Y\rangle \left\{ \frac{\langle X,Y\rangle\langle X,V\rangle}{\sqrt{\langle Y,Y\rangle}} - 4\langle X,V\rangle \right. \\
& \left. \left. - \frac{\langle X,Y\rangle^2\langle V,Y\rangle}{\langle Y,Y\rangle^{3/2}} + \frac{4\langle X,Y\rangle\langle V,Y\rangle}{\langle Y,Y\rangle} \right\} \right].
\end{aligned}$$

Similarly, we can write

$$(4.8) \qquad
\begin{aligned}
g_Y\left([Z,V]_{\mathfrak{m}}, U\right) = & \frac{\langle X,Y\rangle^3}{\left(\langle X,Y\rangle - \sqrt{\langle Y,Y\rangle}\right)^4}\left[\langle [Z,V]_{\mathfrak{m}}, U\rangle \left\{ \frac{\langle X,Y\rangle^2}{\sqrt{\langle Y,Y\rangle}} - \langle X,Y\rangle \right\} \right. \\
& + \langle [Z,V]_{\mathfrak{m}}, Y\rangle \left\{ \frac{\langle X,Y\rangle\langle X,U\rangle}{\sqrt{\langle Y,Y\rangle}} - 4\langle X,U\rangle \right. \\
& \left. \left. - \frac{\langle X,Y\rangle^2\langle U,Y\rangle}{\langle Y,Y\rangle^{3/2}} + \frac{4\langle X,Y\rangle\langle U,Y\rangle}{\langle Y,Y\rangle} \right\} \right].
\end{aligned}$$

Next, Cartan tensor is given by

$$C_Y\left(Z,U,V\right) = \frac{1}{2}\frac{d}{dt}\left[g_{Y+tV}\left(Z,U\right) \right]\Big|_{t=0}.$$

After computations, we get

(4.9)
$$
\begin{aligned}
2C_Y\left([Z,Y]_{\mathfrak{m}}, U, V\right) = & \frac{\langle X,Y\rangle^3}{\left(\langle X,Y\rangle - \sqrt{\langle Y,Y\rangle}\right)^4} \times \\
& \left[\langle [Z,Y]_{\mathfrak{m}}, U\rangle \left\{ \frac{\langle X,Y\rangle\langle X,V\rangle}{\sqrt{\langle Y,Y\rangle}} - 4\langle X,V\rangle - \frac{\langle X,Y\rangle^2\langle V,Y\rangle}{\langle Y,Y\rangle^{3/2}} + \frac{4\langle X,Y\rangle\langle V,Y\rangle}{\langle Y,Y\rangle} \right\} \right. \\
& \left. + \langle [Z,Y]_{\mathfrak{m}}, V\rangle \left\{ \frac{\langle X,Y\rangle\langle X,U\rangle}{\sqrt{\langle Y,Y\rangle}} - 4\langle X,U\rangle - \frac{\langle X,Y\rangle^2\langle U,Y\rangle}{\langle Y,Y\rangle^{3/2}} + \frac{4\langle X,Y\rangle\langle U,Y\rangle}{\langle Y,Y\rangle} \right\} \right].
\end{aligned}$$

Adding equations (4.7), (4.8) and (4.9), we get

$$g_Y\left([Z,U]_\mathfrak{m}, V\right) + g_Y\left([Z,V]_\mathfrak{m}, U\right) + 2C_Y\left([Z,Y]_\mathfrak{m}, U, V\right)$$

$$= \frac{\langle X,Y\rangle^3}{\left(\langle X,Y\rangle - \sqrt{\langle Y,Y\rangle}\right)^4} \times$$

$$\left[\left\{\langle [Z,U]_\mathfrak{m}, V\rangle + \langle [Z,V]_\mathfrak{m}, U\rangle\right\}\left\{\frac{\langle X,Y\rangle^2}{\sqrt{\langle Y,Y\rangle}} - \langle X,Y\rangle\right\}\right.$$

$$+ \left\{\langle [Z,U]_\mathfrak{m}, Y\rangle + \langle [Z,Y]_\mathfrak{m}, U\rangle\right\}\left\{\frac{\langle X,Y\rangle\langle X,V\rangle}{\sqrt{\langle Y,Y\rangle}} - 4\langle X,V\rangle - \frac{\langle X,Y\rangle^2\langle V,Y\rangle}{\langle Y,Y\rangle^{3/2}} + \frac{4\langle X,Y\rangle\langle V,Y\rangle}{\langle Y,Y\rangle}\right\}$$

$$+ \left\{\langle [Z,V]_\mathfrak{m}, Y\rangle + \langle [Z,Y]_\mathfrak{m}, V\rangle\right\}\left\{\frac{\langle X,Y\rangle\langle X,U\rangle}{\sqrt{\langle Y,Y\rangle}} - 4\langle X,U\rangle - \frac{\langle X,Y\rangle^2\langle U,Y\rangle}{\langle Y,Y\rangle^{3/2}} + \frac{4\langle X,Y\rangle\langle U,Y\rangle}{\langle Y,Y\rangle}\right\}\right].$$

$$= \frac{\langle X,Y\rangle^3}{\left(\langle X,Y\rangle - \sqrt{\langle Y,Y\rangle}\right)^4}\left[0 + 0 + 0\right], \text{because } (G/H, \langle\ ,\ \rangle) \text{ is naturally reductive.}$$

Hence, $(G/H, F)$ is naturally reductive.   □

5   Flag curvature of naturally reductive homogeneous space

In ([25]), authors find a formula for flag curvature of naturally reductive homogeneous (α, β)-metric spaces in the sense of Deng and Hou.

Here, we derive formula for flag curvature of naturally reductive homogeneous Finsler space with infinite series (α, β)-metric in the sense of Deng and Hou.

Theorem 5.1. *Let G/H be naturally reductive homogeneous Finsler space with infinite series (α, β)-metric $F = \dfrac{\beta^2}{\beta - \alpha}$, defined by an invariant Riemannian metric $\tilde{a}$ and an invariant vector field $\tilde{X}$ on G/H, Assume that $\tilde{X}_H = X$. and (P,Y) be a flag in $T_H(G/H)$ such that $\{U,Y\}$ is an orthonormal basis of P with respect to $\langle\ ,\ \rangle$. Then the flag curvature of the flag (P,Y) is given by*

(5.1)

$$K(P,Y) = \left(\frac{\langle X,Y\rangle - 1}{\langle X,Y\rangle}\right)^4 \times$$

$$\frac{\left[\left\{\frac{1}{4}\left\langle X, \left[Y, [U,Y]_\mathfrak{m}\right]_\mathfrak{m}\right\rangle + \left\langle X, \left[Y, [U,Y]_\mathfrak{h}\right]\right\rangle\right\}\langle X,U\rangle\left\{\langle X,Y\rangle^2 + 2\right\} + \left\{\frac{1}{4}\left\langle U, \left[Y, [U,Y]_\mathfrak{m}\right]_\mathfrak{m}\right\rangle + \left\langle U, \left[Y, [U,Y]_\mathfrak{h}\right]\right\rangle\right\}\langle X,Y\rangle^2\left\{\langle X,Y\rangle - 1\right\}\right]}{\left[\langle X,Y\rangle^2\left\{\langle X,Y\rangle^3 + \langle X,Y\rangle^2 - 5\langle X,Y\rangle + 3\right\} + 2\langle X,U\rangle^2\left\{\langle X,Y\rangle^2 + 4\langle X,Y\rangle - 5\right\}\right]}.$$

Proof. Since F is naturally reductive, using proposition 3.4 in ([18]) on page 202, we have

$$R(U,Y)Y = \frac{1}{4}\left[Y, [U,Y]_\mathfrak{m}\right]_\mathfrak{m} + \left[Y, [U,Y]_\mathfrak{h}\right], \ \forall\ U,Y \in \mathfrak{m}.$$

Substituting above value in equation (3.1), we get

$$K(P,Y) = \left(\frac{\langle X,Y\rangle - 1}{\langle X,Y\rangle}\right)^4 \times$$

$$\frac{\left[\left\langle X, \frac{1}{4}\big[Y,[U,Y]_{\mathfrak{m}}\big]_{\mathfrak{m}} + \big[Y,[U,Y]_{\mathfrak{h}}\big]\right\rangle \langle X,U\rangle \{\langle X,Y\rangle^2 + 2\} \right.}{\left[\langle X,Y\rangle^2 \{\langle X,Y\rangle^3 + \langle X,Y\rangle^2 - 5\langle X,Y\rangle + 3\} + 2\langle X,U\rangle^2 \{\langle X,Y\rangle^2 + 4\langle X,Y\rangle - 5\}\right]}.$$

Simplifying the above equation, we get equation (5.1). □

If $H = \{e\}$, then we have the following corollary:

Corollary 5.2. *Let $F = \dfrac{\beta^2}{\beta - \alpha}$ be defined by a bi-invariant Riemannian metric $\tilde{a}$ on a Lie group G and a left invariant vector field X on G such that the Chern connection of F coincides with the Riemannian connection of $\tilde{a}$. Assume that (P,Y) be a flag in $T_e(G)$ such that $\{U,Y\}$ is an orthonormal basis of P with respect to $\langle\ ,\ \rangle$. Then the flag curvature of the flag (P,Y) is given by*

(5.2)
$$K(P,Y) = \left(\frac{\langle X,Y\rangle - 1}{\langle X,Y\rangle}\right)^4 \times$$

$$\frac{\left[\langle X, [Y,[U,Y]]\rangle \langle X,U\rangle \{\langle X,Y\rangle^2 + 2\} + \langle U, [Y,[U,Y]]\rangle \langle X,Y\rangle^2 \{\langle X,Y\rangle - 1\}\right]}{4\left[\langle X,Y\rangle^2 \{\langle X,Y\rangle^3 + \langle X,Y\rangle^2 - 5\langle X,Y\rangle + 3\} + 2\langle X,U\rangle^2 \{\langle X,Y\rangle^2 + 4\langle X,Y\rangle - 5\}\right]}.$$

Proof. Since $\langle\ ,\ \rangle$ is bi-invariant, we have

$$R(U,Y)Y = \frac{1}{4}\big[Y,[U,Y]\big].$$

Substituting above value in equation (3.1), we get

$$K(P,Y) = \left(\frac{\langle X,Y\rangle - 1}{\langle X,Y\rangle}\right)^4 \times$$

$$\frac{\left[\left\langle X, \frac{1}{4}[Y,[U,Y]]\right\rangle \langle X,U\rangle \{\langle X,Y\rangle^2 + 2\} + \left\langle U, \frac{1}{4}[Y,[U,Y]]\right\rangle \langle X,Y\rangle^2 \{\langle X,Y\rangle - 1\}\right]}{\left[\langle X,Y\rangle^2 \{\langle X,Y\rangle^3 + \langle X,Y\rangle^2 - 5\langle X,Y\rangle + 3\} + 2\langle X,U\rangle^2 \{\langle X,Y\rangle^2 + 4\langle X,Y\rangle - 5\}\right]}.$$

Simplifying the above equation, we get equation (5.2). □

References

[1] P. L. Antonelli, R. S. Ingarden, and M. Matsumoto, *The Theory of Sprays and Finsler spaces with Applications in Physics and Biology*, Kluwer Academic Publishers, Netherlands, 58, 1993.

[2] D. Bao, S. S. Chern and Z. Shen, *An Introduction to Riemann-Finsler Geometry*, GTM-200, Springer-Verlag 2000.

[3] L. Berwald, *Über parallelübertragung in Räumen mit allgemeiner Massbestimmung*, *Jber. Deutsch. Math.-Verein.*, **34** (1926), 213-220.

[4] S. S. Chern, Finsler geometry is just Riemannian geometry without the quadratic restriction, *Notices Amer. Math. Soc.*, **43** (9) (1996), 959-963.

[5] S. S. Chern and Z. Shen, *Riemann-Finsler Geometry*, Nankai Tracts in Mathematics, Vol. 6, World Scientific Publishers, 2005.

[6] E. B. Christoffel, *Über die Transformation der homogenen Differentialausdrücke zweiten Grades*, *J. Riene. Angew. Math.*, **70** (1869), 46-70.

[7] S. Deng, *Homogeneous Finsler Spaces*, Springer Monographs in Mathematics, New York, 2012.

[8] S. Deng and Z. Hou, The group of isometries of a Finsler space, *Pacific J. Math*, **207** (2002), 149-155.

[9] S. Deng and Z. Hou, Invariant Finsler Metrics on Homogeneous Manifolds, *J. Phys. A: Math. Gen.*, **37** (2004), 8245–8253.

[10] S. Deng and Z. Hou, Invariant Randers metrics on Homogeneous Riemannian manifolds, *J. Phys. A: Math. Gen.* **37** (2004), 4353-4360; Corrigendum, *ibid*, **39** (2006), 5249-5250.

[11] S. Deng and Z. Hou, Naturally reductive homogeneous Finsler spaces, *Manuscripta Math.*, **131** (2010), 215-229.

[12] S. Deng and Z. Hu, On flag curvature of homogeneous Randers spaces, *Canad. J. Math.*, **65** (2013), 66-81.

[13] E. Esrafilian and H. R. Salimi Moghaddam, Flag curvature of invariant Randers metrics on homogeneous manifolds, *J. Phys. A: Math. Gen.*, **39** (2006), 3319-3324.

[14] E. Esrafilian and H. R. Salimi Moghaddam, Induced Invariant Finsler Metrics on Quotient Groups, *Balkan Journal of Geometry and Its Applications*, **11** (1) (2006), 73–79.

[15] P. Finsler, *Über Kurven und Flächen in allgemeinen Räumen*, (Dissertation, Göttingen, 1918), *Birkhäuser Verlag*, Basel, 1951.

[16] L. Huang, Flag curvatures of homogeneous Finsler spaces, *European Journal of Mathematics*, **3** (4) (2017), 1000–1029.

[17] F. Klein, Vergleichende Betrachtungen über neuere geometrische Forschungen, Andreas Deichert, Erlangen, 1872; English trans. M. W. Haskell, with additional footnotes by the author, A comparative review of recent researches in geometry, *Bulletin of the New York Mathematical Society*, **2** (1892–1893), 215-249.

[18] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Interscience Publishers, Vol. 1, 1963; Vol. 2, 1969.

[19] D. Latifi, Homogeneous geodesics in homogeneous Finsler spaces, *J. Geom. Phys.*, **57** (2007), 1421-1433.

[20] M. Matsumoto, On a C-reducible Finsler space, *Tensor N. S.*, **24** (1972), 29-37.

[21] M. Matsumoto, Theory of Finsler spaces with (α, β) metric, *Rep. on Math. Phys.*, **31** (1992), 43-83.

[22] J. Milnor, Curvature of left invariant Riemannian metrics on Lie groups, *Adv. Math.*, **21** (1976), 293-329.

[23] S. B. Myers and N. Steenrod, The group of isometries of a Riemannian manifold, *Ann. Math.*, **40** (1939), 400-416.

[24] K. Nomizu, Invariant affine connections on homogeneous spaces, *Amer. J. Math.*, **76** (1954), 33-65.

[25] M. Parhizkar and H. R. Salimi Moghaddam, Naturally reductive homogeneous (α, β)-metric spaces, *Preprint*, arXiv:1305.5994[math.DG].

[26] M. Parhizkar and D. Latifi, On the flag curvature of invariant (α, β)-metrics, *International Journal of Geometric Methods in Modern Physics*, **13** (4) (2016), 1650039(1-11).

[27] T. Püttmann, Optimal pinching constants of odd dimensional homogeneous spaces, *Invent. Math.*, **138** (1999), 631-684.

[28] G. Randers, On an asymmetric metric in the four-space of general relativity, *Phys. Rev.*, **59** (1941), 195-199.

[29] B. Riemann, On the hypotheses which lie at the foundations of geometry, translated from the German by Henry S. White In: A Source Book in Mathematics by David E. Smith, Dover edition, **2** (1959), Dover, New York, 411-425.

[30] H. R. Salimi Moghaddam, On the flag curvature of invariant Randers metrics, *Math. Phys. Anal. Geom.*, **11** (2008), 1-9.

[31] H. R. Salimi Moghaddam, The flag curvature of invariant (α, β)-metrics of type $\dfrac{(\alpha + \beta)^2}{\alpha}$, *J. Phys. A: Math. Theor.*, **41** (2008).

[32] H. R. Salimi Moghaddam, On the curvature of invariant Kropina metrics, *Int. Electron. J. Geom.*, **4** (2) (2011), 136-140.

[33] H. R. Salimi Moghaddam, Invariant Matsumoto Metrics on Homogeneous Spaces, *Osaka J. Math.*, **51** (2014), 39–45.

[34] G. Shanker and K. Kaur, Naturally reductive homogeneous space with an invariant (α, β)-metric, *Lobachevskii Journal of Mathematics*, **40** (2) (2019), 210-218.

[35] G. Shanker and K. Kaur, Homogeneous Finsler space with infinite series (α, β)-metric, *Applied Sciences*, **21** (2019), 219-235.

[36] G. Shanker and K. Kaur, Homogeneous Finsler space with exponential metric, *Advances in Geometry*, **20** (3) (2020), 391-400.

[37] G. Shanker and Ravindra, On Randers change of exponential metric, *Applied Sciences*, **15** (2013), 94-103.

Authors' addresses:

Kirandeep Kaur
Department of Mathematics,
Punjabi University College Ghudda, Bathinda, Punjab, India.
E-mail: kirandeepiitd@gmail.com

Gauree Shanker (orresponding author)
Department of Mathematics and Statistics,
School of Basic and Applied Sciences,
Central University of Punjab,
Bathinda, Punjab-151001, India.
E-mail: gshankar@cup.ac.in