

Subgroups of the projective special linear group $PSL_2(K)$ that contain a projective root subgroup

S. Sabbar, A. Widodo, N. Hidayat, A. Alghofari

Abstract. Suppose k is a subfield of the field K has characteristic different from 2, such that K is an algebraic extension over k . In the currently study we try to describe irreducible subgroups of $PSL_2(K)$ that contains a projective root subgroup is conjugate in $GL_2(K)/Z$, where Z is the center of $GL_2(K)$ consists of all scalar matrices to an intermediate group consisting of all matrices $diag\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, 1, \dots, 1\right)$, $a \in K$.

M.S.C. 2010: 14Lxx, 20G40.

Key words: Special Linear Group; Projective transvection; projective root; k -subgroup; Projective Special Linear Group.

1 Introduction

The study of any group includes the investigation of its quotient groups and its subgroups, since this allows to gain more information about the group itself and the structure of the group. Our study will focus on dealing with the special linear groups $SL_2(K)$ especially of degree 2 over various fields concerning some results of the subgroup structure of the $PSL_2(K)$. In [22], Zalesskii and Serezhkin classified of the irreducible linear groups generated by transvections over finite fields. In [1], Bashkirov were described subgroups of $SL_2(K)$ over infinite fields. In [2] described irreducible subgroups that containing a root k -subgroup of $GL_2(K)$. Bashkirv's theorem can be formulated as follows.

Theorem 1.1. *Suppose the field K is an algebraic extension of the subfield k , when k has char $\neq 2$ and $k \neq GF(3), GF(9)$. If an irreducible group G generated by transvections containing a root k -subgroup such that $G \leq GL_n(K)$, and then, G contains a normal subgroup which conjugate in $GL_n(K)$ to a group in $SL_n(L)$, where L is a subfield of K such that $(k \leq L \leq K)$.*

The objective of display our study is to acquire some additional developments of Bashkirov's theorem with regard to the case of projective linear groups, also regard transvections as a particular case of more general matrices or linear transformations, namely as a particular case of quadratic unipotent elements. In [4, 5] the author

dealing with irreducible linear groups of degree three and four respectively over a quaternion division algebra D . The exceedingly important matrix groups of classical groups are generated by quadratic unipotent elements (see [7]). In the present paper, we consider such groups not only for linear groups but also for projective linear groups thereby developing the materials that are contained in [13, 14, 20]. Transvections being the simplest example of quadratic unipotent elements have a crucial role in the theory of matrix groups to description subgroups of linear groups over various fields, and there are many articles and books, see for instance [10, 11, 15, 17, 18, 19]. In ([3], Theorem 1.1), the author described subgroups be an irreducible contains a transvection is conjugate in $GL_n(K)$ a member of the set consisting $SL_n(L)$, where L is a subring of quaternion division ring D , so in [8, 6] he studied completely reducible subgroups of $GL_n(K)$ that conjugate in $GL_n(K)$ to a group of all matrices $\text{diag}\left(\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}, 1, \dots, 1\right)\right) \in GL_n(K)$, $a \in k$. This paper expands our conception of the projective linear group, and we can form our main result by the next theorem.

Theorem 1.2. *Let k be a field of characteristic $\neq 2$ and containing more than 9 elements. Let K be an algebraic extension of k . Then every subgroup G of $PSL_2(K)$ that contain a projective root k -subgroup $T = t_{12}(k)Z$ either normalizes in $PSL_2(K)$ the projective root K -subgroup $t_{12}(K)Z$ or is conjugate in $GL_2(K)/Z$ to a group that contains the group $PSL_2(L)$, where L is a subfield of K containing k , as a normal subgroup.*

Let K with identity 1 is an arbitrary associative ring and n is a natural number. Recall that $M_n(K)$ denotes the associative ring of all n by n matrices with entries in K . The set of all invertible elements (units) of the ring $M_n(K)$ forms a group $GL_n(K)$ which is called the general linear group of n by n matrices with entries in K . The set of all $n \times n$ matrices M with entries in K such that $\det M = 1$ forms a subgroup of $GL_n(K)$. This is called the special linear group $SL_n(K)$ of n by n matrices over K .

Let $s \in K^n$ (column) and $\psi \in {}^nK$ (row). Then both product $s\psi, \psi s$ are defined. The product of ψ and s , ψs is an element of the field K , whereas $s\psi$ is an $n \times n$ matrix with entries from K . Assume $\psi s = 0$. Then the $n \times n$ matrix $g = I_n + s\psi$ is called a **transvection**. More precisely, we say that g is the transvection corresponding to the pair $(s, \psi) \in K^n \times {}^nK$.

This transvection is $I_n + \alpha e_{ij}$, where e_{ij} is the standard matrix unit which has 1 in its (i, j) position and elsewhere equal zeros. This transvection is called an elementary transvection and denoted by $t_{ij}(\alpha)$. Thus

$$t_{ij}(\alpha) = I_n + \alpha e_{ij} \quad (i \neq j, \alpha \in K).$$

The determinant of each elementary transvection is 1, and so all elementary transvection is in the $SL_n(K)$. The subgroup of the $SL_n(K)$ is called the elementary subgroup generated by all elementary transvection denoted by $E_n(K)$. Thus by definition

$$E(K) = \langle t_{ij}(\alpha) \mid \alpha \in K, 1 \leq i \neq j \leq n \rangle.$$

If g is a transvection, that is if $g = I_n + s\psi$, where $s \in K^n$, $\psi \in {}^nK$ and $\psi s = 0$, then

$$g - I_n = I_n + s\psi - I_n = s\psi,$$

and hence

$$(g - I_n)^2 = s\psi s\psi.$$

But since $\psi s = 0$, have been obtaining that

$$(g - I)^2 = s \cdot 0 \cdot \psi = 0_n,$$

where 0_n designates the zero n by n matrix.

Any matrix $x \in M_n(K)$ satisfying the condition $(x - I_n)^2 = 0$, this matrix is called a quadratic unipotent. Thus transvections give a particular example of quadratic unipotent matrices. Let x be a quadratic unipotent matrix of $M_n(K)$, then the matrix $x - I_n$, satisfying the condition $(x - I_n)^2 = 0$ is a quadratic unipotent matrix of index 2. If the ring K is a field, then according to the well-known description of the nilpotent matrix (see for instance, [16]) any nilpotent matrix of index 2 is conjugate in the group $GL_n(K)$ to the matrix

$$\text{diag} \left(\left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right), \dots, \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right), 0, \dots, 0 \right)$$

(all empty positions should be read as zero). Thus every quadratic unipotent $n \times n$ matrix x over the field K conjugate in $GL_n(K)$ to the matrix

$$\text{diag} \left(\left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right), \dots, \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right), 1, \dots, 1 \right).$$

Here the number of blocks $\left(\begin{array}{cc} 1 & r \\ 0 & 1 \end{array} \right)$, say r , is referred to as the residue of the quadratic unipotent element x . From the definition of transvection, one can deduce that every transvection over the field K is conjugate in the group $GL_n(K)$ to the matrix

$$\begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} = t_{12}(1)$$

and thus any transvection over a field is a quadratic unipotent element of residue 1. Note in passing that this implies, in particular, then the determinant of every transvection is 1, and so each transvection is an element of the $SL_n(K)$.

2 Preliminary Results

One of the basic problems with which we are concerned is the stems from a strong classical result dealing with the generation of the $SL_2(K)$ over K . The proof of this result appears in ([1] Theorem 4) author has described subgroups of the $SL_2(K)$ generated by transvection. This theorem is written as follows

Theorem 2.1. ([1]) *If k is a field of char $\neq 2$, and if K is an algebraic extension of k , such that*

$$SL_2(k) \leq G \leq SL_2(K)$$

then G contains the group $SL_2(L)$ as a normal subgroup, such that L is a subfield of K contains k , ($k \leq L \leq K$). If we have G generated by the transvections, then $G = SL_2(L)$.

The intention of the present study is to gain some additional results of the Theorem 2.1 with regard to the case of projective linear groups

Let k be an arbitrary field and $n \geq 2$ an integer. We know that the center of the special linear group $SL_n(k)$ consists of all scalar matrices with determinant 1. In other words, a matrix $g \in SL_n(k)$ belongs to the center of $SL_n(k)$ if and only if g is of the form αI_n , where α is an element of k such that $\alpha^n = 1$. In particular, if $n = 2$, the center of $SL_2(k)$ is the subgroup of all matrices αI_2 with $\alpha^2 = 1$ (recall,

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

the identity matrix). Now suppose the characteristic of k is not equal to 2. In any such a field, the equation $\alpha^2 = 1$ has exactly two roots, ± 1 , and therefore, in this case, the center of $SL_2(k)$ is the subgroup

$$Z = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\} = \{\pm I_2\}$$

isomorphic to a cyclic group of order 2. From this, one can conclude, that the center of the group SL_2 does not depend on the field over which this group is considered. Now assume K be an extension field of a subfield k , then by the above discussion the center of $SL_2(K)$ is the same subgroup Z that containing all scalar matrices. Any coset of $SL_2(k)$ by the subgroup Z , gZ , with $g \in SL_2(k)$ is simultaneously the coset of $SL_2(K)$ by Z . Thus the set of all cosets of $SL_2(k)$ is contained in the set of all cosets of $SL_2(K)$ by Z . But the set of all cosets of Z in $SL_2(k)$ forms the projective special linear group $PSL_2(k)$ over k , whereas the set of all cosets of Z in $SL_2(K)$ forms the projective special linear group over K , $PSL_2(K)$. Thus we obtain that if the field k is a subfield of the field K , then the group $PSL_2(k)$ is a subgroup of $PSL_2(K)$. So it is reasonable to consider intermediate groups lying between $PSL_2(k)$ and $PSL_2(K)$, that is, groups G such that

$$PSL_2(k) \leq G \leq PSL_2(K)$$

By Theorem 2.1 we try to conclude theorem to describe intermediate subgroups between $PSL_2(k)$ and $PSL_2(K)$. But before that, we will introduce elements which are the main tool of our investigation, namely, projective transvections.

Definition 2.1. An element $x \in PSL_2(k)$, that is, a coset

$$gZ = \{g, -g\},$$

where g is some element in $SL_2(k)$ will be called a projective transvection if x as a coset of Z in $SL_2(k)$ has a representative which is a transvection of $SL_2(k)$.

Thus an element

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} Z \in PSL_2(k)$$

is a projective transvection because the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is a transvection of the group $SL_2(k)$ (an elementary transvection). We have

$$\begin{aligned} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} Z &= \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, -\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \right\}. \end{aligned}$$

Also

$$\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} Z = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} Z,$$

and therefore the coset

$$\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} Z \in PSL_2(k)$$

is also a projective transvection. Note, however, that the matrix

$$\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$$

is not a transvection. To see this, we recall that every transvection $g \in SL_2(K)$ is a quadratic unipotent element, that is, $(g - I_2)^2 = 0$. But

$$\begin{aligned} &\left(\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & +1 \end{pmatrix} \right)^2 \\ &= \begin{pmatrix} -2 & -1 \\ 0 & -2 \end{pmatrix}^2 = \begin{pmatrix} 4 & 4 \\ 0 & 4 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

because by our assumption the field k is of characteristic, not equal 2, and hence $4 \neq 0$. Thus a coset

$$gZ \in PSL_2(k) \quad (g \in SL_2(k))$$

can be a projective transvection even in the case when g is not a transvection. Also, the group $PSL_2(k)$ contains elements that are not transvection at all. As a concrete example of such elements, we assume that $\text{char } k \neq 3$ (for instance, we can take $k = \mathbb{Q}$) and consider the coset

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix} Z = \left\{ \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -2 \end{pmatrix} \right\} \in PSL_2(k).$$

Then

$$\left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)^2$$

$$= \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix}$$

which is distinct from the zero matrix because $1 \neq 0$ and $\text{char } k \neq 2, 3$; and

$$\begin{aligned} & \left(\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)^2 \\ &= \begin{pmatrix} -\frac{3}{2} & 0 \\ 0 & -3 \end{pmatrix}^2 = \begin{pmatrix} \frac{9}{4} & 0 \\ 0 & 3 \end{pmatrix} \end{aligned}$$

which is distinct from the zero matrix in view of our assumption $\text{char } k \neq 2, 3$. Thus both of

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -2 \end{pmatrix}$$

are not quadratic unipotent elements and hence they are not transvections. This means that the coset

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix} Z$$

is not projective transvection.

By Theorem 2.1 has been concluded our theorem in [21] to describe intermediate subgroups lying between $PSL_2(k)$, and $PSL_2(K)$. This description is given by next theorem.

Theorem 2.2. *Assume k is a subfield of a field K of characteristic not equal 2 containing more than 9 elements, and let K is an algebraic extension of k . If*

$$PSL_2(k) \leq X \leq PSL_2(K),$$

then X as a normal subgroup contains the group $PSL_2(L)$, where L containing k is a subfield of K . If X is generated by projective transvections, then $X = PSL_2(L)$.

Now let us recall that $GL_n(F)$ is the general linear group of degree n over a field F can be viewed as the group $GL(V)$ of all invertible linear transformations (automorphism) of an n -dimensional vector space V over the field F . Thus any subgroup of $GL_n(F)$ (including the special linear group $SL_n(F)$) can be considered as a group of invertible linear transformations of V .

Now let C be a subgroup of $GL_n(F)$ viewed as the automorphism group of the n -dimensional vector space V , and let U be a subspace of V . Then U is said to be invariant under C (or C -invariant) if $g(u) \in U$ for all $u \in U$ and for all $g \in C$.

The group C is said to be irreducible if the space V has no nonzero proper subspaces that are invariant under C . In other words, C is irreducible if and only if the zero subspace $\{0\}$ and the space V itself are the only subspaces of V invariant under C . Now have been needing to form the following auxiliary assertion involving the notion of an irreducible linear group.

Lemma 2.3. *Let k is a subfield of a field K . A subgroup H of $GL_2(K)$ that contain a root k -subgroup*

$$t_{12}(k) = \left\{ \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \mid r \in k \right\}$$

and a matrix

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K)$$

with $c \neq 0$, is irreducible.

Proof. We identify any 2 by 2 matrix whose entries lie in K with a linear transformation of a 2-dimensional vector space V over K having a fixed basis e_1, e_2 . Let U be a nonzero subspace of V such that U is invariant under H . We show that $U = V$.

Assume first that $e_2 \in U$. Then

$$t_{12}(1)(e_2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} (e_2) = e_1 + e_2 \in U$$

Hence

$$e_1 = (e_1 + e_2) - e_2 \in U$$

because U is a subspace and so closed under subtraction (addition). Thus both of vectors e_1, e_2 belong to U . But e_1, e_2 is a basis of V and therefore every vector of V can be written as a linear combination of e_1, e_2 . This together with fact that U is a subspace of V implies that every vector of V is in U , and thus $U = V$.

Now suppose that $e_2 \notin U$. Hence if $e = \beta e_1 + \gamma e_2$ ($\beta, \gamma \in K$) is an arbitrary nonzero vector in U , then $\beta \neq 0$ (indeed, if β were zero, then $e = \gamma e_2 \in U$, where $\gamma \neq 0$ since $e \neq 0$, and so we would have

$$\gamma^{-1}e = \gamma^{-1}(\gamma e_2) = (\gamma^{-1}\gamma)e_2 \in U$$

which is false). Hence the vector $\beta^{-1}e$ is defined, and since this vector is in U (the subspace U is closed under multiplication by elements of the field K), we may replace e by

$$\beta^{-1}e = \beta^{-1}(\beta e_1 + \gamma e_2) = \beta^{-1}\beta e_1 + \beta^{-1}\gamma e_2 = e_1 + \beta^{-1}\gamma e_2,$$

and assume $\beta = 1$, that is $e = e_1 + \gamma e_2$. If $\gamma = 0$, that is, if $e = e_1 \in U$, then U contains the vector

$$t_{12}(1)(e_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} (e_1) = e_1 + e_2,$$

and hence the vector

$$(e_1 + e_2) - e_1 = e_2$$

which contradicts our assumption $e_2 \notin U$. Therefore, $\gamma \neq 0$. Then U contains the vector

$$\begin{aligned} t_{12}(1)(e) &= t_{12}(1)(e_1 + \gamma e_2) \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} (e_1 + \gamma e_2) = e_1 + \gamma(e_1 + \gamma e_2) = e_1(1 + \gamma) + \gamma e_2, \end{aligned}$$

and so the vector

$$\begin{aligned} [e_1(1 + \gamma) + \gamma e_2] - e &= e_1(1 + \gamma) + \gamma e_2 - (e_1 + \gamma e_2) \\ &= e_1(1 + \gamma - 1) + \gamma e_2 - \gamma e_2 = e_1\gamma. \end{aligned}$$

But $\gamma \neq 0$, so U contains $\gamma^{-1}(\gamma e_1) = e_1$ which is impossible as we have already seen. The lemma is proved. \square

Next, will be introduced a family of subgroups of the PSL_2 . Each transvection in $SL_2(K)$ has the form

$$\begin{pmatrix} 1 + \lambda_1\mu_1 & \lambda_1\mu_2 \\ \lambda_2\mu_1 & 1 - \lambda_1\mu_1 \end{pmatrix},$$

where the elements $\lambda_1, \lambda_2, \mu_1, \mu_2 \in K$ satisfy the equation $\lambda_1\mu_1 + \lambda_2\mu_2 = 0$. For any $r \in K$, the matrix

$$\begin{pmatrix} 1 + \lambda_1\mu_1r & \lambda_1\mu_2r \\ \lambda_2\mu_1r & 1 - \lambda_1\mu_1r \end{pmatrix}$$

is a transvection too, and the set of all these transvections forms a subgroup of $SL_2(K)$ which is called a root k -subgroup of $SL_2(K)$ corresponding to the given transvection

$$\begin{pmatrix} 1 + \lambda_1\mu_1 & \lambda_1\mu_2 \\ \lambda_2\mu_1 & 1 - \lambda_1\mu_1 \end{pmatrix}.$$

Therefore, the set of all cosets

$$\begin{aligned} & \begin{pmatrix} 1 + \lambda_1\mu_1r & \lambda_1\mu_2r \\ \lambda_2\mu_1r & 1 - \lambda_1\mu_1r \end{pmatrix} Z \\ &= \left\{ \begin{pmatrix} 1 + \lambda_1\mu_1r & \lambda_1\mu_2r \\ \lambda_2\mu_1r & 1 - \lambda_1\mu_1r \end{pmatrix}, - \begin{pmatrix} 1 + \lambda_1\mu_1r & -\lambda_1\mu_2r \\ \lambda_2\mu_1r & 1 - \lambda_1\mu_1r \end{pmatrix} \right\} \end{aligned}$$

with r is a subgroup of $PSL_2(K)$, ranging over k , and we shall refer to this subgroup as a *projective root k -subgroup* of $PSL_2(K)$. A concrete example of a projective root k -subgroup is provided by the group of all cosets

$$\begin{aligned} t_{12}(r)Z &= \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} Z \\ &= \left\{ \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -r \\ 0 & -1 \end{pmatrix} \right\} \end{aligned}$$

with r ranging over k

Note that Z , being a normal subgroup of the center of $GL_2(K)$, and so the quotient group $GL_2(K)/Z$ is defined.

3 Proof of the Main Result

Proof. Assume that G does not normalize $t_{12}(K)Z$. Again let

$$\varphi : SL_2(K) \rightarrow SL_2(K)/Z$$

be the canonical homomorphism. Denote by H the full preimage of G under φ . Since $G \geq T$, the full preimage of T under φ consists the of all matrices

$$\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -r \\ 0 & -1 \end{pmatrix}$$

with $r \in k$. In particular, H contains a root k -subgroup

$$t_{12}(k) = \left\{ \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \mid r \in k \right\},$$

that is, we have

$$t_{12}(k) \leq H \leq SL_2(K).$$

Assume that H normalizes $t_{12}(K)$, that is, for any $h \in H$ and for any $r \in k$,

$$h^{-1}t_{12}(r)h \in t_{12}(K).$$

Then

$$\begin{aligned} & (hZ)^{-1}(t_{12}(r)Z)(hZ) \\ &= (h^{-1}Z)(t_{12}(r)Z)(hZ) \end{aligned}$$

By ([12], Theorem 22.1)

$$= h^{-1}t_{12}(r)hZ \in t_{12}(K)Z$$

which shows that

$$hZ \in \varphi(H) = G$$

must normalize $t_{12}(K)Z$. Since any element in G has the form hZ with $h \in H$, this implies that G normalizes $t_{12}(K)Z$ which contradicts the assumption made at the beginning of the proof. Thus H does not normalize $t_{12}(K)$ and hence H contains a matrix, h say, whose (21)-entry is different from 0. Indeed, assume that each matrix in H has 0 as its (21)-entry. This means that each matrix $h \in H$ can be written as

$$h = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

with $a, b, c \in K$. Moreover, $\det h = 1$, so $c = a^{-1}$ and

$$h = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}.$$

Therefore, for any each $r \in K$, we have

$$\begin{aligned} h^{-1}t_{12}(r)h &= \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}^{-1} \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \\ &= \begin{pmatrix} a^{-1} & -b \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \\ &= \begin{pmatrix} a^{-1} & a^{-1}r - b \\ 0 & a \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & a^{-1}b + a^{-2}r - ba^{-1} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & a^{-2}r \\ 0 & 1 \end{pmatrix} \in t_{12}(K). \end{aligned}$$

This shows that h normalizes $t_{12}(K)$ and since h is an arbitrary element of H , we obtain that H normalizes $t_{12}(K)$ which is false. Thus H contains

$$h_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $c \neq 0$. By Lemma 2.3, the group

$$\langle t_{12}(k), h_0 \rangle,$$

which is a subgroup of H , is irreducible and hence H is irreducible itself. Thus H is an irreducible subgroup of $SL_2(K)$ containing a root k -subgroup $t_{12}(k)$. By ([1], Theorem 5), there exists a subfield L of a field K whereas $k \leq L$ and H is conjugate in $GL_2(K)$ to a subgroup containing the group $SL_2(L)$ as a normal subgroup. In other words, for some $\sigma \in GL_2(K)$,

$$SL_2(L) \trianglelefteq \sigma^{-1}H\sigma.$$

Applying the homomorphism φ to both sides of the last relation, we obtain

$$\varphi(SL_2(L)) \trianglelefteq \varphi(\sigma^{-1}H\sigma),$$

that is,

$$\varphi(SL_2(L)) \trianglelefteq \varphi(\sigma)^{-1}\varphi(H)\varphi(\sigma).$$

But

$$\varphi(SL_2(L)) = PSL_2(L) \text{ and } \varphi(H) = G.$$

Also, we have $\varphi(\sigma) = \sigma Z \in GL_2(K)/Z$, whence we conclude that

$$PSL_2(L) \trianglelefteq \varphi(\sigma)^{-1}G\varphi(\sigma)$$

which accomplish the proof of the theorem □

References

- [1] E. L. Bashkirov, *On subgroups of the special linear group of degree 2 over an infinite field*, Mat. Sb. 187 (1996), 19-36.
- [2] E. L. Bashkirov, *Linear groups that contain a root subgroup*, Siberian Math. J. 37 (1996), 754-759.
- [3] E. L. Bashkirov, *Some completely reducible linear groups over a quaternion division ring containing a root subgroup*, Comm. Algebra. 31 (2003), 5727-5754.
- [4] E. L. Bashkirov, *Irreducible linear groups of degree 3 over a quaternion division ring containing a root subgroup*, Comm. Algebra, 32 (2004), 1747-1761.
- [5] E. L. Bashkirov, *Irreducible linear groups of degree four over a quaternion division algebra that contains root subgroup*, Comm. Algebra, 34 (2005), 1931-1948.
- [6] E. L. Bashkirov, *Completely reducible linear groups over a quaternion division algebra that contain a root subgroup*, Comm. Algebra. 35(2007), 1019-1054.
- [7] E. L. Bashkirov, *On subgroups of the group GL_7 over a field that contains a Chevalley group of type G_2 over a subfield*, to appear in Journal of Pure and Applied Algebra, 219 (2015), 1992-2014.
- [8] E. L. Bashkirov, C. K. Gupta, *Linear groups over locally finite extensions of infinite fields*, Internat. Journal Algebra and Comp. 17 (2007), 905-922.
- [9] E. L. Bashkirov, C. K. Gupta, *Linear groups over integral extensions of semilocal commutative rings*, Communications in Algebra, 42 (2014), 4149-4171.

- [10] P. M. Cohn, *On the structure of GL_2 over a ring*, Inst. Hautes Études Sci. Publ. Math. 30 (1966), 5-53.
- [11] H. Cuypers, A. Steinbach, *Linear transvection groups and embedded polar spaces*, Invent. Math. 137 (1999), 169–198.
- [12] J. R. Durbin, *Modern algebra, an Introduction*, John Weyley and Sons Inc, 2009.
- [13] R. H. Dye, *Maximal subgroups of $GL_{2n}(K)$, $SL_{2n}(K)$, $PGL_{2n}(K)$, $PSL_{2n}(K)$ associated with symplectic polarities*, J. Algebra, 66 (1980), 1-11.
- [14] R. H. Dye, *The conjugacy classes of fixed point free elements of order three or four in $GL_{2n}(K)$, $SL_{2n}(K)$, $PGL_{2n}(K)$, $PSL_{2n}(K)$* , J. Algebra, 169 (1994), 399-417.
- [15] E. W. Ellers, J. Malzan, *The length problem for the special unitary group generated by positive transvections*, J. Univ. Kuwait Sci. 20 (1993), 1–24.
- [16] D.T. Finkbeiner, *Introduction to matrices and linear transformations*, Freeman and Company, San Fransisco and London, 1960.
- [17] S. P. Humphries, *Some subgroups of $SL(3, Z)$ generated by transvections*, Comm. Algebra, 20 (1992), 1675-1703.
- [18] J. McLaughlin, *Some groups generated by transvections*, Arch. Math. (Basel) 18 (1967), 364-368.
- [19] A. I. Steinbach, *Subgroups of classical groups generated by transvections or Siegel transvections*, I, II, Geom. Dedicata, 8 (1997), 281-322, 323–357.
- [20] S. M. Sabbar, *Certain subgroups of the projective special linear $PSL_2(K)$ over a field K* , Master Dissertation: Fatih University, Istanbul, Turkey, 2015.
- [21] S. M. Sabbar, *Intermediate subgroups of the projective special linear group $PSL_2(K)$ over various fields*, Baghdad Sciences Journal, 2019
- [22] A. E. Zalesskii, V. N. Serezhkin, *Linear groups generated by transvections*, Izv. Akad. Nauk SSSR. Ser. Mat. 40 (1976), 26-49.

Author's address:

Shakir Sabbar (correspondent author)
 Brawijaya University, Department of Mathematics and natural Sciences,
 ca. Main Road 12, Alanbar, Iraq.
 E-mail: shamartr@gmail.com

Agus Widodo
 Brawijaya University, Department of Mathematics and natural Sciences,
 Jl. Raya ITS, Keputih, Malang, Indonesia.
 E-mail: prof.agus_widodo@yahoo.com

Noor Hidayat
 Brawijaya University, Department of Mathematics and natural Sciences,
 Jl. Satsui Tubun II, Malang, Indonesia.
 E-mail: noorh@ub.ac.id

Abdul Rouf Alghofari
 Brawijaya University, Department of Mathematics and natural Sciences,
 Jl. Satsui Tubun II, Malang, Indonesia.
 E-mail: abdul_rouf@ub.ac.id