Combinatorial Geometry and its Algorithmic Applications
by Janos Pach and Micha Shahir

(The Alcalá Lectures), American Mathematical Society,
Math. Surveys and Monographs 152, 2009, ISBN 978-0-8218-4691-9; viii+235 pages

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For the hurried reader, we present the abridged Contents of the book: Preface and Apology; **Chapter 1.** Sylvester-Gallai problem: The Beginnings of Combinatorial Geometry; **Chapter 2.** Arrangements of Surfaces: Evolution of the Basic Theory; **Chapter 3.** Davenport-Schinzel Sequences: The Inverse Ackermann Function in Geometry; **Chapter 4.** Incidences and Their Relatives: From Szemeredi and Trotter to Cutting Lenses; **Chapter 5.** Crossing Numbers of Graphs: Graph Drawing and its Applications; **Chapter 6.** Extremal Combinatorics: Repeated Patterns and Pattern Recognition; **Chapter 7.** Lines in Space: From Ray Schooting to Geometric Transversals; **Chapter 8.** Geometric Coloring Problems: Sphere Packings and Frequency Allocation; **Chapter 9.** From Sam Loyd to Laszlo Fejes Toth: The 15 Puzzle and Motion Planning; Bibliography; Index.

For the interested reader, we give here some more details about the book. The starting point was a series of lectures given by the authors in Alcalá (Spain), in 2006, covering some of their team work and completed with historical and state-of-the-art notes.

The "Big Bang" of Combinatorial Geometry seems to be an apparently innocent question put by James Sylvester in the *Educational Times*, in 1893: *Is it true that any finite set of points in the Euclidean plane, not all on a line, has two elements whose connecting line does not pass through a third* ? The answer is affirmative, but the first published proof appeared only after half a century. Meanwhile, new exciting problems arose, about incidence between points and lines (in the plane or in space); about incidence between points and circles or spheres; concerning arrangements of curves in the plane ("Graph drawing") or surfaces (spheres, boxes,…) in higher dimensions.

Applications of Combinatorial geometry include but are not limited to: pattern matching and recognition; ray shooting and hidden surface removal in Computer graphics; chromatic numbers of graphs used for frequency allocation in cellular telephone networks; motion planning (variations of the "piano movers" problem on graphs and grids); reconfiguration of robotic systems.

The book under review may be used at many levels and for various purposes. Firstly, it may be considered as a course in Computational geometry for a Master degree, due to the pedagogic style of the authors and to the big number of problems within. Secondly, this book would be a gem for young mathematicians, looking for an "Ariadne's string" to guide them in the labyrinth of modern Computational geometry; the state-of-the-art comments and the 760 items in the references recommend it as a valuable research resource. Thirdly, we belive that many computer scientists will find exciting ideas for new algorithms, in order to solve problems arising from the domains of applications previously quoted.

Last but not least, lovers of Geometry as an Art will discover the flavor of Harmony, Beauty and Mistery, hidden in every quest, revealed from pictures and singing in proofs the Eternal Glory of the human spirit.

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Riemannian Holonomy Groups and Calibrated Geometry
by Dominic D. Joyce

Oxford University Press, Oxford, New York, 2007

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The book is based on some previous research projects of the author. More than half the book is a revised version of parts of his monograph *Compact Manifolds with Special Holonomy*, Oxford University Press, 2000 – which is the core of the Riemannian holonomy material (Chapters one to seven and Chapter ten); here are included, besides the classical topic, some own results of the author concerning the G_2 and *Spin*(7) holonomy of compact manifolds. But the monograph under review has added more very interesting areas and new results of the author in the field, regarding quaternionic Kahler manifolds in Chapter 10, exceptional holonomy groups in Chapter 11 and, and, most important, four new chapters on calibrated geometry (Chapters 4, 8, 9 and 12).

The stated intention of the author, made in the Preface, is that the book be not *a vehicle for publishing my own research*, but for selecting *material based on how I see the field and what I think it would be useful for a new researcher in the subject to know*. This goal of the author is excellently accomplished, in our opinion.

The content of the first three chapters covers the basic differential geometry and the analysis on manifolds used in the paper. The first chapter, entitled *Background material*, includes exterior forms on manifolds, de Rham cohomology, Hodge theory, Sobolev and Holder spaces. The second chapter is entitled *Introduction to connections, curvature and holonomy groups*. Here are considered in brief connections on vector bundles (in particular on the tangent bundle of a manifold) and on principal bundles, as well as *G*-structures. *Riemannian holonomy groups* is the title of the third chapter. Marcel Berger's theorem (concerning the list of holonomy groups that are possible on simply-connected manifolds with irreducible and non-symmetric metric) is explained, as well as the relation of the holonomy group with the topology of the underlying manifold.

The definition of *Calibrated Geometry* and its link to holonomy groups is performed in the fourth chapter. The theory was invented by Harvey and Lawson in 1982. The calibrated submanifolds are a special kind of minimal submanifolds defined by a closed form, called *calibration*. The relation between calibrated geometry and holonomy groups, central to the book, is explained. Constant calibrations and the natural integral currents (which involve geometric measure theory) are also studied.

The fifth chapter is dealing with *Kahler manifolds*. One begins with basic definitions, tensors on complex manifolds and their decomposition. Then holomorphic vector bundles, Kahler metrics, Kahler potential, curvature of a Kahler metric and further exterior forms are discussed, first from the perspective of Riemannian geometry and then using complex algebraic geometry. The link to calibrated geometry is finally performed.

The sixth chapter is devoted to *The Calabi Conjecture*. The proof is performed reformulating it as a nonlinear, elliptic partial differential equation in an unknown real function. The proof guarantees the existence of a Kahler metric with zero Ricci form.

Calabi-Yau manifolds are studied in the seventh chapter. From the enormous material in the literature, the author gives to the reader a doubtless path: the basic differential geometry of Calabi-Yau manifolds, then orbifolds and crepant resolutions and finally ways to construct Calabi-Yau manifolds. The holonomy groups $Sp(k)$ and $SU(m)$, as well as compact Ricci-flat manifolds are very important ingredients, basically used in the exposition.

The eighth chapter is entitled *Special Lagrangian geometry*. Here one studies special Lagrangian submanifolds (*SL* m-folds for short) in $\mathbf{C}^n$, or in a Calabi-Yau m-fold or almost Calabi-Yau m-fold, calibrated by the real part of the holomorphic volume form. The basic theory, constructions and examples are given on: *SL* cones, asymptotically conical *SL* m-folds, then m-folds in (almost) Calabi-Yau m-folds, compact, nonsingular and singular *SL* m-folds.

The *Mirror symmetry and the SYZ Conjecture* are studied in the ninth chapter. The mirror symmetry is a non-classical relationship between pairs of Calabi-Yau 3-folds. It was discovered by physicist working in string theory. The importance of special Lagrange geometry in this theory is via SYZ conjecture, that explains mirror symmetry in terms of dual fibrations with special Lagrangian fibres. Two rather different 'explanations' of mirror symmetry are discussed: the homological mirror symmetry conjecture and the SYZ Conjecture.

Hyperkahler and quaternionic Kahler manifolds, are studied in the tenth chapter. Their holonomy lies in $Sp(m)$ (thus the metric is Kahler and Ricci-flat) and in $Sp(m)\ Sp(1)$ (they are Einstein manifolds of positive or negative scalar curvature, never Kahler, but having a twistor space). Some interesting examples and problems are discussed here.

The exceptional holonomy groups G_2 in 7 dimensions and $Spin(7)$ in 8 dimensions are discussed in the eleventh chapter. A reading list of exceptional holonomy, inciting examples and applications are given. A motivation of the particular interest in this study comes from physics (M-theory in string theory).

Associative, coassociative and Cayley submanifolds are defined and studied in the last chapter. These are calibrated submanifolds: two in 7-manifolds with holonomy G_2 (called *associative* 3-*folds* and called *coassociative* 4-*folds*, respectively) and one in 8-manifold with holonomy $Spin(7)$ (called *Cayley* 4-*folds*). Coassociative 4-folds can be defined using a closed form, but the other two cannot, their study requiring different technques. The discussion is given in the same fair spirit as in the whole book.

In our opinion, this is a well written book by an eminent specialist in the field, that provides generous guidance to the reader. We recommend the book to specialists as well as to beginners, which can find here not only a brought up-to-date source in the field, but also invitations to understand and to approach some deep and difficult problems from mathematics and physics.

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Fourier--Mukai Transforms in Algebraic Geometry
by D. Huybrechts

Oxford Mathematical Monographs, 2006,viii+307 pp.,
Oxford University Press, ISBN 0-19-929686-3 (978-0-19-929686-6)

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The book is based on a course given by the author at the Institut de Mathématiques de Jussieu in 2004 and 2005. Conceived as a first specialized course in algebraic geometry, it is focused into the subject of derived category of coherent sheaves on a smooth projective variety. The study is motivated by the results by Mukai in 80's years about equivalences between derived categories of non-isomorphic varieties, and by that of Moscow school in the last 15 years, by Kontsevich's homological mirror theory and by Bondal and Orlov's results concerning the role of derived categories in classification of varieties etc.

The background of the first three chapters of R. Hartshorne's Algebraic geometry book (Springer, 1977) suffices to follow the lectures. Other notions, occasionally needed in the book (e.g. singular cohomology, Hodge theory, abelian varieties, K3 surfaces) are then presented in a suitable manner at their places.

The first three chapters of the book give the basic background on derived categories and derived functors. They are entitled: 1. Triangulated categories, 2. Derived categories: a quick tour and 3. Derived categories of coherent shaves.

The next chapters are: 4. Derived category and canonical bundle - I, 5. Fourier--Mukai transforms, 6. Derived category and canonical bundle - II, 7. Equivalence criteria for Fourier--Mukai transforms, 8. Spherical and exceptional objects, 9. Abelian varieties, 10. K3 surfaces, 11. Flips and flops, 12. derived categories of surfaces and 13. Where to go from here. Chapter 4 is devoted to results by Bondal and Orlov concerning classification of derived categories of smooth projective varieties in dimension one and the description of the group of autoequivalences of the bounded derived categories of such varieties. In Chapter 6 one refine these results, showing that Kodaira dimension and canonical ring (Orlov) as well as nefness of the canonical bundle and the numerical Kodaira dimension (Kawamata) are preserved under derived equivalence. Fourier--Mukai transform is defined and first studied in Chapter 5, then in Chapter 7 one study when it is in fact an equivalence. The consideration of spherical objects in Chapter 8 give rise to braid group (Seidel and Thomas); some extensions using spectral sequences, or a more general geometric context are also presented here. The general context of abelian varieties is considered in Chapter 9, based on results by Mukai and Orlov. A next step is performed in Chapter 10, where following Mukai and Orlov, a derived version of the global Torelli theorem (that two K3 surfaces are isomorphic if and only if they have isomorphic periods) is given. The changes under a blow-up is studied in Chapter 11. The situation of isomorphic surfaces that follows from their isomorphic derived categories is studied in Chapter 12. It is proved, following Bridgeland, Maciocia and Kawamata, that it is true for surfaces that are neither elliptic, nor K3, nor abelian. Pointers to more advanced topics are given in Chapter 13, where a deeper preparation is needed for the reader to understand the material, only sketched and proofs are missing; here, the author proposes a deeper look inside some hot problems, as an invitation for an interested reader.

To conclude with, we think that the monograph under review is very helpful to all the people interested to study in algebraic geometry the derived category of coherent sheaves on a smooth projective variety. The author is a very good specialist, having original contributions in the field. The book is well written and self contained, modulo some basic facts covered by an introductory course in algebraic geometry (as pointed in our beginning). The definitions are very clear and the proofs are generously explained. The bibliography has 118 titles, covering the basic and related sources, in many directions.

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Algebraic Geometry and Arithmetic Curves, by Quing Liu

**Oxford Graduate Texts in Mathematics 6, 2006, xv+577 pp.,
Oxford University Press, ISBN 0-19-920249-4 (978-0-19-920249-2)**

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The monograph under review is intended as an introduction to Algebraic and Arithmetic Geometry, in the language of schemes, dealing with both arithmetic surfaces and the reduction of algebraic curves.

The ten chapters are organized in two parts. In the first part, the author treats: **I.** Some topics in commutative algebra (tensor products, flatness, formal completion, the Artin-Rees lemma)**; II.** General properties of schemes (spectrum of a ring, ringed topological spaces, schemes, reduced schemes and integral schemes,dimension); **III.** Morphisms and base change (the technique of base change, applications to algebraic varieties, some global properties of morphisms); **IV.** Some local properties (normal schemes, regular schemes, flat morphisms and smooth morphisms, Zariski's "Main Theorem" and applications); **V.** Coherent sheaves and Cech cohomology (coherent sheaves on a scheme, Cech cohomology,cohomology of projective schemes); **VI**. Sheaves of differentials (Kahler differentials, differential study of smooth morphisms, local complete intersection, duality theory, Grothendieck duality); **VII.** Divisors and applications to curves (Cartier divisors, Weil divisors, Van der Waarden's purity theorem, Riemann-Roch theorem, algebraic curves, Hurewitz formula, singular curves and the structure of $Pic^0(X)$).

The second part contains: **VIII.** Birational geometry of surfaces (blowing-ups, excellent schemes, fibered surfaces); **IX.** Regular surfaces (intersection theory on a regular surface, intersection and morphisms, minimal surfaces, Castelnuovo's criterion, applications to contraction, Artin's contractability criterion); **X.** Reduction of algebraic curves (models and reductions, reduction of elliptic curves, Neron models of elliptic curves, stable reduction of algebraic curves, Deligne-Mumford theorem, proof of Artin-Winters).

The book ends with a bibliography containing 97 references; an index of symbols and an index of notions allow an easier and quicker recovering of the key information.

The exposition is clear, with a good balance between rigor and intuitive hints, between formal reasoning and insightful examples. More than 600 exercises help the reader to interact actively with the text. All these valuable features make this monograph a good reference for graduate students and young researchers.

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Quasiconformal Maps and Teichmüller Theory
by A. Fletcher and V. Markovic

Oxford Graduate Texts in Mathematics 11, 2007, viii+189 pp.,
Oxford University Press, ISBN 0-19-856926-2 (978-0-19-856926-8)

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Riemann had proved that the set M_g of equivalence classes of closed Riemann surfaces of genus g>1 can be parametrized locally with 6g-6 real parameters. The set M_g is named a moduli space.

Teichmüller published in the period 1938-1944 six articles in which he studied Riemann's moduli problem by introducing the notion of quasiconformal maps between two Riemann surfaces S, S' and proving his fundamental theorem: in each homotopy class of homeomorphisms $f: S \to S'$ there exists a unique "extremal" quasiconformal map. To this aim, he used pairs of holomorphic quadratic differential forms on S and S'. The moduli spaces are described as unions of Euclidean metric spaces.

Teichmüller's theory has been improved by many authors, among which we mention the names of Lavrentiev, Ahlfors, Bers, Pflüger and Mori. An important tool introduced by these authors is the Beltrami's equation.

Fletcher and Markovic give in their book of 189 pages a rich survey of Teichmüller's theory, including the latest achievements. The book contains 11 chapters with the following titles: The Grötzsch argument, Geometric definition of quasiconformal maps, Analytic properties of quasiconformal maps, Quasi-isometries and quasisymmetric maps, The Beltrami differential equation, Holomorphic motions and applications, Teichmüller spaces, Extremal quasiconformal mappings, Unique extremality, Isomorphisms of Teichmüller spaces, Local rigidity of Teichmüller spaces.

Fletcher and Markovic's book is an excellent graduate textbook. Each part of the book is rigorously and pedagogically elaborated, but the reviewer considers that, globally, Teichmüller's main results are not sufficiently underlined. There is also a local critical remark: at page 89, in Theorem 7.5.1., the roles of the functions f, g should be reversed.

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Topology: A Geometric Approach
by Terry Lawson

Oxford Graduate Texts in Mathematics 9, 2003, xv+388 pp.,
Oxford University Press, ISBN 0-19-851597-9 (978-0-19-851597-5)

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As the author declares in the very first line of the preface: "the book is intended to introduce advanced undergraduates and beginning graduate students to topology, with an emphasis on its geometric aspects". What is meant by "geometric aspects" is also mentioned, although not very explicit: for example the detailed study of surface classification (and the emphasis on the understanding of low-dimensional spaces), elementary geometric transforms and maybe also the fundamental group. We can accept that geometric intuition is helpful and also, in some sense, basic so we agree with the point of view of the author. The book contains a rich supply of figures which would stress the geometric concept behind and help the reader to assimilate the abstract notions involved.

The book in divided into two parts: the first part is a "Geometric Introduction to Topology" and the second treats "Covering Spaces, CW Complexes and Homology". After a chapter on basic point set topology (with a section on geometric transforms in the plane and another on Jordan curve theorem) the first part of the book contains a chapter dedicated to the classification of surfaces and a chapter about the fundamental group and its applications (for example vector fields on surfaces are treated). The second part starts with a chapter on covering spaces followed by a chapter on CW complexes and finally a (long) chapter on homology put an end to the book.

The style is quite clear, precise and the explanations detailed enough to make the book very readable. The proofs are in general complete and elegant the examples abound (as in the description of the basic surfaces, in the computation of fundamental groups or in the applications of singular homology, to cite only at random). There are also various comments which prove to be very useful, in my opinion, for those at their first contact with such subtle questions as treated in the book.

But the most interesting and stimulating idea of the author seems to be that of adding a huge quantity of exercises, both in the text (at almost every step) as at the end of each chapter (of the first part). I find the exercises very interesting and useful for a good understanding of the notions and a decent master of these notions. New notions are also introduced during these exercises and these notions could give, to the student, a feeling of new things to be studied and, why not, to leave some room for invention. For example, the first chapter has 97 supplementary exercises, the second chapter 109 etc. There are comments to the exercises showing the intentions of the author and explaining the subject and the reasons of these exercises. At the end of the book there are solutions of some selected exercises. As for the difficulty of the exercises we find very easy ones but also quite difficult with deep inside and far reaching conclusions.

As a final conclusion, we have a very nice, well planned and written book which can be used both for individual, active study and for an one year-long course. The teacher has, using the book, many strategies to choose in selecting the material and a big supply of application at hand. The subject of topology will be always of great interest for various reasons and good text books welcome.

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