Analysis of a mathematical model for the pine wilt disease using a graph theoretic approach

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Abstract. In this paper, we study a vector–host mathematical model for the pine wilt disease, considering indirect and direct transmission of the disease. An explicit formula for the basic reproduction number ($\mathcal{R}_0$) is obtained. The global stability of the DFE is proved using a graph theoretic approach and a matrix theoretic method. We show that when $\mathcal{R}_0$ is less than one, the DFE always exists and it is globally asymptotically stable, and the disease persists at an endemic level. Finally, local sensitivity analysis of parameters and numerical experiments are carried out to illustrate the theoretical bases for the prevention and control of the disease.

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1 Introduction

Vector–borne diseases are illnesses that are transmitted by mammals, birds, arthropods and insects. Pests that transmit diseases are called vectors. These vectors receive the pathogenic organism from an infected, animal or human, and transmits it either to an intermediate carrier or directly to a human carrier. The transfer occurs directly by bites, or tissue infection, or indirectly through disease transmission. Although vector–borne diseases are more known in humans, numerous cases of them are also frequent in plants. Among them, some important wilting diseases of trees, such as the pine wilt disease (PWD) and the red ring disease of palms, are caused by nematodes that have intriguing association with insect vectors [20, 11, 1].

In this work, we deal with dynamics of PWD. It is a severe illness that typically kills affected trees within a few weeks. The causal pathogen is the pine wood nematode (PWN), *Bursaphelenchus xylophilus*. This nematode is found in aboveground parts of the tree. Nematodes kill the tree by feeding on the cells surrounding the resin ducts. This causes resin to leak into the tracheids, resulting in *tracheid cavitation* or air pockets in the water transport system, the tree cannot move water upward and consequently wilts and dies. *Bursaphelenchus xylophilus* is transmitted by vector pine sawyer beetles in the genus *Monochamus*. The dispersal stage of the nematode

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is carried in the respiratory system of the insect, and thus spread from tree to tree as the beetles feed on the young shoots of pine trees [14, 24, 26, 38]. Three types of transmission of PWD have been discovered: (a) first, when adult beetles, infected with nematode, flies to healthy pine trees and begin maturation feeding and transmit nematode into the tree, this is known as the primary transmission [23]. (b) Secondary transmission occurs when mature females lay eggs on dead or dying, freshly cut pine trees [36]. (c) Third is the horizontal transmission of the nematode. It happens through the mating of male and female bark beetles [2].

After its appearance in Japan, Korea, China and Taiwan where it has killed millions of plants [22], PWD is internationally considered the most harmful disease of the pine trees worldwide. PWD causes significant damage to forestry, local economies, and the ecologies of affected countries, degrading the quality and decreasing the quantity of pine wood products. In Japan, for example, the annual loss of pine wood as a result of PWD was more than 100 million m^3 across 11 years between 1978 and 1988 and more than 50 million m^3 between 1989 and 2014 [7]. From an ecological point of view, the loss of pine trees also reduces ecosystem functions and services, decreasing habitats for wild animals and affecting soil erosion [7] . Once introduced into a region, PWN spreads rapidly to neighboring areas through vector beetles or accompanying human activity [7]. Control is therefore labor–intensive and costly, highlighting the importance of identifying vulnerable areas and prioritizing control measures.

In the study of causes, patterns and the effects of a disease, mathematical modeling has become a powerful and quite important tool which enables one to understand the mechanisms that mainly stimulate the prevailing of the disease, factors which have a significant influence in the spread of disease and can propose the schemes to control the disease. For PWD, we can find some works in the literature, but we highlight the most recent. Some of these works have been focused on modeling the dynamics of PWD sawyers as vector [37, 33, 30, 19, 3], other works focused on the control of the disease [19, 27, 4, 9, 10] and some works focused on particular aspects of the disease [31, 21, 18, 8, 34, 28].

In this work, a mathematical model for host–vector relationship between pine trees and pine sawyer beetles using ordinary differential equations (ODEs) is formulated. We assume that the exploitation rate of infected pine trees is greater than susceptible or exposed pine trees. First, we formulated a mathematical model by considering indirect and direct transmission of the disease. Then, $\mathcal{R}_0$ is determined and conditions for the existence of equilibrium solutions are established. Although the compartmental structure of mathematical models using ODEs for vector–borne diseases is standard, the mathematical techniques for stability and optimal control analysis may change. Thus, in this work, the global stability of the disease–free equilibrium (DFE) is proved using Perron eigenvectors, Lyapunov theory and a graph theoretic approach. Additionally, sensitivity analysis of parameters is performed by analysis of the sensitivity indices of the basic reproduction number as well as positive constant level of infected compartments. Finally, numerical experiments and conclusions are carried out to support the analytical results.

2 Mathematical model formulation

In this section, we formulate a mathematical model which comprises of susceptible host pine trees $S_h(t)$, exposed pine trees $E_h(t)$, infected host pine trees $I_h(t)$, susceptible vector beetles $S_v(t)$ and infected beetles $I_v(t)$ at time t. The usual transmission of nematodes into pine trees occurs during maturation feeding of infected vectors. The nematodes are carried by the pine sawyers when they emerge from infected pines. However, the beetles may also be directly infected during mating. The population of a host pine trees at time t is denoted by N_h, and the total population of vector adult beetles is denoted by N_v. Thus $N_h = S_h + E_h + I_h$ and $N_v = S_v + I_v$.

Let Π_h the input rate of pine trees and Π_v the constant emergence rate of adult beetles. We assume that δ_1 represents the rate of transmission per contact in maturation feeding. Some parameters involved in the model are described below: β_1 represents the average number of contacts per day with vector adult beetles at the time of maturation feeding, δ_2 represents the rate at which I_v transmit nematode via oviposition, β_2 denotes the average number of contacts per day of bark beetles during oviposition, β_3 represents the rate of adult beetles carrying nematode when they come out of dead pines, α represents the rate at which the exposed pines move to the infectious class, β is transmission rate among beetles during mating, μ_h represents the rate of natural death of susceptible pine trees, σ denotes the felling and isolation rate of I_h, and μ_v represents the death rate of the vector population. The incidence terms relative to the host population during maturation feeding and oviposition are $\beta_1\delta_1 S_h I_v$ and $\beta_2\delta_2\eta S_h I_v$, respectively. For vector population $\beta_3 I_h S_v$ and $\beta S_v I_v$ are the incidence terms. Other parameters involved in the model are specified in Table 1.

Table 1: Description, dimension and values of the parameters of the model (2.1).

Parameter	Description	Dimension	Value	Source
Π_h	Input rate of pine trees	Trees×Day $^{-1}$	0.009041	[17]
Π_v	Emergence rate of adult bark beetles	B.beetles×Day $^{-1}$	0.002691	[17]
μ_h	Natural death rate of healthy pine trees	Day $^{-1}$	0.0000301	[25]
μ_v	Mortality rate of bark beetles	Day $^{-1}$	0.011764	[34]
σ	Exploitation rate of infected pine trees	Day $^{-1}$	0.004	[3]
β_1	Rate of transmission per contact during maturation feeding	Day $^{-1}$	0.00166	[15]
δ_1	Average number of contacts of adult beetles through maturation feeding	Dimensionless	0.2	[12]
β_2	The rate at which bark beetles transmit nematode during oviposition	Day $^{-1}$	0.0004	[15]
δ_2	Average number of contacts during the oviposition period	Dimensionless	0.41	[17]
η	The probability that susceptible pine trees is not infectious by nematode and ceases oleoresin exudation naturally	Dimensionless	0.0000301	[17]
α	Transmission rate of exposed pine trees into infectious pine trees	Dimensionless	0.002	[13]
β_3	Rate of adult beetles carrying nematode when they emerge from dead pines	Day $^{-1}$	0.00305	[13]
β	Transmission rate among beetles during mating	Day $^{-1}$	0.00305	[3]

With the above assumptions, the vector–host mathematical model can be written by the following system of non–linear ODEs:

$$
\left\{
\begin{aligned}
\dot{S}_h &= \Pi_h - \beta_1\delta_1 S_h I_v - \beta_2\delta_2\eta S_h I_v - \mu_h S_h,\\
\dot{E}_h &= \beta_1\delta_1 S_h I_v - \alpha E_h - \mu_h E_h,\\
\dot{I}_h &= \beta_2\delta_2\eta S_h I_v + \alpha E_h - \sigma I_h,\\
\dot{S}_v &= \Pi_v - \beta_3 S_v I_h - \beta S_v I_v - \mu_v S_v,\\
\dot{I}_v &= \beta_3 S_v I_h + \beta S_v I_v - \mu_v I_v.
\end{aligned}
\right.
\tag{2.1}
$$

Let us denote as $\mathbf{P} = (S_h, E_h, I_h, S_v, I_V)$. The set

$$
\xi = \left\{ \mathbf{P} \in \mathbb{R}^5_+ : \frac{\Pi_h}{\sigma} \leq S_h + E_h + I_h \leq \frac{\Pi_h}{\mu_h}, 0 \leq S_v + I_v \leq \frac{\Pi_v}{\mu_v} \right\},
\tag{2.2}
$$

is a positively invariant set. Figure 1 shows the flow diagram associated to the mathematical model (2.1).

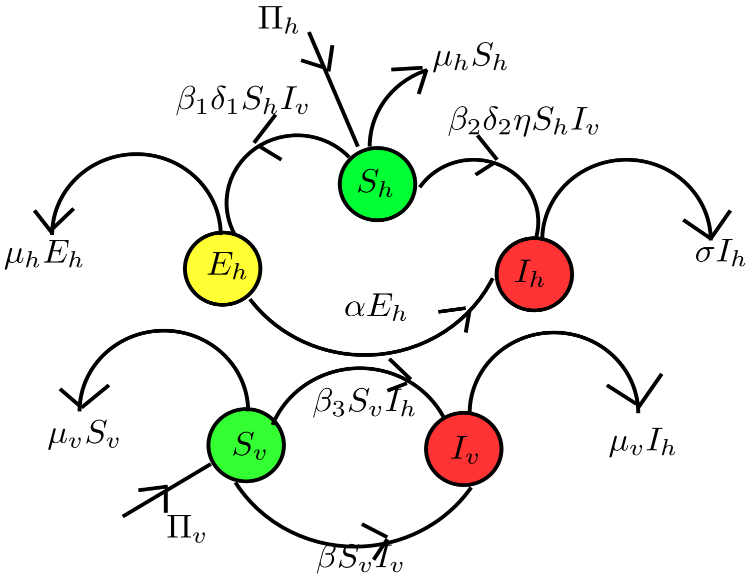

Figure 1: Flow diagram of the mathematical model (2.1).

3 Basic reproduction number

Direct calculations show that the model (2.1) always has a DFE given by

$$\mathbf{E}_0 = \left(\frac{\Pi_h}{\mu_h}, 0, \frac{\Pi_v}{\mu_v}, 0, 0 \right).$$

The dynamics of a disease is characterized by the basic reproduction number that is defined to be *the average number of secondary infections, which are produced by an infected individual in a totally susceptible population.* This enables us to find whether an infectious disease will prevail through population or not. We used the next–generation operator [6] to calculate it . $\mathcal{R}_0$ is obtained by choosing the dominant eigenvalue (spectral radius) of $\hat{\mathbf{F}}\hat{\mathbf{V}}^{-1}$, where $\hat{\mathbf{F}} = \left[\frac{\partial(\check{F}_i(\mathbf{E}_0))}{\partial(\mathbf{x}_j)}\right]$ and $\hat{\mathbf{V}} = \left[\frac{\partial(\check{V}_i(\mathbf{E}_0))}{\partial(\mathbf{x}_j)}\right]$, being E_h, I_h and I_v the infected compartments. Therefore,

$$\begin{bmatrix} \check{F}_1 \\ \\ \check{F}_2 \\ \\ \check{F}_3 \end{bmatrix} = \begin{bmatrix} \beta_1 \delta_1 S_h I_v \\ \\ \beta_2 \delta_2 \eta S_h I_v \\ \\ \beta_3 S_v I_h + \beta S_v I_v - \mu_v I_v \end{bmatrix}, \text{ and } \begin{bmatrix} \check{V}_1 \\ \\ \check{V}_2 \\ \\ \check{V}_3 \end{bmatrix} = \begin{bmatrix} \alpha E_h + \mu_h E_h \\ \\ -\alpha E_h + \sigma I_h \\ \\ \mu_v I_v \end{bmatrix}.$$

Thus

$$\hat{\mathbf{F}} = \begin{pmatrix} 0 & 0 & \beta_1 \delta_1 \frac{\Pi_h}{\mu_h} \\ 0 & 0 & \beta_2 \delta_2 \eta \frac{\Pi_h}{\mu_h} \\ 0 & \frac{\beta_3 \Pi_v}{\mu_v} & \frac{\beta \Pi_v}{\mu_v} \end{pmatrix} \text{ and } \hat{\mathbf{V}} = \begin{pmatrix} \mu_h & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \mu_v \end{pmatrix},$$

which gives

$$\hat{\mathbf{F}}\hat{\mathbf{V}}^{-1} = \begin{bmatrix} 0 & 0 & \beta_1 \delta_1 \frac{\Pi_h}{\mu_h \mu_v} \\ \\ 0 & 0 & \beta_2 \delta_2 \eta \frac{\Pi_h}{\mu_h \mu_v} \\ \\ 0 & \frac{\beta_3 \Pi_v}{\mu_v \sigma} & \frac{\beta \Pi_v}{\mu_v^2} \end{bmatrix}.$$

Since $\mathcal{R}_0$ is the spectral radius of the above matrix, we have that

$$\mathcal{R}_0 = \frac{\beta \Pi_V}{2\mu_v^2} + \sqrt{\left(\frac{\beta \Pi_V}{2\mu_v^2} \right)^2 + \frac{\Pi_V \Pi_h \eta \beta_2 \beta_3 \delta_2}{\sigma \mu_h \mu_v^2} + \frac{\Pi_h \Pi_V \alpha \beta_1 \beta_3 \delta_1}{\sigma \mu_h \mu_v^2 (\alpha + \mu_h)}}. \tag{3.1}$$

4 Existence and stability of the DFE

Now, we analyze the global behavior of the DFE for the system (2.1). For this end, we construct a Lyapunov function using the techniques and the notation described in [32]. We have the following results.

Theorem 4.1. *If $\mathcal{R}_0 \leq 1$, the DFE of the system (2.1) is globally asymptotically stable (GAS) in ξ.*

Proof. Let $\mathbf{x} = (E_h, I_h, I_v)^T$ and the function

$$Q(t) = \mathbf{w}^T \hat{\mathbf{V}}^{-1} \mathbf{x}, \tag{4.1}$$

where

$$\mathbf{w}^T = \begin{bmatrix} 0 & \dfrac{\frac{\beta_3}{\mu_v^2}\Pi_v}{\sqrt{L}} & \dfrac{\frac{\beta}{\mu_v^2}\Pi_v + \sqrt{L}}{2\sqrt{L}} \end{bmatrix},$$

is the left eigenvector of $\hat{\mathbf{V}}^{-1}\hat{\mathbf{F}}$ and

$$L = \left(\frac{\beta}{\mu_v^2}\Pi_v\right)^2 + 4\left(\eta\beta_2\frac{\delta_2}{\sigma\mu_h}\Pi_h + \frac{1}{(\alpha+\mu_h)}\alpha\beta_1\frac{\delta_1}{\sigma\mu_h}\Pi_h\right)\frac{\beta_3}{\mu_v^2}\Pi_v.$$

Substituting above values in (4.1) and after simplifications, we get

$$Q(t) = \frac{\beta_3\Pi_v}{\mu_v^2\sigma\sqrt{L}}I_h + \frac{\beta_3\Pi_v\eta}{\mu_v^2\left(\alpha+\mu_h\right)\sigma\sqrt{L}}E_h + \frac{\beta\Pi_v + \mu_v^2\sqrt{L}}{2\mu_v^3\sqrt{L}}I_v.$$

Taking the orbital derivative of $Q(t)$ along of the trajectories of the system (2.1), we obtain

$$\begin{aligned}
\dot{Q}(t) &= \frac{\beta_3\Pi_v}{\mu_v^2\sigma\sqrt{L}}\dot{I}_h + \frac{\beta_3\Pi_v\eta}{\mu_v^2\left(\alpha+\mu_h\right)\sigma\sqrt{L}}\dot{E}_h + \frac{\beta\Pi_v + \mu_v^2\sqrt{L}}{2\mu_v^3\sqrt{L}}\dot{I}_V \\
&= \frac{\beta_3\Pi_v}{\mu_v^2\sigma\sqrt{L}}\left(\beta_2\delta_2\eta S_h I_v + \eta E_h - \sigma I_h\right) + \frac{\beta_3\Pi_v\eta}{\mu_v^2\left(\alpha+\mu_h\right)\sigma\sqrt{L}}\left(\beta_1\delta_1 S_h I_v - \left(\alpha+\mu_h\right)E_h\right) \\
&+ \frac{\beta\Pi_v + \mu_v^2\sqrt{L}}{2\mu_v^3\sqrt{L}}\left(\beta_3 S_v I_h + \beta S_v I_v - \mu_v I_v\right) \\
&= \frac{\beta_3\Pi_v}{\mu_v^2\sigma\sqrt{L}}\beta_2\delta_2\eta S_h I_v + \frac{\beta_3\Pi_v\eta}{\mu_v^2\left(\alpha+\mu_h\right)\sigma\sqrt{L}}\beta_1\delta_1 S_h I_v + \frac{\beta\Pi_v + \mu_v^2\sqrt{L}}{2\mu_v^3\sqrt{L}}\beta S_v I_v \\
&- \frac{\beta\Pi_v + \mu_v^2\sqrt{L}}{2\mu_v^2\sqrt{L}}I_v + \frac{k+\sqrt{L}}{2\mu_v\sqrt{L}}\beta_3 S_v I_h - \frac{\beta_3\Pi_v}{\mu_v^2\sqrt{L}}I_h \\
&\leq \frac{\beta_3\Pi_v}{\mu_v^2\sigma\sqrt{L}}\beta_2\delta_2\eta\frac{\Pi_h}{\mu_h}I_v + \frac{\beta_3\Pi_v\eta}{\mu_v^2\left(\alpha+\mu_h\right)\sigma\sqrt{L}}\beta_1\delta_1\frac{\Pi_h}{\mu_h}I_v + \frac{\beta\Pi_v + \mu_v^2\sqrt{L}}{2\mu_v^3\sqrt{L}}\beta\frac{\Pi_v}{\mu_v}I_v \\
&- \frac{\beta\Pi_v + \mu_v^2\sqrt{L}}{2\mu_v^2\sqrt{L}}I_v + \frac{k+\sqrt{L}}{2\mu_v\sqrt{L}}\beta_3\frac{\Pi_v}{\mu_v}I_h - \frac{\beta_3\Pi_v}{\mu_v^2\sqrt{L}}I_h \\
&\leq \frac{1}{\sqrt{L}}R_0^2 I_v - \frac{1}{\sqrt{L}}R_0 I_v + \frac{1}{\sqrt{L}}\left(R_0 - 1\right)I_h \\
&\leq \frac{1}{\sqrt{L}}R_0^2\left(1 - \frac{1}{R_0}\right)I_v + \frac{1}{\sqrt{L}}\left(R_0 - 1\right)I_h.
\end{aligned}$$

Thus, $\dot{Q}(t) \leq 0$ if and only if $\mathcal{R}_0 \leq 1$. Furthermore, $\dot{Q}(t) = 0$ if and only if $I_h = I_v = 0$. Hence, it is easy to check that the largest compact invariant set for the system (2.1) is $\mathbf{E}_0$. Thus, by LaSalle's Invariance Principle [16], the equilibrium $\mathbf{E}_0$ is GAS in ξ. □

5 Endemic equilibrium and its stability

We determine the endemic equilibrium solutions of the system (2.1), which are obtained by solving the following system of algebraic equations

$$\begin{cases} \Pi_h - \beta_1\delta_1 S_h I_v - \beta_2\delta_2\eta S_h I_v - \mu_h S_h = & 0, \\ \beta_1\delta_1 S_h I_v - \alpha E_h - \mu_h E_h = & 0, \\ \beta_2\delta_2\eta S_h I_v + \alpha E_h - \sigma I_h = & 0, \\ \Pi_v - \beta_3 S_v I_h - \beta S_v I_v - \mu_v S_v = & 0, \\ \beta_3 S_v I_h + \beta S_v I_v - \mu_v I_v = & 0. \end{cases}$$

After calculations, we get that the solutions of the above system are given by

$$S_h^* = \frac{\Pi_h}{\left(\left(\beta_1\delta_1 + \beta_2\delta_2\eta\right) I_v^* + \mu_h\right)},$$

$$E_h^* = \frac{\beta_1\delta_1\Pi_h I_v^*}{\left(\left(\beta_1\delta_1 + \beta_2\delta_2\eta\right) I_v^* + \mu_h\right)\left(\alpha + \mu_h\right)},$$

$$I_h^* = \frac{\left(\beta_2\delta_2\eta\Pi_h\left(\alpha + \mu_h\right) + \alpha\beta_1\delta_1\Pi_h\right) I_v}{\left(\sigma\left(\beta_1\delta_1 + \beta_2\delta_2\eta\right) I_v + \sigma\mu_h\right)\left(\alpha + \mu_h\right)},$$

$$S_v^* = \frac{\Pi_v\sigma\left(\left(\beta_1\delta_1 + \beta_2\delta_2\eta\right) I_v + \mu_h\right)\left(\alpha + \mu_h\right)}{\Pi_h\beta_3 I_v\left(\beta_2\delta_2\eta\left(\alpha + \mu_h\right) + \alpha\beta_1\delta_1\right) + \sigma\left(\alpha + \mu_h\right)\left(\beta I_v + \mu_v\right)\left(\left(\beta_1\delta_1 + \beta_2\delta_2\eta\right) I_v + \mu_h\right)},$$

and I_v^* is uniquely determined from the following quadratic equation

$$aI_v^{*2} + bI_v^* + c = 0, \tag{5.1}$$

where

$$a = \sigma\beta\mu_v\left(\beta_1\delta_1 + \beta_2\delta_2\eta\right),$$

$$\begin{aligned} b = {} & \sigma\beta\mu_h\mu_v + \sigma\left(\beta_1\delta_1 + \beta_2\delta_2\eta\right)\mu_v^2 - \Pi_v\beta\sigma\left(\beta_1\delta_1 + \beta_2\delta_2\eta\right) \\ & + \frac{\Pi_h\beta_3\left(\beta_2\delta_2\eta\left(\alpha + \mu_h\right) + \alpha\beta_1\delta_1\right)}{\left(\alpha + \mu_h\right)}\mu_v, \end{aligned}$$

$$c = \sigma\mu_v^2\mu_h\left(1 - \frac{\Pi_v\beta}{\mu_v^2} - \left(\frac{\Pi_v\Pi_h\beta_3\beta_2\delta_2\eta}{\sigma\mu_v^2\mu_h} + \frac{\Pi_v\Pi_h\beta_3\alpha\beta_1\delta_1}{\sigma\mu_v^2\mu_h\left(\alpha + \mu_h\right)}\right)\right).$$

Note that $a > 0$ while the sign of c depends on $\mathcal{R}_0$. If $\mathcal{R}_0 > 1$ then $c < 0$. Thus, the above quadratic equation has only one positive root for $R_0 > 1$. Hence, the unique endemic equilibrium exists whenever the reproduction number exceeds unity. Now, we will prove the global stability of the endemic equilibrium $\mathbf{E}^* = (S_h^*, E_h^*, I_h^*, S_v^*, I_v^*)$, by using a graph–theoretic approach [32]. To this end, we will consider matrix tree theorem of Kirchoff [29] and the results which are given in [32]. We have the following results.

Theorem 5.1. *The unique positive endemic equilibrium point of the system (2.1), $\mathbf{E}^*$ is GAS in ξ.*

Proof. Let

$$\begin{aligned}
&\mathbb{D}_1 = (S_h - S_h^* - S_h^* \ln \tfrac{S_h}{S_h^*}) + \left(E_h - E_h^* - E_h^* \ln \tfrac{E_h}{E_h^*}\right) \quad && \mathbb{D}_2 = (E_h - E_h^* - E_h^* \ln \tfrac{E_h}{E_h^*}) \\
&\mathbb{D}_3 = \left(I_h - I_h^* - I_h^* \ln \tfrac{I_h}{I_h^*}\right) && \mathbb{D}_4 = \left(S_v - S_v^* - S_v^* \ln \tfrac{S_v}{S_v^*}\right) \\
&\mathbb{D}_5 = \left(I_v - I_v^* - I_v^* \ln \tfrac{I_v}{I_v^*}\right).
\end{aligned}$$

Differentiating and using the inequality $1 - x + \ln x \leq 0$ for $x > 0$, we have

$$\begin{aligned}
\dot{\mathbb{D}_1} =& \left(1 - \frac{S_h^*}{S_h}\right) \dot{S}_h + \left(1 - \frac{E_h^*}{E_h}\right) \dot{E}_H \\
=& \left(1 - \frac{S_h^*}{S_h}\right) \left(\left(\beta_1\delta_1 + \beta_2\delta_2\eta\right) S_h^* I_v^* - \left(\beta_1\delta_1 + \beta_2\delta_2\eta\right) S_h I_v\right) - \mu_h \left(1 - \frac{S_h^*}{S_h}\right)(S_h - S_h^*) \\
&+ \left(1 - \frac{E_h^*}{E_h}\right)\left(\beta_1\delta_1 S_h I_v - \frac{\beta_1\delta_1 S_h^* I_v^*}{E_h^*} E_h\right) \\
=& \left(\beta_1\delta_1 + \beta_2\delta_2\eta\right) S_h^* I_v^* \left(1 - \frac{S_h^*}{S_h}\right)\left(1 - \frac{S_h I_v}{S_h^* I_v^*}\right) - \mu_h \left(\frac{S_h - S_h^*}{S_h}\right)(S_h - S_h^*) \\
&+ \beta_1\delta_1 S_h^* I_v^* \left(1 - \frac{E_h^*}{E_h}\right)\left(\frac{S_h I_v}{S_h^* I_v^*} - \frac{E_h}{E_h^*}\right) \\
&+ \beta_1\delta_1 S_h^* I_v^* \left(\frac{S_h I_v}{S_h^* I_v^*} - \frac{E_h}{E_h^*} - \frac{E_h^*}{E_h}\frac{S_h I_v}{S_h^* I_v^*} + 1\right) \\
\leq& \left(\beta_1\delta_1 + \beta_2\delta_2\eta\right) S_h^* I_v^* \left(\ln \frac{S_h I_v}{S_h^* I_v^*} - \frac{S_h I_v}{S_h^* I_v^*} - \ln \frac{I_v}{I_v^*} + \frac{I_v}{I_v^*}\right) + \\
&\beta_1\delta_1 S_h^* I_v^* \left(\frac{S_h I_v}{S_h^* I_v^*} - \frac{E_h}{E_h^*} + \ln \frac{E_h}{E_h^*} - \ln \frac{S_h I_v}{S_h^* I_v^*}\right) := b_{12}\mathbf{G}_{12} + b_{13}\mathbf{G}_{13},
\end{aligned}$$

and similarly

$$\begin{aligned}
\dot{\mathbb{D}_2} \leq& \beta_1\delta_1 S_h^* I_v^* \left(\frac{S_h I_v}{S_h^* I_v^*} - \ln \frac{S_h I_v}{S_h^* I_v^*} - \frac{E_h^*}{E_h} + \ln \frac{E_h^*}{E_h}\right) := b_{25}\mathbf{G}_{25}, \\
\dot{\mathbb{D}_3} \leq& \beta_2\delta_2\eta S_h^* I_v^* \left(\frac{S_h I_v}{S_h^* I_v^*} - \ln \frac{S_h I_v}{S_h^* I_v^*} - \frac{I_h^*}{I_h}\right) + \ln \frac{I_h^*}{I_h} := b_{31}\mathbf{G}_{31}, \\
\dot{\mathbb{D}_4} \leq& \beta_3 S_v^* I_h^* \left(\frac{I_h^*}{I_h} - \ln \frac{I_h^*}{I_h} - \frac{S_v I_h}{S_v^* I_h^*} + \ln \frac{S_v I_h}{S_v^* I_h^*}\right) \\
&+ \beta S_v^* I_v^* \left(\frac{I_v}{I_v^*} - \ln \frac{I_v}{I_v^*} - \frac{S_v I_v}{S_v^* I_v^*} - \ln \frac{S_v I_v}{S_v^* I_v^*}\right) := b_{43}\mathbf{G}_{43} + b_{42}\mathbf{G}_{42}, \\
\dot{\mathbb{D}_5} \leq& \beta_3 S_v^* I_h^* \left(\frac{S_v I_h}{S_v^* I_h^*} - \ln \frac{S_v I_h}{S_v^* I_h^*} - \frac{I_v^*}{I_v} + \ln \frac{I_v^*}{I_v}\right) \\
&+ \beta S_v^* I_v^* \left(\frac{S_v I_v}{S_v^* I_v^*} + \ln \frac{I_v}{I_v^*} - \frac{I_v}{I_v^*} - \ln \frac{S_v I_v}{S_v^* I_v^*}\right) := b_{54}\mathbf{G}_{54} + b_{53}\mathbf{G}_{53}.
\end{aligned}$$

We observe that there are four cycles and along each cycle $\mathbf{G}_{31} + \mathbf{G}_{13} = 0$, $\mathbf{G}_{42} + \mathbf{G}_{54} + \mathbf{G}_{25} = 0$, $\mathbf{G}_{31} + \mathbf{G}_{43} + \mathbf{G}_{54} + \mathbf{G}_{25} + \mathbf{G}_{12} = 0$, $\mathbf{G}_{31} + \mathbf{G}_{53} + \mathbf{G}_{25} + \mathbf{G}_{12} = 0$. Thus, there exists c_1, c_2, c_3, c_4, c_5 such that $\mathbb{D} = c_1\mathbb{D}_1 + c_2\mathbb{D}_2 + c_3\mathbb{D}_3 + c_4\mathbb{D}_4 + c_5\mathbb{D}_5$.

Besides

$$c_3 b_{31} = c_1(b_{12} + b_{13}) \quad c_2 b_{25} = c_1 b_{12} + c_4 b_{42}$$

$$c_5 a_{54} = c_4(b_{42} + b_{43}) \quad c_2 b_{25} = c_5(b_{53} + b_{54}).$$

Thus,

$$\begin{aligned}
\mathbb{D} =& \frac{\beta^2 S_v^{2*} I_v^{2*} + \beta\beta_3 S_v^{2*} I_v^* I_h^* + \beta_3^2 S_v^{2*} I_h^{2*}}{(\beta_1\delta_1 + \beta_2\delta_2\eta)(\beta_3 S_v^* I_h^* + \beta S_v^* I_h^*)}\mathbb{D}_1 + \frac{\beta S_v^* I_v^* + \beta_3 S_v^* I_h^*}{\beta_1\delta_1 S_h^* I_h^*}\mathbb{D}_2 \\
&+ \frac{(\beta^2 S_v^{2*} I_v^{2*} + \beta\beta_3 S_v^{2*} I_h^* I_v^* + \beta_3^2 S_v^{2*} I_h^{2*})(\beta_1\delta_1 + \beta_2\delta_2\eta + \beta_1\delta_1 S_h^* I_v^*)}{\beta_1\delta_1 S_h^{2*} I_v^{2*}(\beta_1\delta_1 + \beta_2\delta_2\eta)(\beta_3 S_v^* I_h^* + \beta S_v^* I_h^* + \beta S_v^* I_v^*)}\mathbb{D}_3 \\
&+ \frac{\beta_3 S_v^* I_h^*}{\beta_3 S_v^* I_h^* + \beta S_v^* I_v^*}\mathbb{D}_4 + \mathbb{D}_5,
\end{aligned}$$

is a Lyapunov function for the system (2.1). Using this Lyapunov function and LaSalle's invariance principle [16], it follows that $\mathbf{E}^*$ is GAS in ξ. □

6 Local sensitivity analysis of parameters and numerical experiments

To check which factors are responsible for the expanse and the existence of the disease, we shall carry out a sensitivity analysis. Our main objective is to control the disease. It is only possible if we reduce the basic reproduction number below unity. For this purpose, we calculate the ratio of relative change in the parameter to relative change in the reproduction number which is called sensitivity index. We identify the most influential factors for the spread of infectious disease by observing the sensitivity index. The *normalized forward sensitivity of a variable to the parameter* is defined as the ratio of relative change in the variable to the relative change in the parameter [5]. When a variable is a differentiable function of the parameter, the sensitivity index may be defined in terms of partial derivatives as

$$\Gamma_p^x = \frac{\partial x}{\partial p} \times \frac{p}{x}.$$

We use the numerical values of the parameters given in Table 1, and we see that the sensitivity index of $\mathcal{R}_0$ to the parameter σ is -150.818. It means that when we increase the value of a by 10%, the value of $\mathcal{R}_0$ increases by almost 151%. Looking at the sensitivity indices of $\mathcal{R}_0$ with respect to all the model parameters given in Table 2, we observe that the most sensitive parameter is μ_h (exploitation rate of susceptible pine trees), which has the highest value -20339.471. Negative sign shows that $\mathcal{R}_0$ is a decreasing function of μ_h. This index is illogical because we cannot increase the exploitation rate of healthy pine trees in order to decrease the value of the basic reproduction number below unity. However, we should focus on the parameter β_1 which is the second one sensitive parameter for $\mathcal{R}_0$.

Although we do not have an explicit expression for the endemic equilibrium, by using the values of parameters given in Table 1, we can calculate the sensitivity indices

Table 2: Values of sensitivity indices of $\mathcal{R}_0$, ${I_h}^*$ and ${I_v}^*$.

Parameters	Sensitivity index of $\mathcal{R}_0$	Sensitivity index of ${I_h}^*$	Sensitivity index of ${I_v}^*$
Π_h	66.726	2.438	2.829
Π_v	235.472	2.961	2.221
μ_h	-20339.471	-1.865	-2.164
μ_v	-107.727	-5.532	-4.150
σ	-150.818	-2.438	-2.829
β_1	363.412	1.829	2.122
δ_1	3.016	1.829	2.122
β_2	0.022	0.2×10^{-4}	0.3×10^{-4}
δ_2	0.2×10^{-4}	0.2×10^{-4}	0.3×10^{-3}
η	0.302	0.2×10^{-4}	0.3×10^{-4}
α	4.472	0.036	0.041
β_3	197.794	2.438	1.829
β	9.961	0.132	0.099

of ${I_h}^*$ and ${I_v}^*$. The sensitivity indices of the endemic level of infectious bark beetles ${I_v}^*$, given in Table 2, show that the parameters μ_v, σ and Π_h play a vital role in the enhancement or reduction in the values of ${I_v}^*$. Besides, ${I_v}^*$ is a decreasing function of μ_v andσ and a increasing function of Π_h. An increment in μ_v of 10%, implies a reduction in the value of ${I_v}^*$ of 41%. Similarly, by increasing the value of σ in 10%, implies a reduction of ${I_v}^*$ of 28%. We should focus on the mortality rate of bark beetles as well as the felling rate of infectious pines in order to achieve the decreased endemic level of infectious pines.

The sensitivity indices for ${I_h}^*$ given in Table 2, express that the most sensitive parameter for the endemic level of infectious pines I_h^* is μ_v. An increment of μ_v in 10%, implies that ${I_h}^*$ would decrease almost 55%. The endemic level of infectious hosts is also an increasing function of Π_v. By decreasing the value of Π_v by 10%, ${I_h}^*$ would decrease almost 29%.

The above results are corroborated in Figures 2 and 3. Figure 2 shows the endemic level of infectious hosts for different values of the parameters. We can see that parameters Π_h, Π_V, β_1, δ_1, β_2, δ_2, η, β_3 and β are directly related to ${I_h}^*$ and inversely related to μ_h, μ_v, σ and α. It can also be observed that a considerable change occurs in the value of ${I_h}^*$ by an increment or reduction of Π_h, Π_v, μ_v, β_1, δ_1 and α. On the other hand, we can see from Figure 3 that the endemic level of infectious bark beetles (vectors) is greatly influenced by the parameters Π_v and μ_v.

7 Discussion

In this paper, we proposed and analyzed a PWD mathematical model, considering both direct and indirect transmission of the disease. For the model, the basic reproduction number was calculated using the next generation operator, and we discussed the asymptotic behavior of the equilibrium solutions using a graph theoretic approach. More specifically, we constructed a suitable Lyapunov function and we proved that if $\mathcal{R}_0 < 1$, then the DFE is GAS, and thus the disease always dies out, while if $\mathcal{R}_0 > 1$, the unique endemic equilibrium $\mathbf{E}^*$ exists and is GAS, so that the disease persists at the endemic equilibrium.

For the reduction of the endemic level of infected classes, we applied a useful tech-

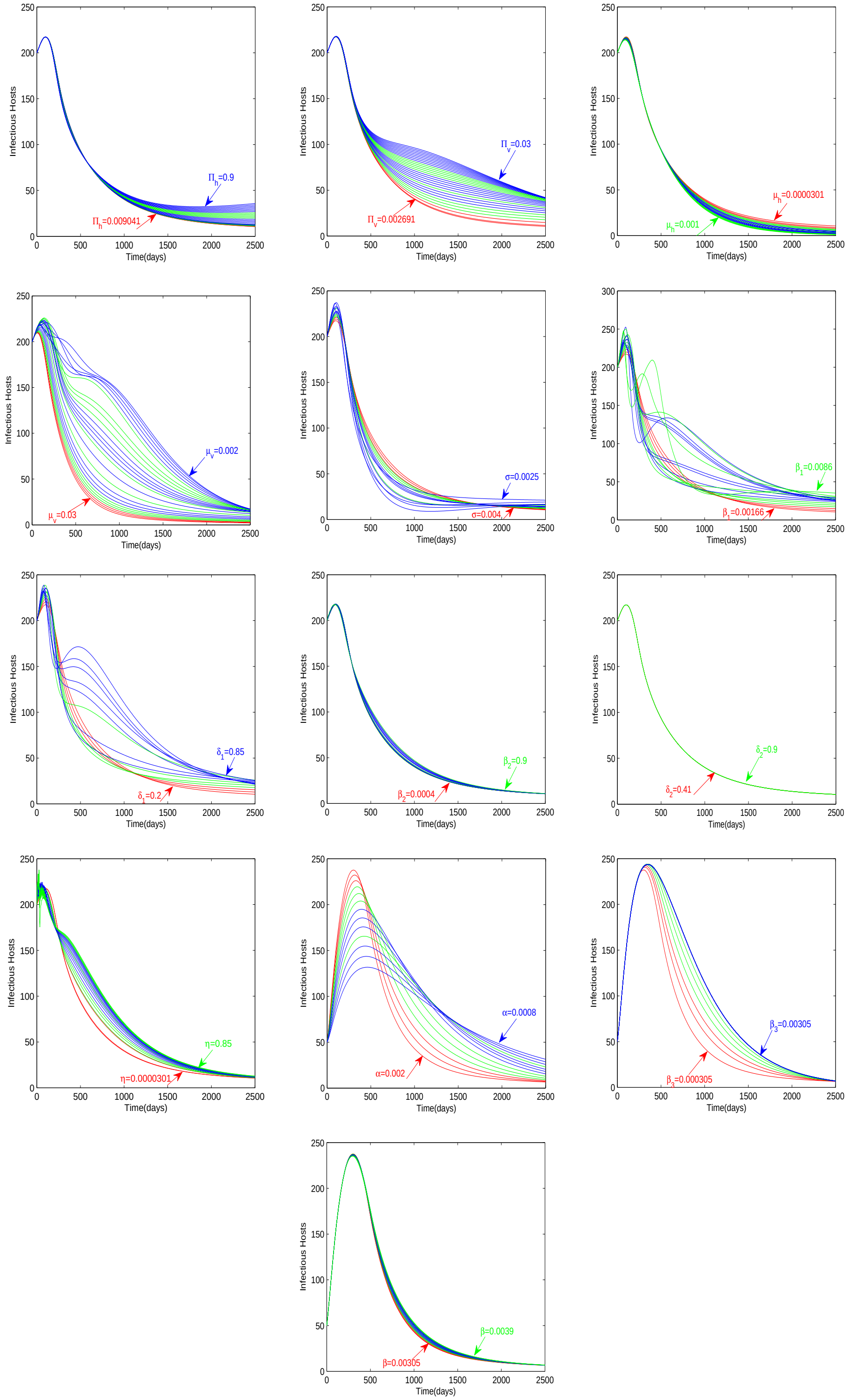

Figure 2: Endemic level of infected hosts for different values of the parameters.

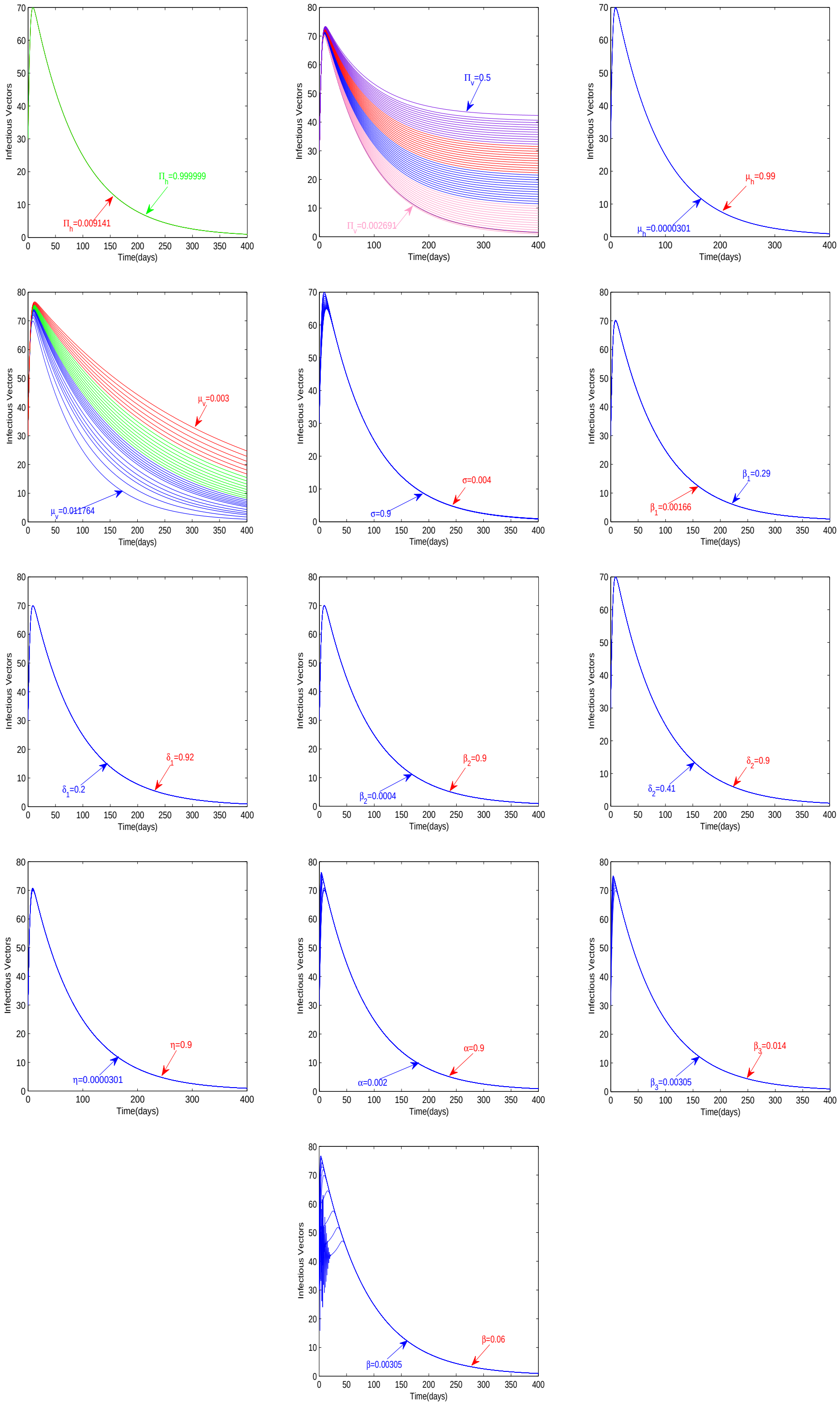

Figure 3: Endemic level of infected vectors for different values of the parameters.

nique of local sensitivity analysis of parameters. Firstly, we calculated the sensitivity indices of the basic reproduction number and the endemic levels of infectious hosts and vectors. We computed the ratio of relative change in the parameters to the relative change in the variables. The parameter which gives the highest value of the variable is the most sensitive parameter. We observed that the numerical value of exploitation rate of susceptible pine trees is maximum but we could not exploit susceptible pine trees for the complete eradication of the disease. However, we should focus on the second sensitive parameter to reduce the reproduction number below unity. The calculations tell us that β_1 is the parameter which can help us to reduce the value of reproduction number. Similarly by the analysis of sensitivity indices of I_h^* and I_v^*, we found that the mortality of bark beetles is essential to reduce the infection from the community. Secondly, we observed how the endemic level of infectious hosts and vectors is effected by changing the values of parameters. We performed a variation in parameter values and observed the corresponding change in the endemic level of infectious hosts and vectors.

Sensitivity analysis help us to design effective control strategies for the eradication of the disease or to reduce the endemic level of infected classes because it best describes the crucial factors that play vital role in the transmission of disease. An open problem in this work is to incorporate those parameters as functions of control, and analyze the corresponding optimal control problem.

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