On the Geometries of Superior Order

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The geometries of higher order, defined in the present paper as the study of the category of jet bundles $(J_o^k M, \pi, M)$, were suggested by the old problem of the prolongations to $J_o^k M$ of Riemannian structures g apriori given on the base manifold M. In the case $k = 1$ there are very good examples: the geometry of Finsler spaces and the geometry of Lagrange spaces, [3].

For $k > 1$ there are some geometrical results concerning the analytical Mechanics of superior order and there are some important studies on the total space of jet bundles. Recently, together with Gh.Atanasiu, we have considered the notion of k-osculator bundle $(Osc^k M, \pi, M)$, which can be identified to the bundle $(J_0^k M, \pi, M)$ and, for it, we have studied:

a. The Geometry of the total space $Osc^k M$: vertical distributions, Liouville vector fields, k-tangent structure, sprays, nonlinear connections, N-linear connections, curvatures, torsions, structure equations etc.

b. Taking this oportunity we solved the problem of the prolongation to $Osc^k M$ of the Riemannian structure g apriori given on M. The same problem for Finsler and Lagrange structures can be formulated.

c. We have defined the notion of higher order Lagrange spaces and studied for them the most important geometrical object fields.

d. Recently, to the Tsagas' workshop, in December 1995, at the "Aristoteles" University of Thessaloniki, the author presented the theory of subspaces in the higher Lagrange spaces.

In the present paper I shall describe, in synthesis, the construction of these geometries.

1. Let M be a real n-dimensional C^∞-manifold and $(Osc^k M, \pi, M)$ its k-osculator bundle. We briefly describe the manifold $E = Osc^k M$. Two curves in M, $\sigma, \rho : I \to M$, with $x_0 \in M$ as common point, $x_0 = \sigma(0) = \rho(0), 0 \in I$, have at x_0 a "contact of order k", $(k \in N^\star)$, if for any function $f \in \mathcal{F}(\mathcal{U})$, $x_0 \in U$, (U being an open set in M), satisfy

$$\frac{d^\alpha}{dt^\alpha}(f \circ \sigma)(t) \mid_{t=0} = \frac{d^\alpha}{dt^\alpha}(f \circ \rho)(t) \mid_{t=0}, \quad (\alpha = 1, ..., k).$$

The relation "contact of order k" is a relation of equivalence. We denote by $[\rho]_{x_0}$ a class of equivalence and named it a "k-osculator space" in the point x_0 to the manifold M.

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Let $Osc^k_{x_0}$ be the set of k-osculator spaces in x_0 to M and let us consider the set

$$Osc^k M = \bigcup_{x_0 \in M} Osc^k_{x_0}$$

and the mapping

$$\pi : ([\rho]_{x_0}) \in Osc^k M \longrightarrow x_0 \in M.$$

On the set $Osc^k M$ there is a natural differentiable structure induced by that of M such that π becomes a differentiable submersion.

If (U, φ) is a local chart in M, $x_0 \in U$ and $\rho : I \to M$ is a curve represented by

$$x^i = x^i(t),\ t \in I,\ x^i_0 = x^i(0),\ (0 \in I).$$

The class $[\rho]_{x_0}$ has a representative element given by

$$x^{*i}(t) = x^i(0) + t\frac{dx^i}{dt}(0) + ... + t^k \frac{1}{k!}\frac{d^k x^i}{dt^k}(0), \quad t \in (-\varepsilon, \varepsilon) \subset I.$$

The previous polynomial functions are defined by the coefficients:

$$x^i_0 = x^i(0),\ y^{(1)i}_0 = \frac{dx^i}{dt}(0), ..., y^{(k)i}_0 = \frac{1}{k!}\frac{d^k x^i}{dt^k}(0).$$

Therefore, the pair $(\pi^{-1}(U), \phi)$, with

$$\phi : \pi^{-1}(\mathcal{U}) \subset \mathcal{E} \longrightarrow \varphi(\mathcal{U}) \times \mathcal{R}^{\|\backslash}$$

$$\phi([\rho]_{x_0}) = (x^i_0, y^{(1)i}_0, ..., y^{(k)i}_0)$$

is a local chart on E, $(i, j, h, ... = 1, ..., n)$

So that, a differentiable atlas $\mathcal{A}_{\mathcal{M}}$ on M determines a differentiable atlas $\mathcal{A}_{\mathcal{E}}$ on E. The triple $(Osc^k M, \pi, M)$ is a differentiable bundle.

The previous considerations show that the transformations of the local coordinates $(x^i, y^{(1)i}, ..., y^{(k)i}) \longrightarrow (\tilde{x}^i, \tilde{y}^{(1)i}, ..., \tilde{y}^{(k)i})$ have the form

$$(1.1) \qquad \begin{cases} \tilde{x}^i &= \tilde{x}^i(x^1, ..., x^n),\ det \parallel \frac{\partial \tilde{x}^i}{\partial x^j} \parallel \neq 0 \\ \tilde{y}^{(1)i} &= \frac{\partial \tilde{x}^i}{\partial x^j} y^{(1)j} \\ \dots\dots\dots & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ k\tilde{y}^{(k)i} &= \frac{\partial \tilde{y}^{(k-1)i}}{\partial x^j} y^{(1)j} + ... + k\frac{\partial \tilde{y}^{(k-1)i}}{\partial y^{(k-1)j}} y^{(k)j}, \end{cases}$$

where we have

$$\frac{\partial \tilde{y}^{(\alpha)i}}{\partial x^j} = \frac{\partial \tilde{y}^{(\alpha+1)i}}{\partial y^{(1)j}} = ... = \frac{\partial \tilde{y}^{(k)i}}{\partial \tilde{y}^{(k-\alpha)i}}, \quad (\alpha = 0, 1, ..., k-1, y^{(0)} = x) \quad etc.$$

2. Let us consider the natural basis

$$\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^{(1)i}}, ..., \frac{\partial}{\partial y^{(k)i}}\right),$$

of the tangent space T_uE, $\quad u = (x, y^{(1)}, ..., y^{(k)}) = (x^i, y^{(1)i}, ..., y^{(k)i})$.

On E there exist k-vertical distributions $V_1, ..., V_k$. The distribution V_1 is locally generated by the vector fields $\left(\frac{\partial}{\partial y^{(1)i}}, ..., \frac{\partial}{\partial y^{(k)i}}\right)$; V_2 is a subdistribution of V_1, locally generated by $\left(\frac{\partial}{\partial y^{(2)i}}, ..., \frac{\partial}{\partial y^{(k)i}}\right)$ etc ..., V_k is a subdistribution of V_{k-1} locally generated by the vector fields $\left(\frac{\partial}{\partial^{(k)i}}\right)$.

All these distributions are integrable.

Also, on E there exist k-independent Liouville vector fields given by

$$\overset{1}{\Gamma} = y^{(1)i}\frac{\partial}{\partial y^{(k)i}},\ \overset{2}{\Gamma} = y^{(1)i}\frac{\partial}{\partial y^{(k-1)i}} + 2y^{(2)i}\frac{\partial}{\partial y^{(k)i}}, ...,$$

$$\overset{k}{\Gamma} = y^{(1)i}\frac{\partial}{\partial y^{(1)i}} + ... + ky^{(k)i}\frac{\partial}{\partial y^{(k)i}}. \tag{2.1}$$

On the manifold E there exists a k-tangent structure, $J : \mathcal{X}(E) \to \mathcal{X}(E)$, such that

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^{(1)i}},\quad J\left(\frac{\partial}{\partial y^{(1)i}}\right) = \frac{\partial}{\partial y^{(2)i}}, ..., \tag{2.2}$$

$$J\left(\frac{\partial}{\partial y^{(k-1)i}}\right) = \frac{\partial}{\partial y^{(k)i}},\quad J\left(\frac{\partial}{\partial y^{(k)i}}\right) = 0.$$

Of course, we have $J(\overset{k}{\Gamma}) = \overset{k-1}{\Gamma}, ..., J(\overset{2}{\Gamma}) = \overset{1}{\Gamma}$, $J(\overset{1}{\Gamma}) = 0$ and J is an integrable structure.

A k-spray on E is a vector field $S \in \mathcal{X}(E)$ with the property

$$J(S) = \overset{k}{\Gamma}.$$

Then, S can be uniquely represented in the form:

$$S = y^{(1)i}\frac{\partial}{\partial x^i} + ... + ky^{(k)i}\frac{\partial}{\partial y^{(k-1)i}} - (k+1)G^i\frac{\partial}{\partial y^{(k)i}}, \tag{2.3}$$

$G^i(x, y^{(1)}, ..., y^{(k)})$ being the coefficients of the spray S.

A nonlinear connection on $E = Osc^kM$ is a supplementary regular distribution $N : u \in E \to N_u \subset T_uE$, to the vertical distribution $V : u \in E \to V_u \subset T_uE$:

$$T_uE = N_u \oplus V_u, \quad \forall u \in E.$$

Setting $N_0 = N$, $N_1 = J(N_0), ..., N_{k-1} = J(N_{k-2})$ we get from the previous direct decomposition:

$$T_uE = N_0(u) \oplus N_1(u) \oplus ... \oplus N_{k-1}(u) \oplus V_u, \quad \forall u \in E. \tag{2.4}$$

Consequently, to study the geometry of total space of the k-osculator bundle $E = Osc^kM$ we shall use a fixed nonlinear connection N and we shall express the geometrical object fields on E by means of the direct decomposition (2.4). Also,

N can be determined by some important geometric objects on E, like k-sprays or higher order Lagrangians.

A local basis adapted to (2.4) is given by

$$\left\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, ..., \frac{\delta}{\delta y^{(k)i}}\right\} \tag{2.5}$$

where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \underset{(1)}{N}{}^j{}_i \frac{\partial}{\partial y^{(1)j}} - ... - \underset{(k)}{N}{}^j{}_i \frac{\partial}{\partial y^{(k)j}} \tag{2.5'}$$

and

$$\frac{\delta}{\delta y^{(1)i}} = J\left(\frac{\delta}{\delta x^i}\right), ..., \frac{\delta}{\delta y^{(k)i}} = J\left(\frac{\delta}{\delta y^{(k-1)i}}\right),$$

$$\frac{\delta}{\delta y^{(k)i}} = \frac{\partial}{\partial y^{(k)i}}.$$

The set of functions $\left\{\underset{(1)}{N}{}^j{}_i, ..., \underset{(k)}{N}{}^j{}_i\right\}$ are called the *coefficients of the nonlinear connection N.*

The dual adapted basis is as follows

$$\left\{\delta x^i, \delta y^{(1)i}, ..., \delta y^{(k)i}\right\}, \tag{2.6}$$

where

$$\delta x^i = dx^i, \delta y^{(1)i} = dy^{(1)i} + \underset{(1)}{M}{}^i{}_j dx^j, ..., \delta y^{(k)i} = \tag{2.6'}$$

$$= dy^{(k)i} + \underset{(1)}{M}{}^i{}_j dy^{(k-1)i} + ... + \underset{(k)}{M}{}^i{}_j dx^j.$$

The set of functions $\left\{\underset{(1)}{M}{}^i{}_j, ..., \underset{(k)}{M}{}^i{}_j\right\}$ are the *dual coefficients of the nonlinear connection N.*

The relationships between the direct coefficients $\left(\underset{(1)}{N}{}^i{}_j, ..., \underset{(k)}{N}{}^i{}_j\right)$ and the dual coefficients of the nonlinear connection N are the following:

$$\begin{cases} \underset{(1)}{M}{}^i{}_j &= \underset{(1)}{N}{}^i{}_j, \\ \underset{(2)}{M}{}^i{}_j &= \underset{(2)}{N}{}^i{}_j - \underset{(1)}{N}{}^i{}_m \underset{(1)}{M}{}^m{}_j \\ \ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \underset{(k)}{M}{}^i{}_j &= \underset{(k)}{N}{}^i{}_j - \underset{(k-1)}{N}{}^i{}_m \underset{(1)}{M}{}^m{}_j - ... - \underset{(1)}{N}{}^i{}_m \underset{(k-1)}{M}{}^m{}_j. \end{cases} \tag{2.7}$$

Here we get an important results:

Theorem 2.1. (Miron-Atanasiu). *A k-spray S with the coefficients G^i determines a nonlinear connection N, whose dual coefficients are*

$$\underset{(1)}{M}{}^i{}_j = \frac{\partial G^i}{\partial y^{(k)j}}, \underset{(2)}{M}{}^i{}_j = \frac{1}{2}\left(S\underset{(1)}{M}{}^i{}_j + \underset{(1)}{M}{}^i{}_m \underset{(1)}{M}{}^m{}_j\right), ...,$$

$$\underset{(k)}{M}{}^i{}_j = \frac{1}{k}\left(S\underset{(k-1)}{M}{}^i{}_j + \underset{(1)}{M}{}^i{}_m \underset{(k-1)}{M}{}^m{}_j\right). \tag{2.8}$$

3. As a first application we solve the problem of prolongation of order k of a Riemann space $R^n = (M, g)$.

The problem is to determine a Riemannian metric G on $E = Osc^k M$ depending only on the given metric g on the base manifold M.

We shall give here, without demonstrations, the solution of this old and difficult problem.

Let $g_{ij}(x)$ be the local components of the given Riemannian metric g of the space R^n and $\gamma^i_{jh}(x)$ be its Christoffel symbols.

Theorem 3.1. *There exist the nonlinear connections N on E determined only by the given Riemannian structure g. One of them has the dual coefficients*

$$\begin{cases} \underset{(1)}{M}{}^i{}_j & = \gamma^i_{jm}(x) y^{(1)}{}_m, \\ \underset{(2)}{M}{}^i{}_j & = \frac{1}{2}\left(\Gamma \underset{(1)}{M}{}^i{}_j + \underset{(1)}{M}{}^i{}_m \underset{(1)}{M}{}^m{}_j\right), \\ \cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \underset{(k)}{M}{}^i{}_j & = \frac{1}{k}\left(\Gamma \underset{(k-1)}{M}{}^i{}_j + \underset{(1)}{M}{}^i{}_m \underset{(k-1)}{M}{}^m{}_j\right), \end{cases} \tag{3.1}$$

where Γ *is the operator*

$$\Gamma = y^{(1)i}\frac{\partial}{\partial x^i} + ... + k y^{(k)i}\frac{\partial}{\partial y^{(k-1)}}. \tag{3.2}$$

This nonlinear connection N, depending only on g, will be called *canonical.*

Let us consider the dual basis (2.6) adapted to the canonical nonlinear connection N and the tensor field G on E:

$$G = g_{ij}(x)dx^i \otimes dx^j + g_{ij}(x)\delta y^{(1)i} \otimes \delta y^{(1)j} + ... + g_{ij}(x)\delta y^{(k)i} \otimes \delta y^{(k)j}. \tag{3.3}$$

We can formulate the following important result:

Theorem 3.2. *The pair $Prol^k R^n = (Osc^k M, G)$ is a Riemann space of dimension $(k+1)n$, whose metric G depends only on the given Riemannian metric g of the space $R^n = (M, g)$.*

With this Theorem the problem of prolongation to $Osc^k M$ of the Riemannian space R^n is solved.

An important remark: If we represent the Liouville vector fields $\overset{1}{\Gamma}, ..., \overset{k}{\Gamma}$ in the adapted basis (2.5) of the canonical nonlinear connection N, we get, for instance

$$\overset{k}{\Gamma} = z^{(1)i}\frac{\delta}{\delta y^{(1)i}} + ... + k z^{(k)i}\frac{\delta}{\delta y^{(k)i}}.$$

In this case $z^{(1)i}, ..., z^{(k)i}$ are d-vector fields. They have the expressions

$$z^{(1)i} = y^{(1)i}, 2z^{(2)i} = 2y^{(2)i} + \underset{(1)}{M}{}^i{}_m y^{(1)m},$$

$$kz^{(k)i} = ky^{(k)i} + (k-1)\underset{(1)}{M}{}^i{}_m y^{(k-1)m} + ... + \underset{(k-1)}{M}{}^i{}_m y^{(1)m}. \tag{3.4}$$

Therefore, the following functions

$$L(x, y^{(1)}, ..., y^{(k)}) = g_{ij}(x) z^{(k)i} z^{(k)j} \tag{3.5}$$

is an important Lagrangian of order k, which depends on the Riemannian structure g, only.

Therefore, the geometry of the space $Prol^k R^k$ can be identified to the geometry of the higher order Lagrange spaces $L^{(k)n} = (M, L)$, in which the fundamental function is given by (3.5).

Of course, we can formulate the same problem for the prolongation of order k of the Finsler spaces F^n or for the Lagrange space L^n.

4. A second application can be made in the problem of the geometrization of the variational calculus concerning the functionals of the form:

$$I(c) = \int_0^1 L\left(x, \frac{dx}{dt}, ..., \frac{1}{k!}\frac{d^k x}{dt^k}\right) dt, \tag{4.1}$$

where $L : Osc^k M \to R$ is a differentiable Lagrangian of order k.

Remarking that

$$\overset{0}{E}_i (L) \overset{def}{=} \frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial y^{(1)i}} + ... + (-1)^k \frac{1}{k!}\frac{d^k}{dt^k}\frac{\partial L}{\partial y^{(k)i}} \tag{4.2}$$

is a d-covector field along the curve $c : [0,1] \to M$ we prove that *the Euler-Lagrange equations* derived from (4.1) are given by

$$\overset{0}{E}_i (L) = 0, \quad y^{(1)i} = \frac{dx^i}{dt}, ..., y^{(k)i} = \frac{1}{k!}\frac{d^k x^i}{dt^k}. \tag{4.2$'$}$$

Now, we can prove:

Lemma 4.1. *For any differentiable Lagrangian $L(x, y^{(1)}, ..., y^{(k)})$ and for any function $\phi(t)$, along a smooth curve c we have*

$$\overset{0}{E}_i (\phi L) = \phi \overset{0}{E}_i (L) + \frac{d\phi}{dt} \overset{1}{E}_i (L) + ... + \frac{d^k \phi}{dt^k} \overset{k}{E}_i (L), \tag{4.3}$$

where $\overset{0}{E}_i (L)$ is the d-covector field (4.2) and $\overset{1}{E}_i (L), ..., \overset{k}{E}_i (L)$ are d-covector fields which depend on the Lagrangian L only.

The d-covectors $\overset{1}{E}_i (L), ..., \overset{k}{E}_i (L)$ are called the *Craig-Synge covectors.*

An important result is expressed by:

Theorem 4.1. *If the Lagrangian $L(x, y^{(1)}, ..., y^{(k)})$ is regular, then the equations $\overset{k-1}{E}{}_i (L) = 0$ determines a k-spray whose coefficients depend on the Lagrangian L only.*

Now let us consider the Lie derivation operators with respect to the Liouville vector fields

$$I^1 = \pounds_{\overset{1}{\Gamma}}, ..., I^k = \pounds_{\overset{k}{\Gamma}}. \tag{4.4}$$

We call *energy of order* k of the Lagrangian $L(x, y^{(1)}, ..., y^{(k)})$ with respect to a smooth curve c the scalar field

$$\mathcal{E}_c^k(L) = I^k(L) - \frac{1}{2!}\frac{dI^{(k-1)}(L)}{dt} + ... + (-1)^{(k-1)}\frac{1}{k!}\frac{d^{(k-1)}I^1(L)}{dt^{(k-1)}} - L. \tag{4.5}$$

M. de Leon, D.Krupka et alt, [2], prove the following theorems:

Theorem 4.2. *For any differentiable Lagrangian* $L(x, y^{(1)}, ..., y^{(k)})$ *along a smooth curve* $c : [0, 1] \to (x^i(t)) \in M$ *we have*

$$\frac{d\mathcal{E}_c^k(L)}{dt} = -\overset{0}{E_i}(L)\frac{dx^i}{dt}. \tag{4.6}$$

Theorem 4.3. *For any differentiable Lagrangian* $L(x, y^{(1)}, ..., y^{(k)})$ *the energy of order* k, $\mathcal{E}_c^k(L)$, *is conserved along of every solution curve* c *of the Euler-Lagrange equation* $\overset{0}{E_i}(L) = 0$.

There are the energies of order $k-1, ..., 1$. They are used in a Nöther theorem concerning the higher order Lagrangians, [2].

5. In the category of the geometries of superior order there are these called by us the geometries of the higher order Lagrange spaces, [2].

Let us consider the manifold $\tilde{E} = \widetilde{Osc^k} M$ formed by all points $(x, y^{(1)}, ..., y^{(k)}) \in E$ for which $y^{(1)} \neq 0$.

Thus, we can define a Lagrange space of order k, as a pair $L^{(k)n} = (M, L)$, where M is a real n-dimensional C^∞-manifold and $L(x, y^{(1)}, ..., y^{(k)})$ is a regular Lagrangian on $\tilde{E}$ for which the d-tensor field

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^{(k)i}\partial y^{(k)j}} \tag{5.1}$$

has a constant signature.

L is called the *fundamental function* of the space $L^{(k)n}$ and g_{ij} its fundamental tensor field.

The condition of regularity is expressed by

$$rank \parallel g_{ij} \parallel = n \quad on \tilde{E}. \tag{5.1$'$}$$

An example of Lagrange space of order k is given by the space $L^{(k)n}$ endowed with the fundamental function L from (3.5)

Its fundamental tensor field is just the metric tensor $g_{ij}(x)$ of the Riemannian space $R^n = (M, g)$.

More general, the Lagrangian

$$L(x, y^{(1)}, ..., y^{(k)}) = g_{ij}(x) z^{(k)i} z^{(k)j} +$$

$$+a_i(x,y^{(1)},...,y^{(k-1)})z^{(k)i}+b(x,y^{(1)},...,y^{(k-1)}), \tag{5.2}$$

where the first term is from (3.5) and $a_i(x,y^{(1)},...,y^{(k-1)})$ is a covector field with the property $\frac{\partial a_i}{\partial y^{(k)j}}=0$ and $b(x,y^{(1)},...,y^{(k-1)})$ is a scalar field with the same property $\frac{\partial b}{\partial y^{(k)i}}=0$, determines a space $L^{(k)n}$.

The Lagrangian (5.2) is a direct extenssion to Osc^kM of that from the classical electrodynamics, [1,2,3].

The geometry of a space $L^{(k)n}=(M,L)$ is based on the canonical k-spray S, determined by the fundamental function L. It is given by theorem 4.1.

Namely, we have:

Theorem 5.1. *In a Lagrange space of order k, $L^{(k)n}=(M,L)$ there exist the k-sprays which depend only by the fundamental function L. One of them has the coefficients:*

$$(k+1)G^i=\frac{1}{2}g^{ij}\left\{\Gamma\left(\frac{\partial L}{\partial y^{(k)j}}\right)-\frac{\partial L}{\partial y^{(k-1)j}}\right\} \tag{5.3}$$

where Γ is the operator (3.2).

The k-spray S with the coefficients in (5.3) is called *canonical*

Using this k-spray, by means of the theorem 2.1, we determine the canonical nonlinear connection N of the space $L^{(k)n}$.

Therefore, we can prove:

Theorem 5.2. 1). *There exist N-linear connections D on Osc^kM, which depend by the fundamental function L only and have the properties*

$$g_{ij|m}=0,\quad g_{ij}\overset{(\alpha)}{|}_m=0,\quad (\alpha=1,...,k).$$

2). There exists only one metrical N-connection D in $L^{(k)n}$ for which $h-$ and v^α - torsions vanish. Its coefficients $C\Gamma(N)=\left(L^i_{jk},\underset{(\alpha)}{C}{}^i_{jk}\right)$ are given by

$$L^m_{ij}=\frac{1}{2}g^{ms}\left(\frac{\delta g_{is}}{\delta x^j}+\frac{\delta g_{js}}{\delta x^l}-\frac{\delta g_{ij}}{\delta x^s}\right), \tag{5.4}$$

$$\underset{(\alpha)}{C}{}^m_{ij}=\frac{1}{2}g^{ms}\left(\frac{\delta g_{is}}{\delta y^{(\alpha)j}}+\frac{\delta g_{js}}{\delta y^{(\alpha)i}}-\frac{\delta g_{ij}}{\delta y^{(\alpha)s}}\right),(\alpha=1,...,k).$$

So, $C\Gamma(N)$, with the coefficients (5.4) is the *canonical metrical N-linear connection* of the Lagrange space of order k, $L^{(k)n}$.

The main geometrical properties of the space $L^{(k)n}$ can be derived from the canonical k-spray, canonical nonlinear connection N and from the canonical metrical N-linear connection, $C\Gamma(N)$.

Finally, we remark that a good geometrical model for the higher order Lagrange space $L^{(k)n}$ is a $(k-1)n$-almost contact Riemannian space $H^{(k-1)n}=(Osc^kM,G,F,\alpha,a,\eta(\alpha)a)$ intrinsicly associated to $L^{(k)n}$, [2]. This space is used in the study of the problem of generalization of physical fields which depend on the higher order velocities.

References

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