Lie symmetries and exact solutions for the one-dimensional Kuramoto-Sivashinsky equation

M. Nadjafikhah and S. Dodangeh

Abstract. In this paper, the classical and non-classical symmetries of the one dimensional Modified Kuramoto-Sivashinsky equation (MKS) as well as its similarity solutions using infinitesimal criterion and compatibility condition approaches, have been discussed. Looking at the adjoint representation of the obtained symmetry group on its Lie algebra, we find the preliminary classification of its group-invariant solutions which provides new exact solutions to MKS equation. Also, some aspects of its symmetry properties are given. The latter provides other new exact solutions to MKS equation.

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1 Introduction

The symmetry group method plays an important role in the analysis of differential equations. The history of group classification methods goes back to Sophus Lie. The first paper on this subject is [8], where Lie proves that a linear two-dimensional second-order PDE may admit at most a three-parameter invariance group (apart from the trivial infinite parameter symmetry group, which is due to linearity). He computed the maximal invariance group of the one-dimensional heat conductivity equation and utilized this symmetry to construct its explicit solutions. The theory of Lie systems [9, 20] deals with non-autonomous systems of first order ordinary differential equations [4] and then for partial differential equations [5] such that all their solutions can be written in terms of generic sets of particular solutions and some constants, by means of a time-independent function. Such functions are called *superposition rules* and the systems admitting this mathematical property are called *Lie systems*. Lie succeeded in characterizing systems admitting a superposition rule. Saying it the modern way, he performed symmetry reduction of the heat equation. Nowadays symmetry reduction is one of the most powerful tools for solving nonlinear partial differential equations (PDEs). Recently, there have been several generalizations of the classical Lie group method for symmetry reductions. Ovsiannikov has developed the

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method of partially invariant solutions, [15, 16]. His approach is based on the concept of an equivalence group, which is a Lie transformation group acting in the extended space of independent variables, functions and their derivatives, and preserving the class of partial differential equations under study. The original form of the Kuramoto-Sivashinsky equation in one dimensional framework is $u_t = -u_{xxxx} - u_{xx} - u_x^2$, where has been derived e.g. in the context of chemical turbulence [7]. It is also important because it can describe the flow of a falling fluid film [18].

In this paper, we deal with the following equation:

$$\Delta_1 \ : \ u_t + u_{xx} + u_{xxxx} + (\lambda - 1)u_x^2 - \sigma\lambda u_{xx}^2 = 0, \tag{1.1}$$

where λ and σ are arbitrary constants and $\lambda \neq 1$. This equation is a Kuramoto-Sivashinsky type and called one dimensional Modified Kuramoto-Sivashinsky equation [1]. Here $u(x,t)$ is an unknown function and x, t are space and time variables respectively. This equation apply to model the various physical phonomania, e.g. problems of thermodynamic phase transition arise naturally in solidification, combustion. It is a host of other fields, also, appeared to describe interfaces which are marginally long-wave unstable (See more information in [1, 17]).

Physical applications and mathematical properties of this type of equations have been a motivation for some other papers (See e.g. [17, 19, 7, 18]). Here, symmetry classification for similarity solutions of its one dimensional modified type is focused.

This paper organized as follows: Section 2 is devoted to perform the basic similarity reductions for this system. Reduced equations and exact solutions associated with the symmetries are also obtained in this section. We find an optimal system of one-dimensional sub-algebras for symmetry algebra of this system, in Section 3. We then compute, in Section 4, the invariants associated with the symmetry operators by integrating the characteristic equations. Section 5 provides to obtain some new non-classical symmetries and then some new exact solutions of MKS equation corresponding obtained non-classical symmetries are obtained. We summarize our results in Section 6 and discuss their implications for the full intrinsic

2 Lie symmetry of the system

Let a partial differential equation contains p independent variables and q dependent variables is given. The one-parameter Lie group of transformations

$$\big(x^i,\, u^\alpha\big) \longmapsto \big(x^i,\, u^\alpha\big) + \epsilon\big(\xi^i(x,u),\, \varphi^\alpha(x,u)\big) + O(\epsilon^2),$$

where $i = 1, \ldots, p$ and $\alpha = 1, \ldots, q$. The action of the Lie group can be recovered from that of its associated infinitesimal generators. We consider general vector field

$$X = \sum_{i=1}^{p} \xi^i(x,u)\,\partial_{x_i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x,u)\,\partial_{u^\alpha}. \tag{2.1}$$

on the space of independent and dependent variables. Therefore, the *Lie characteristic* of the vector field X given by (2.7) is the function

$$Q^\alpha(x, u^{(1)}) = \varphi^\alpha(x,u) - \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial u^\alpha}{\partial x_i}, \qquad \alpha = 1, \ldots, q. \tag{2.2}$$

Theorem 2.1. *[14] Let X be a vector field given by (2.1), and let $Q = (Q^1, \ldots, Q^q)$ be its characteristic, as in (2.2). The n^{th} prolongation of X is given explicitly by*

$$X^{(n)} = \sum_{i=1}^{p} \xi^i(x,u)\,\partial_{x^i} + \sum_{\alpha=1}^{q}\sum_{J} \varphi_J^\alpha(x,u^{(n)})\,\partial_{u_J^\alpha}, \tag{2.3}$$

with coefficients $\varphi^\alpha_{J,i} = D_i\varphi^\alpha_J - \sum_{j=1}^{p} D_i\xi^j u^\alpha_{J,j}$. Here, $J = (j_1, \ldots, j_k)$, with $1 \le k \le p$ is a multi-indices, and D_i represents the total derivative with respect to the i-th independent variable x^i the first order differential operator $D_i = \partial/\partial x^i + \sum_{\alpha=1}^{q}\sum_J u^\alpha_{J,i}\partial/\partial u^\alpha_J$.

Theorem 2.2. *[14] A connected group of transformations G is a symmetry group of a differential equation $\Delta = 0$ if and only if the classical infinitesimal symmetry condition*

$$X^{(n)}(\Delta) \equiv 0 \qquad \text{mod} \quad \Delta = 0, \tag{2.4}$$

holds for every infinitesimal generator $X \in \mathfrak{g}$ of G.

As before, in this section, we will have an attempt to perform the basic similarity reductions for the equation (1.1) using infinitesimal criterion method (Find more information in [14, 13]).

Theorem 2.3. *Let $X := \xi^1(x,t,u)\partial_x + \xi^2(x,t,u)\partial_t + \varphi(x,t,u)\partial_u$ be an infinitesimal generator of the classical Lie point symmetry group for equation Δ_1, we then have:*

$$\xi^1 = c_1 + 2t(\lambda-1)c_4, \qquad \xi^2 = c_2, \qquad \varphi = c_3 + c_4 x, \tag{2.5}$$

where c_is for $i = 1,2,3,4$ are arbitrary constants.

Proof: Applying the invariance condition (2.4) in Theorem 2.2 for (1.1) equation, we find

$$X^{(4)}\Big[u_t + u_{xx} + u_{xxxx} + (\lambda-1)u_x^2 - \sigma\lambda u_{xx}^2\Big] \equiv 0 \quad \text{mod (1.1)}, \tag{2.6}$$

where $X^{(4)}$ is the fourth prolongation obtained using (2.3) in Theorem 2.1, with the following form

$$X^{(4)} = \xi^1\partial_x + \xi^2\partial_t + \sum_{\neq J=0}^{4} \varphi^J(x,t,u^{(\neq J)})\partial_{u_J}, \tag{2.7}$$

with coefficients $\varphi^J = D_J Q + \xi^1 u_{J,x} + \xi^2 u_{J,t}$, where $D_J = D_{j_1}\ldots D_{j_k}$ is a multi-index $J = (j_1, \ldots, j_k)$ and $Q = \varphi - \xi^1 u_x - \xi^2 u_t$, depending on x, t, u and first order derivatives of u, is the same Lie characteristic introduced in (2.2). Substituting (2.7) into (2.6), and introducing the relation $u_t = -u_{xx} - u_{xxxx} + (1-\lambda)u_x^2 + \sigma\lambda u_{xx}^2$ to eliminate u_t we are left with a polynomial equation involving the various derivatives of $u(x,t)$ whose coefficients are certain derivatives of ξ^1, ξ^2 and φ. We can equate the individual coefficients to zero, and then it leads to the complete set of determining equations:

$$\xi^1_x, \xi^1_u, \xi^1_{tt}, \xi^2_x, \xi^2_t, \xi^2_u, \varphi_u = 0, \quad 2(\lambda-1)\varphi_x = \xi^1_t, \quad (\lambda \neq 1).$$

Finally, solving this system leads to (2.5). □

Corollary 2.4. *The Lie algebra $\mathfrak{g}$ of infinitesimal projectable symmetries of equation (1.1) is spanned by the four vector fields*

$$X_1 = \partial_t, \qquad X_2 = \partial_x, \qquad X_3 = \partial_u, \qquad X_4 = 2t(\lambda - 1)\partial_x + x\partial_u. \tag{2.8}$$

The commutator table of Lie algebra $\mathfrak{g}$ for (1.1) is given Table 1.

Table 1: Commutation relations satisfied by infinitesimal generators (2.8)

$[\,,\,]$	X_1	X_2	X_3	X_4
X_1	0	0	0	$2(\lambda-1)X_2$
X_2	0	0	0	X_3
X_3	0	0	0	0
X_4	$2(1-\lambda)X_2$	$-X_3$	0	0

To obtain the group transformation which is generated by the infinitesimal generators (2.8), we need to find its integral curves. Consequently, we conclude the following theorem:

Theorem 2.5. *If $g_s^i(x,t,u)$ be the one parameter group generated by (2.8) then*

$$\begin{aligned} &g_s^1 = (x, t+s, u), \quad g_s^2 = (x+s, t, u), \\ &g_s^3 = (x, t, u+s), \quad g_s^4 = \big(x + 2ts(\lambda-1), t, u + ts^2(\lambda-1) + xs\big), \end{aligned} \tag{2.9}$$

In general to each one parameter subgroups of the full symmetry group of a system there will correspond a family of solutions, called *group-invariant solutions*.

Theorem 2.6. *If $u = f(x,t)$ is a solution of (1.1), so are the functions*

$$\begin{aligned} &u^1 = f(x, t-s), \quad u^2 = f(x-s, t), \\ &u^3 = f(x,t) - s, \quad u^4 = f\big(x - 2ts(\lambda-1), t\big) - ts^2(\lambda-1) - xs, \end{aligned} \tag{2.10}$$

where $u^i = g_i^s \cdot f(x,t)$, $i = 1,2,3,4$, and $s \ll 1$ is any positive number.

3 Optimal system of sub-algebras

As it is well known, the Lie group theoretic method plays an important role to find exact solutions of differential equations as well as performing the symmetry reductions. Since any linear combination of infinitesimal generators is also an infinitesimal generator, there are always infinitely many different symmetry subgroups for the essentially different types of solutions is necessary and significant for a complete understanding of the invariant solutions. As any transformation in the full symmetry group maps a solution to another solution, it is sufficient to find invariant solutions which are not related by transformations in the full symmetry group, this has led to the concept of an optimal system [13]. The problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras. For one-dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation. This problem is attacked by the naive approach of taking a general element in the Lie algebra and subjecting it to various adjoint transformations so as to simplify it as much as possible. The idea of using the adjoint representation to classify group-invariant solutions was due to [13] and [14].

The adjoint action is given by the Lie series $\mathrm{Ad}(\exp(sX_i)X_j) = X_j - s\mathbf{ad}_{X_i}(X_j) + (s^2/2)\mathbf{ad}^2_{X_i}(X_j) - \cdots$, where $\mathbf{ad}_X(Y) = [X, Y]$.

Table 2: Adjoint representation of infinitesimal symmetries of the equation (1.1)

Ad	X_1	X_2	X_3	X_4
X_1	X_1	X_2	X_3	$X_4 + 2(1-\lambda)sX_2$
X_2	X_1	X_2	X_3	$X_4 - sX_3$
X_3	X_1	X_2	X_3	X_4
X_4	$X_1 + 2s(\lambda-1)X_2 + (s^2/2)X_3$	$X_2 + sX_3$	X_3	X_4

Theorem 3.1. *A one-dimensional optimal system of (1.1) is provided by those generated by*

$$\begin{array}{llll} Y_1 = X_1, & Y_2 = X_2, & Y_3 = X_3, & Y_4 = X_4, \\ Y_5 = X_4 + X_1, & Y_6 = X_4 - X_1, & Y_7 = X_3 + X_1, & Y_8 = X_3 - X_1, \\ Y_9 = X_2 + X_1, & Y_{10} = X_2 - X_1. & & \end{array} \tag{3.1}$$

Proof: Let $X = \sum_{i=1}^4 a_iX_i$, be a nonzero vector field of $\mathfrak{g}$. We will simplify as many of the coefficients a_i, $i = 1, \cdots, 4$ as possible through proper adjoint applications on X. We follow our aim in the below easy cases:

I. At first, assume that $a_4 \neq 0$. Scaling X if necessary, we can assume that $a_4 = 1$ and so we get $X = a_1X_1 + a_2X_2 + a_3X_3 + X_4$. Using the table of adjoint (Table 2), if we act on X with $\mathrm{Ad}(\exp a_3X_2)$, the coefficient of X_3 can be vanished $X' = a_1X_1 + a_2X_2 + X_4$. Then we apply $\mathrm{Ad}(\exp \frac{\mathrm{a}_2}{2(\lambda-1)}\mathrm{X}_1)$ on X' to cancel the coefficient of X_2, that is

$$X'' = a_1X_1 + X_4, \tag{3.2}$$

which by choosing a_1 equal to ±1 or zero, (3.2) leads to Y_4, Y_5 and Y_6 in (3.1).

II. The remaining one-dimensional subalgebras are spanned by vector fields of the form X with $a_4 = 0$.

   a. If $a_3 \neq 0$ then by scaling X, we can assume that a_3=1. Now by the action of $\mathrm{Ad}(\exp \frac{a_2}{(1-\lambda)a_1}X_4)$ on X, we can cancel the coefficient of X_2, that is

$$\bar{X} = a_1X_1 + X_3, \tag{3.3}$$

which by choosing a_1 equal to ± 1 or zero, (3.3) leads to Y_3, Y_7 and Y_8 in (3.1).

b. Let $a_3 = 0$ and $a_2 \neq 0$, then by scaling X is in the form

$$\tilde{X} = a_1 X_1 + X_2, \tag{3.4}$$

which by choosing a_1 equal to ± 1 or zero, (3.4) leads to Y_2, Y_9 and Y_{10} in (3.1).

There is not any more possible case for studying and the proof is complete. □

According to our optimal system of one-dimensional subalgebras of the full symmetry algebra $\mathfrak{g}$, we need only find group-invariant solutions for four one-parameter subgroups generated by X in (2.9).

4 Lie invariants and similarity solutions

We can now compute the invariants associated with the symmetry operators (3.1). They can be obtained by integrating the characteristic equations. For the operator, $X = \partial_t + \partial_x$, this means $dx/1 = dt/1$. The corresponding invariants of this system have the form $I_1 = x - t$, and $I_2 = u$. Taking into account the last invariant, we obtain a similarity solution of the form $w = w(r) = w(x-t)$, and we substitute it into (1.1) to determine the form of the function w, and then we conclude that $w(x-t)$ is a solution of the following differential equation as similarity reduced equation: $w_{rrrr} + w_{rr} - w_r + (1-\lambda)w_r^2 - \sigma\lambda w_{rr}^2 = 0$,

For other example consider operator $X = \partial_t + \partial_u$. The characteristic equation for this case has the following form: $dt/1 = du/1$. So the corresponding invariants are $r = x$ and $w = u - t$. Taking into account the last invariant, the following similarity solution is obtained: $w = w(r) = w(x)$, where solution satisfies in the similarity reduced equation $w_{rrrr} + w_{rr} + (\lambda - 1)w_r^2 - \sigma\lambda w_r = 0$.

5 Non-classical symmetry of the system

Motivated by the fact that symmetry reductions for many PDEs are known that are not obtained by using the classical symmetries, there have been several generalizations of the classical Lie group method for symmetry reduction. The notion of non-classical symmetries was firstly introduced by Bluman and Cole [2] to study the symmetry reductions of the heat equation. The non-classical symmetries method consists of adding the invariant surface condition to the given equation, and then applying the classical symmetries method. The main difficulty of this approach is that the determining equations are no longer linear.(Find more information in [6, 2, 3, 12]).

For the non-classical method, we seek invariance of both the original equation and its invariant surface condition. This can also be conveniently written as:

$$X^{(4)}\Delta_1 \equiv 0, \qquad \text{mod} \quad \Delta_1 = 0, \Delta_2 = 0, \tag{5.1}$$

where $X = \xi^1(x,t,u)\partial_x + \xi^2(x,t,u)\partial_u + \varphi(x,t,u)\partial_u$, $\Delta_1 := u_t + u_{xx} + u_{xxxx} + (\lambda - 1)u_x^2 - \sigma\lambda u_{xx}^2$, and $\Delta_2 := \xi^2 u_t + \xi^1 u_x - \varphi$.

We must consider two different cases: $\xi^2 = 0$ and $\xi^2 = 1$. In continuation, we discuss about both of these two cases details:

Case I: $\xi^2 = 1$**.** In this case, $\Delta_2 = 0$ will be changed to $u_t = \varphi - \xi^1 u_x$. First, we compute the total derivation with respect to t:

$$D_t(u_t) = D_t((1-\lambda)u_x^2 + \sigma\lambda u_{xx}^2 - u_{xx} - u_{xxxx}), \tag{5.2}$$

and substituting $u_t = \varphi - \xi^1 u_x$ in (5.2) leads to

$$\begin{aligned} D_t(\varphi - \xi^1 u_x) = & \, 2(1-\lambda)u_x D_x(\varphi - \xi^1 u_x) + 2\sigma\lambda u_{xx} D_{xx}(\varphi - \xi^1 u_x) \\ & - D_{xx}(\varphi - \xi^1 u_x) - D_{xxxx}(\varphi - \xi^1 u_x), \end{aligned} \tag{5.3}$$

Therefore solving (5.3), we obtain:

$$\begin{aligned} \varphi^t = & \, 2(1-\lambda)u_x\varphi^x + 2\sigma\lambda u_{xx}\varphi^{xx} - \varphi^{xx} - \varphi^{xxxx} \\ & - 2(1-\lambda)u_x\xi u_{xx} + (1 - 2\sigma\lambda u_{xx})\xi u_{xxx} + \xi u_{xxxxx} + \xi u_{xt} \end{aligned} \tag{5.4}$$

In addition, we have:

$$u_{xt} = 2(1-\lambda)u_x u_{xx} + (2\sigma\lambda u_{xx} - 1)u_{xxx} - u_{xxxxx}, \tag{5.5}$$

Substituting (5.5) into (5.4), we find φ^t with following form:

$$\varphi^t = 2(1-\lambda)u_x\varphi^x + (2\sigma\lambda u_{xx} - 1)\varphi^{xx} - \varphi^{xxxx}, \tag{5.6}$$

where $\varphi^t, \varphi^x, \varphi^{xx}$ and φ^{xxxx} are:

$$\begin{aligned} \varphi^t &= D_t(\varphi - \xi u_x - \xi^2 u_t) + \xi u_{xt} + \xi^2 u_{tt} = D_t(\varphi - \xi u_x) + \xi u_{xt}, \\ \varphi^x &= D_t(\varphi - \xi u_x - \xi^2 u_t) + \xi u_{xx} + \xi^2 u_{xt} = D_x(\varphi - \xi u_x) + \xi u_{xx}, \\ \varphi^{xx} &= D_x^2(\varphi - \xi u_x - \xi^2 u_t) + \xi u_{xxx} + \xi^2 u_{xxt} = D_x^2(\varphi - \xi u_x) + \xi u_{xxx}, \\ \varphi^{xxxx} &= D_x^4(\varphi - \xi u_x - \xi^2 u_t) + \xi u_{xxxxx} + \xi^2 u_{xxxxt} = D_x^4(\varphi - \xi u_x) + \xi u_{xxxxx}, \end{aligned}$$

Applying Lie symmetry method and solving resulted determining equations, we obtain: $\xi^1 = c_1 x/2(1-\lambda) + c_3$ and $\varphi = c_1 t + c_2$. We have the following possibility cases:

- When $c_1 = 1$ and $c_2 \neq c_3 \neq 1$, the symmetries are: $\sigma_1 = h - u_t$, $\sigma_2 = u_x$ and $\sigma_3 = 1$;
- When $c_1 = 1$, $c_2 = c_3$ and $c_2 \neq 1$, the symmetries are: $\sigma_4 = \sigma_1$ and $\sigma_5 = 1 - u_x$;
- When $c_1, c_2 = 1$ and $c_3 \neq 1$, the symmetries are: $\sigma_6 = h - u_t + 1$ and $\sigma_7 = u_x$;
- When $c_1, c_2, c_3 = 1$, the symmetries are: $\sigma_8 = h - u_x - u_t + 1$;
- When $c_1, c_3 = 1$ and $c_2 \neq 1$, the symmetries are: $\sigma_9 = h - u_x - u_t$ and $\sigma_{10} = 1$;
- When $c_2, c_3 = 1$ and $c_1 \neq 1$, the symmetries are: $\sigma_{11} = 1 - u_x - u_t$ and $\sigma_{12} = h$;
- When $c_2 = 1$ and $c_1 \neq c_3 \neq 1$, the symmetries are: $\sigma_{13} = 1 - u_t$, $\sigma_{14} = h$ and $\sigma_{15} = u_x$;

- When $c_2 = 1$ and $c_1 = c_3 \neq 1$, the symmetries are: $\sigma_{16} = 1-u_t$ and $\sigma_{17} = h-u_x$;
- When $c_3 = 1$ and $c_1 \neq c_3 \neq 1$, the symmetries are: $\sigma_{18} = u_x + u_t$, $\sigma_{19} = h$ and $\sigma_{20} = 1$;
- When $c_3 = 1$ and $c_1 = c_2 \neq 1$, the symmetries are: $\sigma_{21} = u_x+u_t$ and $\sigma_{22} = h+1$;
- When $c_1 = c_2 = c_3 \neq 1$, the symmetries are: $\sigma_{23} = u_t$ and $\sigma_{24} = h - u_x + 1$;

Where $h = t+xu_x/2(\lambda-1)$. Since other remained cases don't provide new symmetries, so we ignore them.

Next, using above symmetries, we will solve the group invariant solutions of the equation (1.1):

1. $\sigma_1 = \sigma_4 = h - u_t$. Assuming $\sigma_i(u) = 0$; $i = 1, 4$ and with an integrating gives:

   (5.7) $$u(x,t) = \frac{1}{2}(t^2 - \xi) + f(\xi),$$

   where $\xi = t + 2(\lambda - 1)\ln x$ and f is an arbitrary function. Substituting this obtained $u(x,t)$ in equation (1.1), we can find the indicated reduced similarity equation:

   (5.8) $$f_{\xi\xi\xi} + (3x^2 - 1)f_{\xi\xi} + 2x^2 f_\xi^2 - x^2(4x^2 f + x^2 + 1)f_\xi + 2(x^2 - 2)f^2 + (x^5 + 6x^2 - 3)f + \frac{x^2}{2}(11x^3 + 6x^2 - t) - 1 = 0.$$

2. $\sigma_2 = \sigma_7 = \sigma_{15} = u_x$. Assuming $\sigma_i(u) = 0$; $i = 2, 7, 15$ and with an integrating, we obtain $u = f(t)$, then substituting it into (1.1) gives $u = c$, where c is an arbitrary constant.

3. $\sigma_3 = \sigma_{10} = \sigma_{20} = 1$. Suppose $\sigma_i(u) = 0$; $i = 3, 10, 20$, with an integrating we have $u = 0$.

4. $\sigma_5 = 1-u_x$. Let $\sigma_5(u) = 0$, and then $u = x+f(t)$. Substituting it into (1.1), we can obtain similarity equation with form $f' + \lambda - 1 = 0$, which has the solution $f = (1-\lambda)t + c$ where c is an arbitrary constant. So the MKS equation has the following solution:

   (5.9) $$u(x,t) = x + (1-\lambda)t + c,$$

5. $\sigma_6 = h - u_t + 1$. substituting it into $\sigma(u) = 0$ gives:

   (5.10) $$u(x,t) = \frac{1}{2}(t-\xi)(t+\xi+2) + f(\xi),$$

   where $\xi = t + 2(\lambda-1)\ln x$.

6. $\sigma_8 = h - u_x - u_t + 1$. Solving integral $\sigma_8(u) = 0$ gives:

   (5.11) $$u(x,t) = \frac{1}{2}(t-\xi)(t+\xi+2) + f(\xi),$$

   where $\xi = t + 2(\lambda-1)\ln(2\lambda - x - 2)$.

7. $\sigma_9 = h - u_x - u_t$. Now Solving integral $\sigma_9(u) = 0$ gives:

$$u(x,t) = \frac{1}{2}(t^2 - \xi^2) + f(\xi), \tag{5.12}$$

where $\xi = t + 2(\lambda - 1)\ln(2\lambda - x - 2)$ and f is an arbitrary function.

8. $\sigma_{11} = 1 - u_x - u_t$. Substituting $\sigma_{11}(u) = 0$ leads to $u = x + f(t - x)$, then the indicated reduced similarity equation has the form $f_{\xi\xi\xi\xi} - \sigma\lambda f_{\xi\xi}^2 + f_{\xi\xi} + \lambda f_\xi = 0$.

9. $\sigma_{12} = \sigma_{14} = \sigma_{14} = h$. Therefore we have $u = 2(1 - \lambda)t\ln x + f(t)$, and by substitute it into (1.1), we can obtain indicated reduced similarity equation.

10. $\sigma_{13} = \sigma_{16} = 1 - u_t$. Then $\sigma_i(u) = 0,\ i = 13, 16$ leads to $u = t + f(x)$, and substituting it into (1.1), we can obtain indicated reduced similarity equation as: $f_{\xi\xi\xi\xi} - \sigma\lambda f_{\xi\xi}^2 + f_{\xi\xi} + (1 - \lambda)f_\xi + 1 = 0$.

11. $\sigma_{17} = h - u_x$. With integrating $\sigma_{17}(u) = 0$, we have $u = 2t(1 - \lambda)\ln(2\lambda - x - 2) + f(t)$, as an exact solution.

12. $\sigma_{18} = \sigma_{21} = u_t + u_x$. Therefore we have $u = f(t - x)$, substituting it into (1.1), we obtain $f_{\xi\xi\xi\xi} + f_{\xi\xi} + (\lambda - 1)f_\xi^2 - \sigma\lambda f_{\xi\xi}^2 + f_\xi = 0$.

13. $\sigma_{22} = h + 1$. Integrating $\sigma_{22}(u) = 0$, we find $u = 2(1 - \lambda)(1 + t)\ln x + f(t)$,as an exact solution.

14. $\sigma_{23} = u_t$. Then we have $u = f(x)$, substituting it into (1.1), we can obtain indicated reduced similarity equation as $f_{\xi\xi\xi\xi} + f_{\xi\xi} + (\lambda - 1)f_\xi^2 - \sigma\lambda f_{\xi\xi}^2 = 0$.

15. $\sigma_{24} = h - u_x + 1$. Solving $\sigma_{24}(u) = 0$, we can let $u = 2(1 + t)(1 - \lambda)\ln(2\lambda - x - 2) + f(t)$, by substitute it into (1.1), we can obtain indicated reduced similarity equation.

Case II: $\xi^2 = 0$**.** In this case without loss of generality we can set: $\xi^1 = 1$. So, we have $u_x = \varphi$ and $A(x,t,u) = \varphi_x - \varphi_{xxx} + (1 - \lambda)\varphi^2 - \sigma\lambda\varphi_x^2$. Substituting this expression in determining equation *i.e.* $A\varphi_u + \varphi_t - A_u\varphi - A_x = 0$, we have:

$$\begin{aligned}&\varphi_x\varphi_u - \varphi_{xxx}\varphi_u + 3(1 - \lambda)\varphi^2\varphi_u - \sigma\lambda\varphi_x^2\varphi_u - \varphi\varphi_{xu} - \varphi\varphi_{xxxu}\\ &-2\sigma\lambda\varphi\varphi_x\varphi_{xu} - \varphi_{xx} - \varphi_{xxxx} + 2(1 - \lambda)\varphi\varphi_x - 2\sigma\lambda\varphi\varphi_{xx} + \varphi_t = 0.\end{aligned} \tag{5.13}$$

Assuming $\varphi = \varphi(x,t)$, the (5.13) equation will be changed to:

$$\varphi_t - \varphi_{xx} - \varphi_{xxxx} + 2(1 - \lambda)\varphi\varphi_x - 2\sigma\lambda\varphi\varphi_{xx} = 0. \tag{5.14}$$

This equation is indicated similarity reduced equation in the case $\xi^2 = 0$.

Theorem 2.6 provides a way to generate new solutions using older. For instance, by considering obtained solution $u = (1 - \lambda)t + c$ to MKS equation and using this theorem we can claim the following expressions are solutions of equation (1.1):

$$u^1 = (1 - \lambda)(s - t), \qquad u^2 = (1 - \lambda)(s^2 + 1)t - sx + c.$$

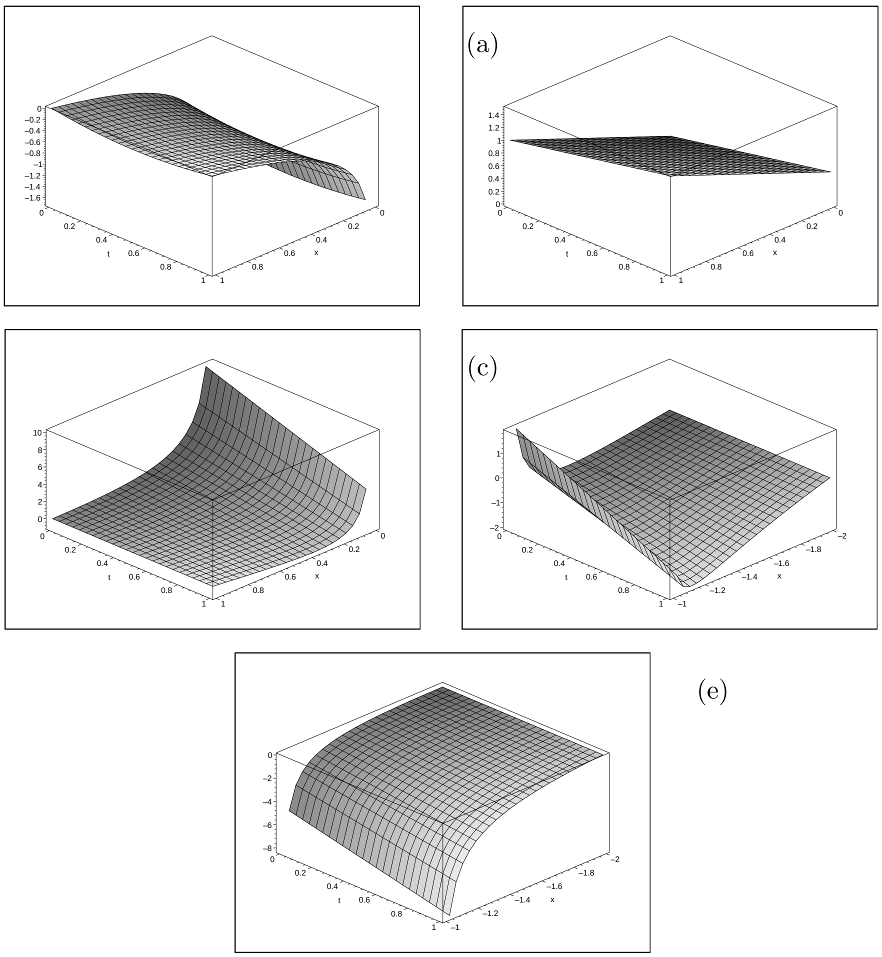

Figure 1: (a) is the figure of solution (5.7), (b) is the figure of solution (5.7), (c) is the figure of solution (5.9), (d) is the figure of solution (5.10), (e) is the figure of solution (5.12), for $f(0) = 0$ and $\lambda = 1/2$.

6 Conclusions

Present paper addresses the classical and non-classical symmetries of the one dimensional Modified Kuramoto-Sivashinsky equation (MKS) as well as its similarity solutions using infinitesimal criterion and compatibility condition approaches, have been discussed. In the last section, we obtain some exact solution for the (1.1) equation. The paper [1] presented calculations of the 1D MKS on a periodic domain and showed that the singularities exhibit all a self-similar structure in u_{xx}. We obtained the some new exact solution for the Modified Kuramoto-Sivashinsky equation (MKS), and provided a comparison between the obtained exact solutions $u(x,t)$. Also, illustrative figures (see Fig. 1) were included for the solutions (5.7–5.12).

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Authors' address:

Mehdi Nadjafikhah, Saeed Dodangeh
Department of Pure Mathematics, School of Mathematics,
Iran University of Science and Technology,
Narmak, Tehran, 16846-13114, Iran.
E-mail: m_nadjafikhah@iust.ac.ir, saeed136541@yahoo.com