The Curvature Theory of Generalized Connection in $Osc^2 M$

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Abstract

In the paper are presented the explicit expressions for the components of the torsion and curvature of a generalized connection on the 2-osculator bundle of a real differentiable manifold; the corresponding Ricci identities for the generalized connection are also derived.

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1 Introduction

Let M be a C^∞, n-dimensional manifold and $E = (Osc^2 M, \pi, M)$ its 2-osculator bundle. The K-osculator bundle was studied among others in [15], [16], [17], where the adapted basis is determined. As we here need more types of indices, the adapted basis of $T(E)$ and $T^*(E)$ will be presented.

Some point $u \in E$ in the local charts (U, φ) and (U', φ') has coordinates (x^i, y^i, z^i) and $(x^{i'}, y^{i'}, z^{i'})$ respectively, $i = 1, 2, \dots, n$. In $U \cap U'$ the allowable coordinate transformations are given by the equations

$$(1.1)\qquad \begin{aligned} x^{i'} &= x^{i'}(x) \qquad rank\left|\frac{\partial x^{i'}}{\partial x^i}\right| = n \\ y^{i'} &= \frac{\partial x^{i'}}{\partial x^j} y^j \\ z^{i'} &= \frac{1}{2}\frac{\partial^2 x^{i'}}{\partial x^k \partial x^j} y^k y^j + \frac{\partial x^{i'}}{\partial x^j} z^j. \end{aligned}$$

It can be shown, that the transformations of type (1.1) form a group.

The adapted basis B of $T(E)$ is

$$(1.2)\qquad B = \left\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^i}, \frac{\partial}{\partial z^i}\right\},$$

where

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(1.3) $$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - (1)\mathcal{N}_i^j \frac{\partial}{\partial y^j} - (2)\mathcal{N}_i^j \frac{\partial}{\partial z^j}$$

(1.4) $$\frac{\delta}{\delta y^i} = \frac{\partial}{\partial y^i} - (1)\mathcal{N}_i^j \frac{\partial}{\partial z^j}$$

Theorem 1.1. *The element of B ((1.2)) with respect to (1.1) are transformed as tensors, i.e.*

(1.5) $$\frac{\delta}{\delta x^i} = \frac{\partial x^{i'}}{\partial x^i}\frac{\delta}{\delta x^{i'}},\ \frac{\delta}{\delta y^i} = \frac{\partial x^{i'}}{\partial x^i}\frac{\delta}{\delta y^{i'}},\ \frac{\partial}{\partial z^i} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial}{\partial z^{i'}}$$

if the nonlinear connections $(1)\mathcal{N}$ *and* $(2)\mathcal{N}$ *are transformed in the following way:*

(1.6) $$(1)\mathcal{N}_{i'}^{j'} = (1)\mathcal{N}_i^j \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^{j'}}{\partial x^j} - \frac{\partial y^{j'}}{\partial x^i}\frac{\partial x^i}{\partial x^{i'}},$$

(1.7) $$(2)\mathcal{N}_{i'}^{j'} = (2)\mathcal{N}_i^j \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^{j'}}{\partial x^j} + (1)\mathcal{N}_i^j \frac{\partial y^{j'}}{\partial x^j}\frac{\partial x^i}{\partial x^{i'}} - \frac{\partial z^{j'}}{\partial x^i}\frac{\partial x^i}{\partial x^{i'}}.$$

The basis of $T^*(E)$ dual to B is

(1.8) $$B^* = \{dx^i, \delta y^i, \delta z^i\},$$

where

(1.9) $$\delta y^i = dy^i + (1)\mathcal{M}_j^i dx^j$$

(1.10) $$\delta z^i = dz^i + (1)\mathcal{M}_j^i dy^j + (2)\mathcal{M}_j^i dx^j.$$

Theorem 1.2. *The elements of B^* ((1.8)) with respect to (1.1) are transformed as*

(1.11) $$dx^{i'} = \frac{\partial x^{i'}}{\partial x^i}dx^i,\ \delta y^{i'} = \frac{\partial x^{i'}}{\partial x^i}\delta y^i,\ \delta z^{i'} = \frac{\partial x^{i'}}{\partial x^i}\delta z^i$$

if for the nonlinear connections $(1)\mathcal{M}$ *and* $(2)\mathcal{M}$ *the following equations are valid*

(1.12) $$(1)\mathcal{M}_j^i = (1)\mathcal{N}_j^i$$

(1.13) $$(2)\mathcal{M}_j^i = (2)\mathcal{N}_j^i + (1)\mathcal{N}_m^i (1)\mathcal{N}_j^m.$$

If we denote as T_H, T_{V_1}, T_{V_2} the subspaces of $T(E)$ spanned by $\{\frac{\delta}{\delta x^i}\}$, $\{\frac{\delta}{\delta y^i}\}$, $\{\frac{\partial}{\partial z^i}\}$ and as T^*_H, $T^*_{V_1}$, $T^*_{V_2}$ the subspaces of $T^*(E)$ spanned by $\{dx^i\}$, $\{\delta y^i\}$, $\{\delta z^i\}$ respectively, then

$$T(E) = T_H \oplus T_{V_1} \oplus T_{V_2}, \qquad T^*(E) = T^*_H \oplus T^*_{V_1} \oplus T^*_{V_2}.$$

For the further examinations it is useful to introduce different kinds of indices. Indices $i, j, h, k, l = \overline{1, n}$ will be used in T_H and T_H^* $a, b, c, d, e, f = \overline{n+1, \ldots, 2n}$ in T_{V_1} and $T_{V_1}^*$, $p, q, r, s, t = \overline{2n+1, \ldots, 3n}$ in T_{V_2} and $T_{V_2}^*$. The Greek letters as indices will take values from 1 to $3n$. Using these notations the adapted basis B and B^* given by (1.2) and (1.8) have the form:

$$B = \{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^a}, \frac{\partial}{\partial z^p}\} = \{\delta_i, \delta_a, \partial_p\} = \{\delta_\alpha\}, \tag{1.14}$$

$$B^* = \{dx^j, \delta y^b, \delta z^p\} = \{\delta^\beta\}, \tag{1.15}$$

where from (1.3), (1.4), (1.9) and (1.10) we get

$$\begin{array}{lrcl} & \text{(a)} \ \frac{\delta}{\delta x^i} & = & \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i} - (1)\mathcal{N}_i^b \frac{\partial}{\partial y^b} - (2)\mathcal{N}_i^q \frac{\partial}{\partial z^q} \\ & \text{(b)} \ \frac{\delta}{\delta y^a} & = & \frac{\partial}{\partial y^a} - (1)\mathcal{N}_a^q \frac{\partial}{\partial z^q} \\ (1.16) & \text{(c)} \ \delta y^a & = & dy^a + (1)\mathcal{N}_i^a dx^i \\ & \text{(d)} \ \delta z^q & = & dz^q + (1)\mathcal{N}_a^q dy^a + ((2)\mathcal{N}_j^q + (1)\mathcal{N}_j^r (1)\mathcal{N}_r^q) dx^j = \\ & & = & dz^q + (1)\mathcal{N}_a^q \delta y^a + (2)\mathcal{N}_j^q dx^j. \end{array}$$

In (1.16)

$$(1)\mathcal{N}_i^j = (1)\mathcal{N}_i^b = (1)\mathcal{N}_a^q, \ (2)\mathcal{N}_i^q = (2)\mathcal{N}_i^j$$

if $i = a(mod n)$ and $j = b = q(mod n)$.

Some vector field $X \in T(E)$ and some 1-form $w \in T^*(E)$ expressed in the bases B and B^* have the form:

$$\begin{array}{rcl} X & = & X^i \frac{\delta}{\delta x^i} + X^a \frac{\delta}{\delta y^a} + X^p \frac{\partial}{\partial z^p} = X^\alpha \delta_\alpha \\ w & = & w_j dx^j + w_b \delta y^b + w_q \delta z^q = w_\beta \delta^\beta. \end{array} \tag{1.17}$$

With respect to (1.1) the coordinates of X and w transform in the following way:

$$\begin{array}{rcl} X^{i'} & = & X^i \frac{\partial x^{i'}}{\partial x^{i'}}, \ X^{a'} = X^a \frac{\partial y^{a'}}{\partial y^a}, \ X^{p'} = X^p \frac{\partial z^{p'}}{\partial z^p}, \\ w_{j'} & = & w_j \frac{\partial x^j}{\partial x^{j'}}, \ w_{b'} = w_b \frac{\partial y^b}{\partial y^{b'}}, \ w_{q'} = w_q \frac{\partial z^q}{\partial z^{q'}}, \end{array}$$

because if $i = a = p(mod n)$ we have

$$\frac{\partial x^{i'}}{\partial x^i} = \frac{\partial y^{a'}}{\partial y^a} = \frac{\partial z^{p'}}{\partial z^p}.$$

2 The generalized connection on $T(E)$

Let $\nabla : T(E) \times T(E) \to T(E)$ be a linear connection such that $\nabla : (X, Y) \to \nabla_X Y \in T(E), \forall X, Y \in T(E)$.

Definition 2.1. The generalized connection on $T(E)$ is a linear connection ∇ determined by:

$$\begin{array}{lrcl} & \text{(a)} \ \nabla_{\delta_i} \delta_\beta & = & F_{\beta\ i}^{\ \kappa} \delta_\kappa, \\ (2.1) & \text{(b)} \ \nabla_{\delta_a} \delta_\beta & = & C_{\beta\ a}^{\ \kappa} \delta_\kappa \\ & \text{(c)} \ \nabla_{\partial_p} \delta_\beta & = & L_{\beta\ p}^{\ \kappa} \delta_\kappa, \end{array}$$

where $\beta = j$ or $\beta = b$ or $\beta = q$ and

$$T^{\cdot\cdot\kappa}_{\cdots}\delta_\kappa = T^{\cdot\cdot k}_{\cdots}\delta_k + T^{\cdot\cdot c}_{\cdots}\partial_c + T^{\cdot\cdot r}_{\cdots}\partial_r. \tag{2.2}$$

We shall use the abbreviated form of (2.1):

$$\nabla_{\delta_\alpha}\delta_\beta = \Gamma_{\beta\ \alpha}^{\ \kappa}\delta_\kappa. \tag{2.3}$$

From (2.1) and (2.3) follows:

If $\alpha = i$, then $\Gamma = F$; if $\alpha = a$, then $\Gamma = C$; if $\alpha = p$, then $\Gamma = L$.

Proposition 2.1. If X is the vector field ((1.17)) defined on E, then the following equations are valid:

$$\begin{array}{lll} \nabla_{\delta_i}X = X^\alpha_{|i}\delta_\alpha & , & X^\alpha_{|i} = \delta_i X^\alpha + F_{\beta\ i}^{\ \alpha}X^\beta \\ \nabla_{\partial_a}X = X^\alpha|_a\delta_\alpha & , & X^\alpha|_a = \partial_a X^\alpha + C_{\beta\ a}^{\ \alpha}X^\beta \\ \nabla_{\partial_p}X = X^\alpha\|_p\delta_\alpha & , & X^\alpha\|_p = \partial_p X^\alpha + L_{\beta\ p}^{\ \alpha}X^\beta, \end{array} \tag{2.4}$$

where

$$\Gamma_{\beta\ .}^{\ \cdot}X^\beta = \Gamma_{j\ .}^{\ \cdot}X^j + \Gamma_{b\ .}^{\ \cdot}X^b + \Gamma_{q\ .}^{\ \cdot}X^q,\ (\Gamma = F \text{ or } \Gamma = C \text{ or } \Gamma = L) \tag{2.5}$$

Theorem 2.1. *If X and Y are vector fields in TE expressed in basis B, ∇ the generalized connection defined by (2.1), then*

$$\nabla_Y X = (X^\alpha_{|\beta})Y^\beta\delta_\alpha, \tag{2.6}$$

where

$$^{\cdots}_{\cdots|\beta}Y^\beta = {}^{\cdots}_{\cdots|j}Y^j + {}^{\cdots}_{\cdots}|_b Y^b + {}^{\cdots}_{\cdots}\|_q Y^q. \tag{2.7}$$

Theorem 2.2. *All covariant derivatives $X^\alpha_{|i}$, $X^\alpha|_a$, $X^\alpha\|_p$ ($\alpha = j$, or $\alpha = b$, or $\alpha = p$) from (2.4) are transformed as tensors with respect to (1.1) if all connection coefficients from (2.1) are transformed as tensors, except the following which have the form:*

$$F_{j\ i}^{\ k} = F_{j'\ i'}^{\ k'}\frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^{j'}}{\partial x^j} + \frac{\partial^2 x^{k'}}{\partial x^i\partial x^j}\frac{\partial x^k}{\partial x^{k'}} \tag{2.8}$$

$$F_{b\ i}^{\ c} = F_{b'\ i'}^{\ c'}\frac{\partial x^{i'}}{\partial x^i}\frac{\partial y^{b'}}{\partial y^b}\frac{\partial y^c}{\partial y^{c'}} + \frac{\partial^2 y^{c'}}{\partial x^i\partial y^b}\frac{\partial z^r}{\partial y^{c'}} \tag{2.9}$$

$$F_{q\ i}^{\ r} = F_{q'\ i'}^{\ r'}\frac{\partial x^{i'}}{\partial x^i}\frac{\partial z^{q'}}{\partial z^q}\frac{\partial z^r}{\partial z^{r'}} + \frac{\partial^2 z^{r'}}{\partial x^i\partial z^q}\frac{\partial z^r}{\partial z^{r'}} \tag{2.10}$$

Theorem 2.3. *The torsion tensor T for the generalized connection ∇ has the form:*

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = T_{\alpha\ \beta}^{\ \kappa} Y^\alpha X^\beta \delta_\kappa, \tag{2.11}$$

where

$$T_{\alpha\ \beta}^{\ \kappa} = \Gamma_{\alpha\ \beta}^{\ \kappa} - \Gamma_{\beta\ \alpha}^{\ \kappa} \tag{2.12}$$

except the following components:

$$\begin{array}{lcl} T_{j\ i}^{\ c} & = & F_{j\ i}^{\ c} - F_{i\ j}^{\ c} + K_{j\ i}^{\ c} \\ T_{j\ b}^{\ c} & = & C_{j\ b}^{\ c} - F_{b\ j}^{\ c} + K_{j\ b}^{\ c} = -T_{b\ j}^{\ c} \\ T_{j\ i}^{\ r} & = & F_{j\ i}^{\ r} - F_{i\ j}^{\ r} + K_{j\ i}^{\ r} \\ T_{j\ b}^{\ r} & = & C_{j\ b}^{\ r} - F_{b\ j}^{\ r} + K_{j\ b}^{\ r} = -T_{b\ j}^{\ r} \\ T_{j\ q}^{\ r} & = & L_{j\ q}^{\ r} - F_{q\ j}^{\ r} + K_{j\ q}^{\ r} = -T_{q\ j}^{\ r} \\ T_{b\ a}^{\ r} & = & C_{b\ a}^{\ r} - C_{a\ b}^{\ r} + K_{a\ b}^{\ r} \end{array} \tag{2.13}$$

where

$$\begin{array}{lcl} K_{i\ j}^{\ c} & = & \delta_j(1)\mathcal{N}_i^c - \delta_i(1)\mathcal{N}_j^c, \\ K_{i\ j}^{\ r} & = & \delta_j(2)\mathcal{N}_i^r - \delta_i(2)\mathcal{N}_j^r + (1)\mathcal{N}_c^r K_{i\ j}^{\ c}, \\ K_{i\ b}^{\ c} & = & \delta_b(1)\mathcal{N}_i^c, \\ K_{i\ b}^{\ r} & = & \delta_b(2)\mathcal{N}_i^r - \delta_i(1)\mathcal{N}_b^r + (1)\mathcal{N}_c^r K_{i\ b}^{\ c}, \\ K_{i\ q}^{\ r} & = & \partial_q(2)\mathcal{N}_i^r, \\ K_{a\ b}^{\ r} & = & \delta_b(1)\mathcal{N}_a^r - \delta_a(1)\mathcal{N}_b^r. \end{array} \tag{2.14}$$

Proof. By direct calculation we obtain

$$[X,Y] = X^\alpha(\partial_\alpha Y^\beta)\partial_\beta - Y^\alpha(\partial_\alpha X^\beta)\partial_\beta + X^\alpha Y^\beta(\partial_\alpha\partial_\beta - \partial_\beta\partial_\alpha), \tag{2.15}$$

$$X^\alpha Y^\beta(\partial_\alpha\partial_\beta - \partial_\beta\partial_\alpha) = X^\alpha Y^\beta K_{\alpha\ \beta}^{\ \kappa}\partial_\kappa = \tag{2.16}$$

$$= X^i Y^j (K_{i\ j}^{\ c}\delta_c + K_{i\ j}^{\ r}\partial_r) + (X^i Y^b - Y^i X^b)(K_{i\ b}^{\ c}\delta_c + K_{i\ b}^{\ r}\partial_r) +$$

$$+ (X^i Y^q - Y^i X^q) K_{i\ q}^{\ r}\partial_r + X^a Y^b K_{a\ b}^{\ r}\partial_r.$$

Substituting (2.15) and (2.16) into (2.11) we obtain (2.12) and (2.13).

3 The curvature theory of ∇

The curvature tensor for the generalized connection ∇ is defined as usual

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{3.1}$$

If we use the notations

$$X = X^\alpha\partial_\alpha,\ Y = Y^\beta\partial_\beta,\ Z = Z^\gamma\partial_\gamma,$$

then we have

$$\nabla_X\nabla_Y Z = \nabla_{X^\alpha\partial_\alpha}\nabla_{Y^\beta\partial_\beta}Z^\gamma\partial_\gamma = \nabla_{X^\alpha\partial_\alpha}[Y^\beta(\partial_\beta Z^\gamma)\partial_\gamma + Y^\beta Z^\gamma\Gamma^{\ \kappa}_{\gamma\ \beta}\partial_\kappa] = X^\alpha(\partial_\alpha Y^\beta)(\partial_\beta Z^\gamma)\partial_\gamma + X^\alpha Y^\beta(\partial_\alpha\partial_\beta Z^\gamma)\partial_\gamma + X^\alpha Y^\beta(\partial_\beta Z^\gamma)\Gamma^{\ \kappa}_{\gamma\ \alpha}\partial_\kappa + X^\alpha(\partial_\alpha Y^\beta)Z^\gamma\Gamma^{\ \kappa}_{\gamma\ \beta}\partial_\kappa + X^\alpha Y^\beta(\partial_\alpha Z^\gamma)\Gamma^{\ \kappa}_{\gamma\ \beta}\partial_\kappa + X^\alpha Y^\beta Z^\gamma(\partial_\alpha\Gamma^{\ \kappa}_{\gamma\ \beta})\partial_\kappa + X^\alpha Y^\beta Z^\gamma\Gamma^{\ \theta}_{\gamma\ \beta}\Gamma^{\ \kappa}_{\theta\ \alpha}\partial_\kappa. \tag{3.2}$$

From (2.15) and (2.16) follows

$$\nabla_{[XY]}Z = X^\alpha(\partial_\alpha Y^\beta)(\partial_\beta Z^\gamma)\partial_\gamma + X^\alpha(\partial_\alpha Y^\beta)Z^\gamma\Gamma^{\ \kappa}_{\gamma\ \beta}\partial_\kappa - Y^\alpha(\partial_\alpha X^\beta)(\partial_\beta Z^\gamma)\partial_\gamma - Y^\alpha(\partial_\alpha X^\beta)Z^\gamma\Gamma^{\ \kappa}_{\gamma\ \beta}\partial_\kappa + X^\alpha Y^\beta[(\partial_\alpha\partial_\beta - \partial_\beta\partial_\alpha)Z^\gamma]\partial_\gamma + X^\alpha Y^\beta Z^\gamma K^{\ \theta}_{\alpha\ \beta}\Gamma^{\ \kappa}_{\gamma\ \theta}\partial_\kappa. \tag{3.3}$$

Substituting (3.2) and (3.3) into (3.1) we obtain

$$R(X,Y)Z = [K^{\ \kappa}_{\gamma\ \beta\alpha}X^\alpha Y^\beta - (K^{\ c}_{i\ j}C^{\ \kappa}_{\gamma\ c} + K^{\ q}_{i\ j}L^{\ \kappa}_{\gamma\ q})X^iY^j - (K^{\ c}_{i\ b}C^{\ \kappa}_{\gamma\ c} + K^{\ r}_{i\ b}L^{\ \kappa}_{\gamma\ r})(X^iY^b - Y^iX^b) - K^{\ r}_{i\ q}L^{\ \kappa}_{\gamma\ r}(X^iY^q - Y^iX^q) - K^{\ r}_{a\ b}L^{\ \kappa}_{\gamma\ r}X^aY^b]Z^\gamma\partial_\kappa, \tag{3.4}$$

where

$$(3.5)K^{\ \kappa}_{\gamma\ \beta\alpha} = (\partial_\alpha\Gamma^{\ \kappa}_{\gamma\ \beta} - \Gamma^{\ \theta}_{\gamma\ \alpha}\Gamma^{\ \kappa}_{\theta\ \beta}) - (\alpha,\beta). \tag{3.5}$$

As the indices $\alpha,\ \beta,\ \gamma,\ \kappa$ belong to one of the sets $\{i,j,k,l,\ldots\}$, $\{a,b,c,d,\ldots\}$, $\{p,q,r,s,t,\ldots\}$ (corresponding to T_H, T_{V_1}, T_{V_2} respectively), so on the TE we have $3^4 = 81$ types of curvature tensors. It is meaningless to introduce different letters as R, P, S for the curvature tensors as in Finsler geometry.

We shall denote

$$R^{\ \kappa}_{\gamma\ \beta\alpha} = K^{\ \kappa}_{\gamma\ \beta\alpha} \tag{3.6}$$

for all (β,α) except when $(\beta,\alpha) = (j,i)$, $(\beta,\alpha) = (i,b)$, $(\beta,\alpha) = (i,q)$ and $(\beta,\alpha) = (b,a)$. In these cases we have

$$\begin{aligned} R^{\ \kappa}_{\gamma\ ji} &= K^{\ \kappa}_{\gamma\ ji} - K^{\ c}_{i\ j}C^{\ \kappa}_{\gamma\ c} - K^{\ q}_{i\ j}L^{\ \kappa}_{\gamma\ q},\\ R^{\ \kappa}_{\gamma\ ib} &= K^{\ \kappa}_{\gamma\ ib} + K^{\ c}_{i\ b}C^{\ \kappa}_{\gamma\ c} + K^{\ r}_{i\ b}L^{\ \kappa}_{\gamma\ r} = -R^{\ \kappa}_{\gamma\ bi},\\ R^{\ \kappa}_{\gamma\ iq} &= K^{\ \kappa}_{\gamma\ iq} + K^{\ p}_{i\ q}L^{\ \kappa}_{q\ p} = -R^{\ \kappa}_{\gamma\ qi},\\ R^{\ \kappa}_{\gamma\ ba} &= K^{\ \kappa}_{\gamma\ ba} - K^{\ r}_{a\ b}L^{\ \kappa}_{\gamma\ r} = -R^{\ \kappa}_{\gamma\ ab}. \end{aligned} \tag{3.7}$$

As $R^{\ \kappa}_{\gamma\ \beta\alpha} = -R^{\ \kappa}_{\gamma\ \alpha\beta}$ we can write (3.4) in the form:

$$R(X,Y)Z = [\tfrac{1}{2}K^{\ \kappa}_{\gamma\ \beta\alpha}(X^\alpha Y^\beta - Y^\alpha X^\beta) - \tfrac{1}{2}(K^{\ c}_{i\ j}C^{\ \kappa}_{\gamma\ c} + K^{\ q}_{i\ j}L^{\ \kappa}_{\gamma\ q})(X^iY^j - Y^iX^j) - \tfrac{1}{2}(K^{\ c}_{i\ b}C^{\ \kappa}_{\gamma\ c} + K^{\ r}_{i\ b}L^{\ \kappa}_{\gamma\ r}(X^iY^b - Y^iX^b) + \tfrac{1}{2}(K^{\ c}_{i\ b}C^{\ \kappa}_{\gamma\ c} + K^{\ r}_{i\ b}L^{\ \kappa}_{\gamma\ r})(Y^iX^b - X^iY^b) - \tfrac{1}{2}K^{\ r}_{i\ q}L^{\ \kappa}_{\gamma\ r}(X^iY^q - Y^iX^q) + \tfrac{1}{2}K^{\ r}_{i\ q}L^{\ \kappa}_{\gamma\ r}(Y^iX^q - X^iY^q)] - \tfrac{1}{2}K^{\ r}_{a\ b}L^{\ \kappa}_{\gamma\ r}(X^aY^b - Y^aX^b)Z^\gamma\delta_\kappa \tag{3.8}$$

For $(\beta,\alpha)=(j,i)$ the sum of the first and the second line in (3.8) is equal to $\frac{1}{2}R_{\gamma\ ji}^{\ \kappa}(X^iY^j-Y^iX^j)$, for $(\beta,\alpha)=(b,i)$ the sum of the first and the third line in (3.8) is equal to $\frac{1}{2}R_{\gamma\ bi}^{\ \kappa}(X^iY^b-Y^iX^b)$ etc.

From (3.4)–(3.8) follows

Theorem 3.1. *The curvature tensor of the generalized connection ∇ has the form*

$$R(X,Y)Z=\frac{1}{2}R_{\gamma\ \beta\alpha}^{\ \kappa}(X^\alpha Y^\beta-Y^\alpha X^\beta)Z^\gamma\delta_\kappa, \tag{3.9}$$

where the components of R are determined by (3.6) and (3.7).

Formula (3.9) is short and elegant, but the explicit form of curvature tensor is much longer, for instance if $(\beta,\alpha)=(b,i)$ from (3.5) and (3.7) we have:

$$\begin{aligned}R_{\gamma\ bi}^{\ \kappa} \;=\;& \delta_iC_{\gamma\ b}^{\ \kappa}-F_{\gamma\ i}^{\ \theta}C_{\theta\ b}^{\ \kappa}-\partial_bF_{\gamma\ i}^{\ \kappa}+C_{\gamma\ b}^{\ \theta}F_{\theta\ i}^{\ \kappa}-K_{i\ b}^{\ c}C_{\gamma\ c}^{\ \kappa}-K_{i\ b}^{\ r}L_{\gamma\ r}^{\ \kappa}=\\ & \delta_iC_{\gamma\ b}^{\ \kappa}-F_{\gamma\ i}^{\ k}C_{k\ b}^{\ \kappa}-F_{\gamma\ i}^{\ c}C_{c\ b}^{\ \kappa}-F_{\gamma\ i}^{\ r}C_{r\ b}^{\ \kappa}-\partial_bF_{\gamma\ i}^{\ \kappa}+\\ & C_{\gamma\ b}^{\ k}F_{k\ i}^{\ \kappa}+C_{\gamma\ b}^{\ c}F_{c\ i}^{\ \kappa}+C_{\gamma\ b}^{\ r}F_{r\ i}^{\ \kappa}-K_{i\ b}^{\ c}C_{\gamma\ c}^{\ \kappa}-K_{i\ b}^{\ r}L_{\gamma\ r}^{\ \kappa}.\end{aligned}$$

4 Ricci identities for ∇

From (2.6) it follows

$$\nabla_X\nabla_YZ=[(Z^\gamma_{\ |\beta})Y^\beta]_{|\alpha}X^\alpha\delta_\gamma=(Z^\gamma_{\ |\beta|\alpha}Y^\beta+Z^\gamma_{\ |\beta}Y^\beta_{\ |\alpha})X^\alpha\delta_\gamma. \tag{4.1}$$

From (2.6), (2.15) and (2.16) we obtain

$$\nabla_{[X,Y]}Z=Z^\gamma_{\ |\beta}[X,Y]^\beta\delta_\gamma=A+B, \tag{4.2}$$

where

$$A=Z^\gamma_{\ |\beta}[X^\alpha(\partial_\alpha Y^\beta)-Y^\alpha(\partial_\alpha X^\beta)]\delta_\gamma= \tag{4.3}$$

$$=Z^\gamma_{\ |\beta}[X^\alpha Y^\beta_{\ |\alpha}-Y^\alpha X^\beta_{\ |\alpha}-(\Gamma_{\theta\ \alpha}^{\ \beta}-\Gamma_{\alpha\ \theta}^{\ \beta})X^\alpha Y^\theta]\delta_\gamma$$

$$\begin{aligned}B \;=\;& X^iY^j[Z^\gamma|_cK_{i\ j}^{\ c}+Z^\gamma\|_qK_{i\ j}^{\ q}]\delta_\gamma+\\ & (X^iY^b-Y^iX^b)[Z^\gamma|_cK_{i\ b}^{\ c}+Z^\gamma\|_rK_{i\ b}^{\ r}]\delta_\gamma\\ & (X^iY^q-Y^iX^q)\delta_\gamma+X^aY^aZ^\gamma\|_rK_{a\ b}^{\ r}\delta_\gamma.\end{aligned} \tag{4.4}$$

Taking into account (2.13) and (2.14) we obtain

$$A+B=[Z^\gamma_{\ |\beta}(X^\alpha Y^\beta_{\ |\alpha}-Y^\alpha X^\beta_{\ |\alpha})-Z^\gamma_{\ |\kappa}T_{\beta\ \alpha}^{\ \kappa}X^\alpha Y^\beta]\delta_\gamma. \tag{4.5}$$

From (4.1), (4.2) and (4.5) we obtain

$$\begin{aligned}R(X,Y)Z \;=\;& (Z^\gamma_{\ |\beta|\alpha}-Z_{|\alpha|\beta}+Z^\gamma_{\ |\kappa}T_{\beta\ \alpha}^{\ \kappa})X^\alpha Y^\beta\delta_\gamma=\\ & \tfrac{1}{2}(Z^\gamma_{\ |\beta|\alpha}-Z^\gamma_{\ |\alpha|\beta}+Z^\gamma_{\ |\kappa}T_{\beta\ \alpha}^{\ \kappa})(X^\alpha Y^\beta-Y^\alpha X^\beta)\delta_\gamma.\end{aligned} \tag{4.6}$$

From (4.6) and (3.9) it follows:

Theorem 4.3. *The Ricci equations for the generalized connection ∇ have the form:*

$$Z^{\gamma}_{\ |\beta|\alpha} - Z^{\gamma}_{\ |\alpha|\beta} + Z^{\gamma}_{\ |\kappa} T_{\beta\ \alpha}^{\ \kappa} = R_{\kappa\ \beta\alpha}^{\ \gamma} Z^{\kappa}. \tag{4.7}$$

(4.7) contains 3^3 types of Ricci equations, because each Greek index may be the element from one of the sets: $\{i, j, h, k, l\}$, $\{a, b, c, d, e\}$, $\{p, q, r, s, t\}$.

For $(\beta, \alpha) = (j, i)$ (4.7) becomes

$$Z^{\gamma}_{\ |j|i} - Z^{\gamma}_{\ |i|j} + Z^{\gamma}_{\ |k} T_{j\ i}^{\ k} + Z^{\gamma}|_c T_{j\ i}^{\ c} + Z^{\gamma}\|_p T_{j\ i}^{\ p} =$$

$$= R_{k\ ji}^{\ \gamma} Z^k + R_{c\ ji}^{\ \gamma} Z^c + R_{p\ ji}^{\ \gamma} Z^p,$$

for $(\beta, \alpha) = (p, i)$ (4.7) takes the form

$$Z^{\gamma}\|_{p|i} - Z^{\gamma}_{\ |i}\|_p + Z^{\gamma}_{\ |k} T_{p\ i}^{\ k} + Z^{\gamma}|_c T_{p\ i}^{\ c} + Z^{\gamma}\|_r T_{p\ i}^{\ r} =$$

$$= R_{k\ pi}^{\gamma} Z^k + R_{c\ pi}^{\ \gamma} Z^c + R_{r\ pi}^{\ \gamma} Z^r, \text{ e.t.c.}$$

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