Transformations of Sheaf Connections

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Abstract

After a brief survey of the primary ideas involved in the *theory of connections on vector and principal sheaves* (studied in [7], [8], [14], [15]), we examine the behaviour of connections under various types of morphisms between sheaves of the considered category. The results thus obtained are useful in the development of a non-smooth geometry in the aforementioned abstract framework and related applications.

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Introduction

The need to extend the traditional methods of differential geometry beyond the context of ordinary differentiable manifolds led to the introduction of the so-called *differential spaces*, that is, spaces in the sense of R.Sikorski, M.A.Mostow, I.Satake and others (for illuminating comments and a relevant literature we refer to [6] and [8]). Over such spaces, a great deal of non-smooth geometry has been developed.

Recently, A.Mallios ([7], [8]) had the idea to replace differential spaces (the structure sheaves of which are mainly functional) with *algebraized spaces* $(X, \mathcal{A})$, where X is a topological space and (the structure sheaf) $\mathcal{A}$ a sheaf of topological algebras. These spaces are the base of *vector sheaves* (:locally free $\mathcal{A}$-modules) on which he built up his theory of $\mathcal{A}$-*connections*, thus extending to a purely algebrotopological context the classical case of linear connections on vector bundles of finite rank.

Inspired by this abstraction, the present author considered *principal sheaves* with structure sheaf an appropriate sheaf of groups, called *Lie sheaf of groups*, and he studied connections in this context. Among other features, this generalization contains the case of vector sheaf connections in the sense that each vector sheaf $\mathcal{E}$ gives rise to a principal sheaf $\mathcal{P}(\mathcal{E})$ (:the *sheaf of frames of* $\mathcal{E}$), thus connections on $\mathcal{E}$ are in bijective correspondance with those of $\mathcal{P}(\mathcal{E})$ (see [14], [15], 17]).

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The main purpose of this note is to examine the behaviour of abstract connections, both on principal and vector sheaves, under various morphisms of sheaves in the category at hand. We also study connections linked together by abstract gauge transformations and pull backs of connections by mappings between the bases. Section 3 contains results related with principal sheaf connections, whereas Section 4 is a shorter account of the vector sheaf counterpart of the former.

The results thus obtained are immediately applicable in classification problems, in the study of moduli spaces, the Chern-Weil homomorphism etc. Besides the previous applications, this investigation is interesting per se, since it exhibits the similarities, as well as the differences, between the present approach and its ordinary bundle analogue. This comparison illuminates, in particular, the rôle of certain entities and emphasizes their importance in the development of the geometry under discussion, whereas in the classical case, with its abundance of means, this rôle is not fully exploited if not overlooked. It is precisely the axiomatization of these entities which lays the foundations of the nonsmooth theory of connection within the framework of sheaves (in this respect see also the comments in [14, Section 5]).

For the convenience of the reader, in the first two sections we include the rudimentary notions of [7], [8], [14], [15], heavily needed in the main part of the paper.

1 Fundamental notions

The starting point of the proposed algebrotopological abstraction is an *algebraized space* $(X, \mathcal{A})$, where X is a topological space and $\mathcal{A}$ a sheaf of commutative, associative and unital linear K- algebras ($K = R, C$) over X. This notion comprises differentiable manifolds and many differential spaces (e.q. spaces in the sense of [10]; for other examples see also [8]).

For an algebraized space $(X, \mathcal{A})$, we also consider a *differential triad* $(\mathcal{A}, \lceil, \otimes^\infty)$, where Ω^1 is an $\mathcal{A}$-*module* over X and $d : \mathcal{A} \to \otimes^\infty$ is an Ω^1-valued derivation of $\mathcal{A}$, i.e. a K-linear (sheaf) morphism satisfying the *Leibniz condition*

$$d(s \cdot t) = s \cdot dt + t \cdot ds, \tag{1.1}$$

for every local sections, $s, t \in \mathcal{A}(\mathcal{U})$ and $U \subseteq X$ open.

A typical example is obtained in case X is a real smooth manifold and $\mathcal{A} = \mathcal{C}^\infty_\mathcal{X}$ (: the sheaf of germs of smooth functions on X). Then Ω^1 is the sheaf of germs of differential 1-forms on X and d is the morphism induced by the sheafification of the ordinary differentiation of smooth functions.

Given an arbitrary algebraized space $(X, \mathcal{A})$, a differential triad can be constructed by the sheafification of Kaehler's theory of differentials (for details cf. [9], where d is denoted by ∂, a symbol reserved here for a different entity below).
Note. In (1.1) we have applied the well known fact that a sheaf can be identified with the corresponding (complete) presheaf of its sections and, accordingly, a morphism of sheaves is identified with the morphism between the corresponding preasheaves of sections. This very useful interplay will be used quite often in the

sequel. Of course, one could have defined the operator d by the analoque of (1.1) at the level of stalks:

$$d(a\cdot b)=a\cdot db+b\cdot da;\quad (a,b)\in\mathcal{A}\times_{\mathcal{X}}\mathcal{A}, \tag{1.1$'$}$$

$\mathcal{A}\times_{\mathcal{X}}\mathcal{A}$ denoting the fiber product of $\mathcal{A}$ with itself over X. (for details regarding sheaf theory one may consult e.g. [3], [4], [5]).

The next basic notion, needed in the geometry of principal sheaves, is that of a Lie sheaf of groups, defined as follows: we consider first a triplet $(\mathcal{G},\mathcal{L},\rho)$ where $\mathcal{G}$ is a *sheaf of groups* over X, $\mathcal{L}$ is an $\mathcal{A}$-*module of Lie algebras* and $\rho:\mathcal{G}\longrightarrow\mathcal{A}\sqcap\sqcup(\mathcal{L})$ a continuous morphism of sheaves of groups. ρ induces a *representation* (using the same symbol) $\rho:\mathcal{G}\longrightarrow\mathcal{A}\sqcap\sqcup(\otimes^{\infty}\otimes_{\mathcal{A}}\mathcal{L})$ given by

$$\rho(g)\cdot(\theta\otimes\ell):=[1\otimes\rho(g)](\theta\otimes\ell), \tag{1.2}$$

for every $g\in\mathcal{G}$, $\theta\otimes\ell\in\Omega^1(U)\otimes_{\mathcal{A}(\mathcal{U})}\mathcal{L}(\mathcal{U})$ and $U\subseteq X$ open. The above formula extends, by linearity, to any non-decomposable element and determines the desired new representation.

Next, we define the *Maurer-Cartan differential* for a triplet $(\mathcal{G},\mathcal{L},\rho)$. This is a morphism of sheaves of sets $\partial:\mathcal{G}\to\otimes^{\infty}\otimes_{\mathcal{A}}\mathcal{L}$ such that condition

$$\partial(g\cdot h)=\rho(h^{-1})\cdot\partial g+\partial h, \tag{1.3}$$

holds for every $g,h\in\mathcal{G}(\mathcal{U})$ and $U\subseteq X$ open.

A *Lie sheaf of groups* is precisely a quadruple $(\mathcal{G},\mathcal{L},\rho,\partial)$ with the previous properties. In order to simplify the notations, we denote a Lie sheaf of groups $(\mathcal{G},\mathcal{L},\rho,\partial)$ simply by $\mathcal{G}$, if there is no danger of confusion.

We illuminate the previous construction by two examples: The first is provided by an ordinary Lie group G. Now X is a smooth manifold (thus $\mathcal{A}$ and Ω^1 are as in the first example of differential triads), $\mathcal{G}$ is the sheaf of germs of smooth G-valued functions on X, $\mathcal{L}$ is the sheaf of germs of smooth G-valued maps on X (G: the Lie algebra of G) and ρ is the morphism obtained by sheafification of the adjoint representation $Ad:G\longrightarrow Aut(G)$. Finally, ∂ is obtained, by the same process, from the *total (or logarithmic) differentiation*

$$C^{\infty}(U,G)\ni f\longrightarrow f^{-1}\cdot df\in\Lambda^1(U,G)\quad (U\subseteq X\text{open}).$$

We recall that $\Lambda^1(U,G)$ is the space of G-valued differential 1-forms on $U\subseteq X$ and

$$(f^{-1}\cdot df)_x(u):=(T_{f(x)}\lambda_{f(x)}-1\circ T_xf)(u);\quad x\in U,u\in T_xM,$$

where λ_g denotes the left translation by $g\in G$. Note that the sheaf of germs of the previous forms can be identified with $\Omega^1\otimes_{\mathcal{A}}\mathcal{L}$ (for details we refer to [15]).

The second, more abstract, example is provided by the sheaf of groups $\mathcal{GL}(\backslash,\mathcal{A})$, over an algebraized space $(X,\mathcal{A})$ with a fixed differential triad $(\mathcal{A},\lceil,\otimes^{\infty})$. As a matter of fact, $\mathcal{GL}(\backslash,\mathcal{A})$ is generated by the complete presheaf $U\longrightarrow GL(n,\mathcal{A}(\mathcal{U}))=\mathcal{M}_{\backslash}(\mathcal{A}(\mathcal{U}))^{\circ}$, if $M_n(\mathcal{A}(\mathcal{U}))$ is the space of $n\times n$ matrices with entries in $\mathcal{A}(\mathcal{U})$ and the dot $\circ$ marks spaces of invertible elements. Each $M_n(\mathcal{A}(\mathcal{U}))$ has the structure of a Lie algebra and we take as $\mathcal{L}$ the sheaf generated by the complete presheaf $U \to M_n(\mathcal{A}(\mathcal{U}))$. It is customary to set $\mathcal{L} = \mathcal{M}_\backslash(\mathcal{A})$; thus, $\mathcal{GL}(\backslash, \mathcal{A}) = \mathcal{M}_\backslash(\mathcal{A})^\circ$ and

$$\mathcal{L}(\mathcal{U}) = \mathcal{M}_\backslash(\mathcal{A})(\mathcal{U}) \cong \mathcal{M}_\backslash(\mathcal{A}(\mathcal{U})), \quad \mathcal{GL}(\backslash, \mathcal{A})(\mathcal{U}) \cong \mathcal{GL}(\backslash, \mathcal{A}(\mathcal{U})),$$

for every open $U \subseteq X$. We induce a representation

$$\rho \equiv \mathcal{A}\lceil : \mathcal{GL}(\backslash, \mathcal{A}) \longrightarrow \mathcal{A}\sqcap\sqcup(\otimes^\infty \otimes_\mathcal{A} \mathcal{M}_\backslash(\mathcal{A}))$$

defined by

$$\mathcal{A}\lceil(\}) \cdot (\theta \otimes \dashv) := \theta \otimes (\} \cdot \dashv \cdot \}^{-\infty}),$$

for every $g \in \mathcal{GL}(\backslash, \mathcal{A})(\mathcal{U})$, $a \in \mathcal{M}_\backslash(\mathcal{A})(\mathcal{U})$, $\theta \in \Omega^1(U)$ and $U \subseteq X$ open. Finally, we define a Maurer-Cartan differential

$$\partial : \mathcal{GL}(\backslash, \mathcal{A}) \longrightarrow \otimes^\infty \otimes_\mathcal{A} \mathcal{M}_\backslash(\mathcal{A})$$

by setting

$$\partial(a) := a^{-1} \cdot da; \quad a \in \mathcal{GL}(\backslash, \mathcal{A})(\mathcal{U}), \tag{1.4}$$

with $da := (da_{ij})$, if $a = (a_{ij})$ (see also [7], [8], where ∂ is denoted by $\tilde{\partial}$). We routinely check that (1.3) is satisfied, therefore $(\mathcal{GL}(\backslash, \mathcal{A}), \mathcal{A}\lceil, \mathcal{M}_\backslash(\mathcal{A}), \partial)$ is indeed a Lie sheaf of groups. It is important in the reduction of the geometry of vector sheaves to that of principal sheaves, as we explain in the next section.

2 Connections on geometrical sheaves

Here we review a few facts concerning the structure of principal and vector sheaves, as well as connections on them.

If $\mathcal{G} \equiv (\mathcal{G}, \rho, \mathcal{L}, \partial)$ is a Lie sheaf of groups (see Section 1), a *principal sheaf of structure type* $\mathcal{G}$ and with *structure sheaf* $\mathcal{G}$, denoted by $(\mathcal{P}, \mathcal{G}, \mathcal{X}, \surd)$, is a sheaf of sets $(\mathcal{P}, \mathcal{X}, \surd)$ such that:

i) $\mathcal{G}$ acts on the right of $\mathcal{P}$,

ii) there exists an open covering $\mathcal{C} = \{\mathcal{U}_\alpha \subseteq \mathcal{X} \mid \alpha \in \mathcal{I}\}$ of X and $\mathcal{G}$-*equivariant isomorphisms* (:coordinate mappings) of sheaves of sets

$$\phi_\alpha : \mathcal{P}\,|_{\mathcal{U}_\alpha} \xrightarrow{\simeq} \mathcal{G}\,|_{\mathcal{U}_\alpha} \quad (\alpha \in \mathcal{I}). \tag{2.1}$$

The previous local structure induces a family of *natural section* of $\mathcal{P}$

$$s_\alpha = \phi_\alpha^{-1} \circ \mathbf{1}\,|_{U_\alpha} \in \mathcal{P}(\mathcal{U}_\alpha); \quad \alpha \in \mathcal{I}, \tag{2.2}$$

where $\mathbf{1}$ is the unit section of $\mathcal{G}$, i.e. $\mathbf{1}$(x) is the unit element of the stalk $\mathcal{G}_\S$. Accordingly, setting $U_{\alpha\beta} := U_\alpha \cap U_\beta$, we obtain the *1-cocycle* $(g_{\alpha\beta}) \in Z^1(\mathcal{C}, \mathcal{G})$ given by

$$g_{\alpha\beta} := (\phi_\alpha \circ \phi_\beta^{-1}) \circ \mathbf{1}\,|_{U_{\alpha\beta}} \in \mathcal{G}(\mathcal{U}_{\alpha\beta}). \tag{2.3}$$

As a result, on $U_{\alpha\beta}$,

$$s_\beta = s_\alpha \cdot g_{\alpha\beta}. \tag{2.4}$$

The cocycle $(g_{\alpha\beta})$ completely determines, up to isomorphism, $\mathcal{P}$.

Setting, for convenience,

$$\Omega^1(\mathcal{L}) := \otimes^\infty \otimes_{\mathcal{A}} \mathcal{L}, \tag{2.5}$$

a *connection* on the principal sheaf $(\mathcal{P}, \mathcal{G}, \mathcal{X}, \surd)$ is a morphism of sheaves of sets

$$D : \mathcal{P} \longrightarrow \otimes^\infty(\mathcal{L})$$

satisfying

$$D(s \cdot g) = \rho(g^{-1}) \cdot Ds + \partial g, \tag{2.6}$$

for every $s \in \mathcal{P}(\mathcal{U}), \} \in \mathcal{G}(\mathcal{U})$ and $U \subseteq X$ open. Here $s \cdot g$ denotes the result of the action of $\mathcal{G}$ on $\mathcal{P}$ on the level of sections.

Theorem 2.1 ([15]). *A connection D corresponds bijectively to a 0-cochain* $(\omega_\alpha) \in C^0(\mathcal{C}, \otimes^\infty(\mathcal{L})), \quad \alpha \in \mathcal{I}$, *satisfying the compatibility condition*

$$\omega_\beta = \rho(g_{\alpha\beta}^{-1}) \cdot \omega_\alpha + \partial g_{\alpha\beta}; \quad \alpha, \beta \in I, \tag{2.7}$$

on the overlappings. Under this correspondance, $\omega_\alpha = D(s_\alpha)$.

In analogy to the classical case of ordinary connections and by abuse of language, we call the sections $\omega_\alpha \in \Omega^1(\mathcal{L})(\mathcal{U}_\alpha), \quad \alpha \in \mathcal{I}$, *local connection forms* of D (with respect to $\mathcal{C}$).

Concerning the existence of connections on principal sheaves, their interpretation as sections of the sheaf of connections (in a generalized version of [1]) and other related results, we refer to [15]. Also, [14] is a detailed treatment of ordinary principal connections in this spirit.

The ″linear″, so to speak, counterpart of a principal sheaf is a *vector sheaf.* The term is refering to a locally a free $\mathcal{A}$-module $(\mathcal{E}, \mathcal{X}, \pi)$ of (finite) rank, say, n. This means there exists an open covering $\mathcal{C} = \{\mathcal{U}_\alpha \subseteq \mathcal{X} \mid \alpha \in \mathcal{I}\}$ of X and $\mathcal{A}$-*isomorphisms*

$$\Phi_\alpha : \mathcal{E}\,|_{\mathcal{U}\alpha} \xrightarrow{\cong} \mathcal{A}^\backslash\,|_{\mathcal{U}\alpha} \cong (\mathcal{A}\,|_{\mathcal{U}})^\backslash, \quad \alpha \in \mathcal{I}. \tag{2.8}$$

The coordinate transformations

$$G_{\alpha\beta} := \Phi_\alpha \circ \Phi_\beta^{-1} \in \mathcal{G}\ell(\backslash, \mathcal{A})(\mathcal{U}_{\alpha\beta}) \cong \mathcal{GL}(\backslash, \mathcal{A}(\mathcal{U}_{\alpha\beta})) \tag{2.9}$$

determine a cocycle $(G_{\alpha\beta}) \in Z^1(\mathcal{C}, \mathcal{G}\ell(\backslash, \mathcal{A}))$ which classifies, up to isomorphism, $\mathcal{E}$. Further details can be found in [8].

An interesting example of a principal sheaf is the *sheaf of frames* $\mathcal{P}(\mathcal{E})$ of a given vector sheaf $\mathcal{E}$. It is generated by the complete presheaf

$$U \longrightarrow Iso_{\mathcal{A}|\mathcal{U}}(\mathcal{A}^\backslash \mid \mathcal{U}, \mathcal{E} \mid \mathcal{U}),$$

with U varying in the base $\mathcal{B}$ of X derived from the open covering $\mathcal{C}$ (i.e. $U \in \mathcal{B}$ if there is $V \in \mathcal{C}$ such that $U \subseteq V$). In this respect, the following result is already known.

Theorem 2.2 ([17]). *$\mathcal{P}(\mathcal{E})$ is a principal sheaf with structure and type sheaf the Lie sheaf of groups $\mathcal{GL}(\backslash, \mathcal{A})$. Moreover, $\mathcal{E}$ is associated with $\mathcal{P}(\mathcal{E})$, that is*

$$\mathcal{E} \cong \mathcal{P}(\mathcal{E}) \times_{\mathcal{X}} \mathcal{A}^{\backslash} / \sim,$$

the equivalence relation "$\sim$" beeing defined by the natural action of $\mathcal{GL}(\backslash, \mathcal{A})$ on $\mathcal{P}(\mathcal{E}) \times_{\mathcal{X}} \mathcal{A}^{\backslash}$.

Now, according to [7] and [8], an $\mathcal{A}$-*connection* on a vector sheaf $\mathcal{E}$ is a K-linear morphism of sheaves

$$\nabla : \mathcal{E} \longrightarrow \otimes^{\infty}(\mathcal{E}) := \mathcal{E} \otimes_{\mathcal{A}} \otimes^{\infty}$$

satisfying the Leibniz formula

$$\nabla(\alpha \cdot s) = \alpha \cdot \nabla(s) + s \otimes da, \tag{2.10}$$

for every $\alpha \in \mathcal{A}(\mathcal{U}), \int \in \mathcal{E}(\mathcal{U})$ and $U \subseteq X$ open.

An $\mathcal{A}$-connection ∇ is completely determined by *local matrices* (analogously to the local connection forms of D above) as it is proved by the following

Theorem 2.3 ([8], [17]). *A connection ∇ on $\mathcal{E}$ corresponds bijectively to a 0-cochain $(\omega_\alpha) \in C^0(\mathcal{C}, \otimes^{\infty} \otimes_{\mathcal{A}} \mathcal{M}_{\backslash}(\mathcal{A}))$* such that

$$\omega_\beta = \mathcal{A}\lceil(\mathcal{G}_{\alpha\beta}^{-\infty}) \cdot \omega_\alpha + \partial \mathcal{G}_{\alpha\beta}; \quad \alpha, \beta \in \mathcal{I}. \tag{2.11}$$

More precisely, it can be shown that

$$\omega_\alpha = (\omega_{ij}^\alpha \in M_n(\Omega^1(U_\alpha)) \cong M_n(\mathcal{A}(\mathcal{U}_\alpha)) \otimes_{\mathcal{A}(\mathcal{U}_\alpha)} \otimes^{\infty}(\mathcal{U}_\alpha), \tag{2.12}$$

where the coefficients of the matrix are determined by

$$\nabla(e_j^\alpha) = \sum_{i=1}^{n} e_i^\alpha \otimes \omega_{ij}^\alpha; \quad 1 \leq j \leq n, \tag{2.13}$$

if $(e_i^\alpha)_{1 \leq i \leq n}$ is the natural basis of $\mathcal{E}(\mathcal{U}_\alpha)$ induced by (2.8), i.e.

$$e_i^\alpha(x) := \Phi_\alpha^{-1}(0_x, ..., 1_x, ..., 0_x), \tag{2.14}$$

where 0_x and 1_x (in the i-th entry) are the neutral elements of the algebra $\mathcal{A}_{\S}$.

Observing that the cocycle $(g_{\alpha\beta})$ of $\mathcal{P}(\mathcal{E})$ coincides with $(G_{\alpha\beta})$ of $\mathcal{E}$ and taking into account that the analogue of (2.7) for $\mathcal{P}(\mathcal{E})$ coincides with (2.11), we prove the following abstraction of a well known classical result.

Theorem 2.4 ([17]). *Connections ∇ on $\mathcal{E}$ are in bijective correspondence with connections D on $\mathcal{P}(\mathcal{E})$.*

Before concluding this brief survey, we add a few comments on some other fundamental notions related with connections.

Naturally, of prime importance is the notion of *curvature*. In both cases of connections outlined here, this is established under one further assumption, namely the differential triad $(\mathcal{A}, \lceil, \otimes^{\infty})$ has to be extended to a quintuple $(\mathcal{A}, \lceil, \otimes^{\infty}, \lceil^{\infty}, \otimes^{\in})$, where $\Omega^2 = \Omega^1 \Lambda \Omega^1$ and $d^1 : \Omega^1 \longrightarrow \Omega^2$ is an appropriate K-linear morphism inducing a sort of "exterior differentiation" on the elements of Ω^1 (for details see [7], [8]). After that, one investigates flat and integrable connections, two notions not necessarily identical in the present context (in contrast to the classical case), unless additional cohomological restrictions on X are imposed. Moreover, a Chern-Weil homomorphism, as well as the Chern classes (for vector sheaves) can be obtained. We do not pursue these matters further, refering instead to [8], [16] for details.

3 Mappings of principal sheaf connections

(A) The first main results of this section is dealing with the global and local behaviour of connections under $\mathcal{G}$-X-morphism. More precisely, we assume that $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, \mathcal{X}, \pi)$ and $\mathcal{P}' \equiv (\mathcal{P}', \mathcal{G}, \mathcal{X}, \pi')$ are two principal sheaves over the same base X and with the same Lie sheaf of groups $\mathcal{G} \equiv (\mathcal{G}, \rho, \mathcal{L}, \partial)$ as type and structure sheaf.

Definition 3.1. A $\mathcal{G}$-X-*morphism* $f : \mathcal{P} \longrightarrow \mathcal{P}'$ is a morphism of sheaves of sets such that

$$f(p \cdot g) = f(p) \cdot g; \quad p \in P, g \in \mathcal{G}. \tag{3.1}$$

Lemma 3.2 *Any $\mathcal{G} - \mathcal{X}$-morphism is a $\mathcal{G} - \mathcal{X}$-isomorphism.*

Proof. We can apply a sheaf theoretic version of the standard arguments concerning $G - B-$(iso)morphisms of principal bundles (in this respect see e.q. [2; n^0 6.3.1]).

Definition 3.3 Two connections D and D', respectively on $\mathcal{P}$ and $\mathcal{P}'$, are said to be f-*conjugate* (or f-*related*) if there exists a $\mathcal{G} - \mathcal{X}$-morphism f such that $D = D' \circ f$.

Theorem 3.4 *D and D', as above, are f-conjugate if and only if there exists an open covering $\mathcal{C} = (\mathcal{U}_\alpha)_{\alpha \in \mathcal{I}}$ of X, cocycles $(g_{\alpha\beta})$, $(g'_{\alpha\beta}) \in Z^1(\mathcal{C}, \mathcal{G})$ of $\mathcal{P}$ and $\mathcal{P}'$ respectively and a 0-cochain $(h_\alpha \in C^0(\mathcal{C}, \mathcal{G})$ such that, for every $\alpha, \beta \in I$,*

$$g'_{\alpha\beta} = h_\alpha \cdot g_{\alpha\beta} \cdot h_\beta^{-1}; \tag{3.2}$$

$$\omega_\alpha = \rho(h_\alpha^{-1}) \cdot \omega'_\alpha + \partial h_\alpha, \quad \alpha \in I, \tag{3.3}$$

if (ω_α) and (ω'_α) are the local connection forms of D and D' respectively.

Proof. We may assume that $\mathcal{P}$ and $\mathcal{P}'$ are of structure type $\mathcal{G}$ over the same open covering, say, $\mathcal{C}$ (otherwise we take a common refinement). Then, it is immediate that there is a 0-cochain $(h_\alpha) \in C^0(\mathcal{C}, \mathcal{G})$ such that

$$f(s_\alpha(x)) = s\prime_\alpha(x) \cdot h_\alpha(x); \quad x \in U_\alpha, \alpha \in I, \tag{3.4}$$

if s_α and s'_α are the natural sections of $\mathcal{P}$ and $\mathcal{P}'$, respectively, over U_α (see (2.2)). Therefore, in virtue of (2.6) and (3.4), we obtain (3.2). Furthermore, by (2.2) and (3.4),

$$\omega_\alpha = D(s_\alpha) = (D' \circ f)(s_\alpha) = D'(s'_\alpha \cdot h_\alpha) = \rho(h_\alpha^{-1}) \cdot \omega'_\alpha + \partial h_\alpha,$$

thus getting (3.3).

Conversely, given a cochain (h_α) as in the statement, we determine a $\mathcal{G} - \mathcal{X}$-morphism f by requiring that (3.4) is satisfied. It is well defined by (3.2). Then, using (3.3) and proceeding as before, we check that

$$D(s_\alpha) = \omega_\alpha = D'(s'_\alpha \cdot h_\alpha) = (D' \circ f)(s_\alpha),$$

for every natural section s_α. The last equality, along with (3.1) and (2.6), implies that $D(s) = (D' \circ f)(s)$, for every local section $s \in \mathcal{P}(\mathcal{U}_\alpha)$. Finally, we show that $D(s) = (D' \circ f)(s)$, for every $s \in \mathcal{P}(\mathcal{U})$ and $U \subseteq X$ open, if we take into account that $U = U_{i \in I}(U \cap U_\alpha)$ and apply the previous result for all the restriction $s \mid U \cap U_\alpha$.

Remark. For the bundle analogue of Theorem 3.4 we refer also to [12, Theorem 2.1].

Corollary 3.5. *For a given $\mathcal{G} - \mathcal{X}$-morphism f, each connection D of $\mathcal{P}$ determines a unique f-conjugate connection D' of $\mathcal{P}'$ and vice versa.*

For the sake of completeness let us examine the case of $\mathcal{G} - \mathcal{X}$-automorphisms of $\mathcal{P}$. They are also called *gauge transformations*, although some physicists would prefer the terms *pure gauge transformations or vertical transformations* (as applied to the case of ordinary bundles) (see e.q. [11]).

It is not hard to see that a gauge transformation $f : \mathcal{P} \longrightarrow \mathcal{P}$ corresponds bijectively to a continuous morphism of sheaves $\tau : \mathcal{P} \longrightarrow \mathcal{G}$ such that

$$\tau(p \cdot g) = g^{-1} \cdot \tau(p) \cdot g, \quad (p, g) \in \mathcal{P} \times_{\mathcal{X}} \mathcal{G}.$$

Moreover, each morphism of this type corresponds, in turn, to a (global) section of the associated sheaf $\mathcal{P} \times_{\mathcal{X}} \mathcal{G}/\sim$, the latter being defined by the following local equivalence relations

$$(s, g) \sim (t, h) \Leftrightarrow \exists! \quad a \in \mathcal{G}(\mathcal{U}_\alpha) : \sqcup = \int \cdot \dashv, \quad \langle = \dashv^{-\infty} \cdot \} \cdot \dashv,$$

for every $U_\alpha \in \mathcal{C}$ and $(s, g), (t, h) \in \mathcal{P}(\mathcal{U}_\alpha) \times \mathcal{G}(\mathcal{U}_\alpha)$. It is clear that gauge transformations induce an equivalence relation in the set of connections on $\mathcal{P}$.

Finally, let D be a connection on $\mathcal{P}$ with local forms $\omega_\alpha = D(s_\alpha), \alpha \in I$, if s_α are the natural sections (2.1), defined with respect to $\mathcal{C}$ and the $\mathcal{G}$-equivariant isomorphisms ϕ_α (2.2). In physics terminology, the selection of $(U_\alpha, \phi_\alpha)_{\alpha \in I}$ determines a *gauge fixing.* If f is a gauge transformation, then the connection $D' := D \circ f^{-1}$ has corresponding local forms $\omega'_\alpha = D'(s_\alpha)$ and (3.3) is satisfied with (h_α) determined by $f \circ s_\alpha = s_\alpha \cdot h_\alpha \quad (\alpha \in I)$. However, if we replace the previous gauge by taking the $\mathcal{G}$-equivariant isomorphisms $\psi_\alpha := \phi_\alpha \circ f$, then the corresponding natural sections are precisely $\sigma_\alpha = f^{-1} \circ s_\alpha$. As a result, the new local connection forms, say, $\bar{\omega}_\alpha = D'(\sigma_\alpha)$ coincide with the original ω_α, s. Therefore, we conclude that *gauge-equivalent connections, under an appropriate change of the local gauges, have the same local connection forms.*

This fact is important in the classification of *physical gauge fields* via gauge transformations (in this respect see also the comments of [10; section V]).

(B) The case of transformations of connections on principal sheaves with the same base but different Lie sheaves of groups needs a particular treatment, since the situation is more complicated than the classical case of principal bundles.

We fix two Lie sheaves of groups over the same base X, namely

$$\mathcal{G} \equiv (\mathcal{G}, \mathcal{L}, \rho, \partial) \quad \text{and} \quad \mathcal{G}' \equiv (\mathcal{G}', \mathcal{L}', \rho', \partial').$$

Definition 3.6. A morphism of $\mathcal{G}$ into $\mathcal{G}'$, compatible with (ρ, ∂) and (ρ', ∂') is a pair $(\phi, \bar{\phi})$, where $\phi : \mathcal{G} \longrightarrow \mathcal{G}'$ and $\bar{\phi} : \mathcal{L} \longrightarrow \mathcal{L}'$ are morphisms of sheaves of groups and Lie algebras respectively, such that the diagrams are commutative.

Here δ and δ' are the actions induced on $\mathcal{L}$ and $\mathcal{L}'$ by ρ and ρ' respectively.

Note that the commutativity of Diagram I is equivalent to

$$\bar{\phi} \circ \rho(g) = \rho'(\phi(g)) \circ \bar{\phi}, \quad g \in \mathcal{G}. \tag{3.5}$$

With $\mathcal{G}$ and $\mathcal{G}'$ as above, we give also the following

Definition 3.7 If $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, \mathcal{X}, \pi)$ and $\mathcal{P}' \equiv (\mathcal{P}', \mathcal{G}', \mathcal{X}, \pi')$ are principal sheaves, then a *morphism* between $\mathcal{P}$ and $\mathcal{P}'$ is a quadruple $(f, \phi, \bar{\phi}, id_X)$, where $(\phi, \bar{\phi})$ is a morphism as in Definition 3.6 and $f : \mathcal{P} \longrightarrow \mathcal{P}'$ is a morphism of sheaves of sets such that

$$f(p \cdot g) = f(p) \cdot \phi(g); \quad p \in \mathcal{P}, \} \in \mathcal{G}. \tag{3.6}$$

$(f, \phi, \bar{\phi}, id_X)$ is an isomorphism if all the morphisms involved are sheaf isomorphisms.

Proposition 3.8. Let $(f, \phi, \bar{\phi}, id_X)$ be a morphism as in Definition 3.7. Then the following conclusions are true:

i) If $\bar{\phi}$ is an isomorphism, then each connection D' on $\mathcal{P}'$ induces a connection D on $\mathcal{P}$ such that

$$(1 \otimes \bar{\phi}) \circ D = D' \circ f. \tag{3.7}$$

ii) If $(f, \phi, \bar{\phi}, id_X)$ is an isomorphism, then each connection D on $\mathcal{P}$ induces a connection D' on $\mathcal{P}'$ satisfying also (3.7).

Proof. For i) it suffices to set $D := (1 \otimes \bar{\phi})^{-1} \circ D' \circ f$. We check that D is a connection by virtue of (3.5) and Diagram II, taking also into account convention about the representation of $\mathcal{G}$ into $\Omega^1(\mathcal{L})$ (see equalities (1.2) and (2.5)).

By the same token, we check that $D' := (1 \otimes \phi) \circ D \circ f^{-1}$ is a connection, thus proving ii).

A generalization of Theorem 3.4, proved by an analogous method, is the next

Theorem 3.9. *Let $(\mathcal{P}, \mathcal{G}, \mathcal{X}, \pi)$ and $(\mathcal{P}', \mathcal{G}', \mathcal{X}, \pi')$ be two principal sheaves endowed with two connections D and D' respectively. Then D and D' are conjugate by means of an isomorphism $(f, \phi, \bar{\phi}, id_X)$, i.e. equality (3.7) holds, if and only if there exists a 0-cochain $(h_\alpha) \in C^0(\mathcal{C}, \mathcal{G}')$ such that conditions*

$$g'_{\alpha\beta} = h_\alpha \cdot \phi(g_{\alpha\beta}) \cdot h_\beta^{-1},$$

$$(1 \otimes \bar{\phi}) \cdot \omega_\alpha = \rho'(h_\alpha^{-1}) \cdot \omega'_\alpha + \partial'(h_\alpha),$$

hold for every $\alpha, \beta \in I$.

(C) The final case is dealing with connections induced on the pull back (or inverse image) of a principal sheaf by a continuous map. Here the differential triads may change since the base spaces are not the same any more. For this purpose we recall a few (standard) notions, refering for more details e.g. to [4].

Given a sheaf $(\mathcal{S}, \mathcal{X}, \pi_f)$, in any category, and a continuous map $f : Y \longrightarrow X$, *the pull back* of $\mathcal{S}$ by f is the sheaf (in the same category) $(f \star \mathcal{S}, \mathcal{Y}, \pi_f^\star)$, where

$$f \star \mathcal{S} := \mathcal{Y} \times_{\mathcal{X}} \mathcal{S}, \quad \pi_f^\star := \surd\nabla_\infty \mid \{ \star \mathcal{S}.$$

The canonical map

$$\bar{f}_s := pr_2 \mid f \star \mathcal{S} : \{ \star \mathcal{S} \longrightarrow \mathcal{S}$$

identifies the fibers over y and $f(y)$, for every $y \in Y$.

Equivalently, $f \star \mathcal{S}$ is obtained by the presheaf of (continuous) *sections of $\mathcal{S}$ along f*.

If $(\mathcal{S}, \mathcal{X}, \pi_f)$ and $(\mathcal{T}, \mathcal{X}, \pi_{\mathcal{T}})$ are sheaves and $\phi : \mathcal{S} \longrightarrow \mathcal{T}$ is a morphism (in a given category), then the *universal property* of pull backs induces a morphism $f \star \phi : f \star \mathcal{S} \longrightarrow \{ \star \mathcal{T}$. As a matter of fact,

$$f \star \phi(y, s) := (y, \phi(s)), \quad (y, s) \in (f \star \phi)_y = \{y\} \times \mathcal{S}_{\{(\dagger)}. \tag{3.8}$$

The situation is depicted in the following useful diagram:

Given now an algebraized space $X \equiv (X, \mathcal{A})$, we fix a principal sheaf $(\mathcal{P}, \mathcal{G}, \mathcal{X}, \pi_{\mathcal{P}})$, where $\mathcal{G} \equiv (\mathcal{G}, \rho, \mathcal{L}, \partial)$ is a Lie sheaf of groups with projection $\pi_G : \mathcal{G} \longrightarrow \mathcal{X}$. We denote by π_A and $\pi_\otimes$ the projections of $\mathcal{A}$ and $\Omega^1(\mathcal{L}) = \otimes^\infty \otimes_{\mathcal{A}} \mathcal{L}$ respectively. For a map f as before, we obtain the pull backs $f \star \mathcal{A}, \{ \star \mathcal{G}, \{ \star (\otimes^\infty(\mathcal{L}))$, and $f \star \mathcal{P}$, with respective projections $\pi_A^\star, \pi_G^\star, \pi_\otimes^\star, \pi_P^\star$.
Lemma 3.10. *There exists an isomorphism of $f \star \mathcal{A}$-modules*

$$F : f \star \Omega^1 \otimes_{f\star\mathcal{A}} f \star \mathcal{L} \cong \{ \star (\otimes^\infty \otimes_{\mathcal{A}} \mathcal{L}).$$

Proof. Let V be any open subset of Y and any decomposable element

$$s \otimes t \in (f \star \Omega^1)(V) \otimes_{(f\star\mathcal{A})(\mathcal{V})} (f \star \mathcal{L})(\mathcal{V}).$$

Then we can set

$$s = (id_V, \sigma) \quad \text{and} \quad t = (id_V, \tau),$$

where $\sigma : V \longrightarrow \Omega^1$ and $\tau : V \longrightarrow \mathcal{L}$ are continuous sections along f, i.e.

$$\sigma(y) \in V_{f(y)}, \tau(y) \in \mathcal{L}_{\{(\dagger)}, \dagger \in \mathcal{V}.$$

Hence,

$$(id_V, \sigma \otimes \tau) \in f \star (\Omega^1 \otimes_{\mathcal{A}} \mathcal{L})(\mathcal{V}),$$

after the identification

$$(\Omega^1 \otimes_{\mathcal{A}} \mathcal{L})_\S \equiv \otimes_\S^\infty \otimes \mathcal{L}_\S, \quad \S \in \mathcal{X}.$$

Therefore, the correspondence

$$s \otimes t = (id_V, \sigma) \otimes (id_V, \tau) \longrightarrow (id_V, \sigma \otimes \tau),$$

extended (by $f\star\mathcal{A}$-linearity) on the whole tensor product, determines a mapping

$$F_V : (f \star \Omega^1)(V) \otimes_{(f\star\mathcal{A})(\mathcal{V})} (f \star \mathcal{L})(\mathcal{V}) \longrightarrow (\{ \star (\otimes^\infty \otimes_{\mathcal{A}} \mathcal{L}))(\mathcal{V}).$$

More precisely, for any tensor $\sum_{ij} a_{ij} \cdot (s_i \otimes t_j)$ in the range, where the coefficients s_j and t_j are as before and $a_{ij} \in (f \star \mathcal{A})(\mathcal{V})$ has the form $a = (id_V, \alpha_{ij})$, with $\alpha_{ij}(y) \in \mathcal{A}_{\{(\dagger)}$, we set

$$F_V(\sum_{ij} a_{ij} \cdot s_i \otimes t_j) := (id_V, \sum_{ij} \alpha_{ij} \cdot \sigma_i \otimes \tau_j).$$

In particular, for every $y \in V$, we see that

$$F_V(\sum_{ij} a_{ij} \cdot (s_i \otimes t_j))(y) = (y, \sum_{ij} \alpha_{ij}(y) \cdot (\sigma_i(y) \otimes \tau_j(y))) =$$

$$= (y, \sum_{ij} \alpha_{ij}(y) \cdot (\bar{f}_{\Omega,y}(s_i(y)) \otimes \bar{f}_{\mathcal{L},\dagger}(t(y))) =$$

$$= (y, \sum_{ij} [\bar{f}_{\Omega,y} \otimes \bar{f}_{\mathcal{L},\dagger}]((a_{ij} \cdot (s_i \otimes t_j))(y)),$$

if $\bar{f}_{\Omega,y}$ (resp. $\bar{f}_{\mathcal{L},\dagger}$) is the restriction of $\bar{f}_{\Omega^1}$ (resp. $\bar{f}_{\mathcal{L}}$) on the stalk $(f\star\Omega^1)_y$ (resp. $(f\star\mathcal{L})_\dagger$). We immediately check that (F_V), with V running in the set of all open subsets of Y, is an injective morphism of presheaves inducing, in turn, an injective morphism F between the sheaves of the statement.

On the other hand, F is onto. Indeed, let an arbitrarily chosen $(y,w)\in f\star(\Omega^1\otimes_{\mathcal{A}}\mathcal{L})$. Since $w\in(\Omega^1\otimes_{\mathcal{A}}\mathcal{L})_{\{(\dagger)}$, there is an open $U\subseteq X$, with $f(x)\in U$, and sections

$$\alpha_{ij}\in\mathcal{A}(\mathcal{U}),\quad \sigma_\rangle\in\otimes^\infty(\mathcal{U}),\quad \tau_\rceil\in\mathcal{L}(\mathcal{U})$$

such that

$$w=(\sum_{ij}\alpha_{ij}\cdot(\sigma_i\otimes\tau_j))(f(y)).$$

Then, for $V:=f^{-1}(U)$, the sections

$$a_{ij}:=(id_V,\alpha_{ij}\circ f\mid V)\in(f\star\mathcal{A})(\mathcal{V}),$$

$$s_i:=(id_V,\sigma_i\circ f\mid V)\in(f\star\Omega^1)(V)$$

and

$$t_j:=(id_V,\tau_j\circ f\mid V)\in(f\star\mathcal{L})(\mathcal{V})$$

determine the element

$$(\sum_{ij}a_{ij}\cdot(s_i\otimes t_j))(y)\in((f\star\Omega^1)\otimes_{f\star\mathcal{A}}(f\star\mathcal{L}))_\dagger$$

mapped by F precisely to (y,w). This completes the proof.

Lemma 3.11. *The following properties hold true:*

i) $(Y,f\star\mathcal{A})$ *is an algebrized space.*

ii) $(f\star\mathcal{A},\{\star\lceil,\{\star\otimes^\infty)$ *is a differential triad.*

iii) $f\star\mathcal{G}$ *admits the structure of a Lie sheaf of groups.*

Proof. Property i) is obvious, whereas ii) is a direct consequence of (1.1′) and (3.9), taking also into account that $f\star\Omega^1$ is an $f\star\mathcal{A}$-module.

For iii) we first observe that the map

$$\rho^\star: f\star\mathcal{G}\longrightarrow\mathcal{A}\sqcap\sqcup(\{\star\mathcal{L}),$$

given by

$$[\rho\star(y,g)](y,u):=(y,[\rho(g)](u));\quad y\in Y, g\in\mathcal{G}_{\{(\dagger)},\sqcap\in\mathcal{L}_{\{(\dagger)},$$

determines a representation, the continuity of which is easily established if we use the corresponding actions instead of ρ and $\rho^\star$. Now $\rho^\star$ induces a representation of $f\star\mathcal{G}$ in

$$f\star\Omega^1\otimes_{f\star\mathcal{A}}f\star\mathcal{L}\cong\{\star(\otimes^\infty\otimes_{\mathcal{A}}\mathcal{L})$$

defined by (recall (1.2) and the ensuing comments).

$$\rho^\star(y,g)\cdot(y,\omega):=(y,\rho(y)\cdot\omega);\quad (y,\omega)\in f\star(\Omega^1\otimes_{\mathcal{A}}\mathcal{L}),\quad \dagger\in\mathcal{Y}. \tag{3.10}$$

On the other hand, for any (y,g), $(y,h) \in f \star (\mathcal{G})$, equalities (1.3) and (3.10) yield

$$(f \star \partial)((y,g)\cdot(y,h)) = (f \star \partial)(y, g\cdot h) = (y, \partial(g\cdot h)) =$$
$$= (y, \rho(h^{-1})\cdot \partial g + \partial h) = (y, \rho(h^{-1})\cdot \partial g) + (y, \partial h) =$$
$$= \rho^{\star}(y, h^{-1})\cdot(y, \partial g) + (y, \partial h) = \rho^{\star}((y,h)^{-1})\cdot (f\star\partial)(g) + (f \star \partial)(h).$$

The previous arguments, together with Lemma 3.7, imply that $(f \star \mathcal{G}, \rho^{\star}, \{ \star \mathcal{L})$ is a Lie sheaf of groups as desired.

Theorem 3.12. *For a given principal sheaf $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, \mathcal{X}, \pi_{\mathcal{P}})$ and a continuous mapping $f : Y \to X$, the quadratic $(f \star \mathcal{P}, \{ \star \mathcal{G}, \mathcal{Y}, \pi^{\star}_{\mathcal{P}})$ is a principal sheaf and connections on $\mathcal{P}$ induce corresponding connections on $f \star \mathcal{P}$.*

Proof. The action of $f \star \mathcal{G}$ on the right of $f \star \mathcal{P}$ is clear. On the other hand, the $\mathcal{G}$-equivariant morphisms (2.1) determine the $f \star \mathcal{G}$-equivariant morphisms

$$f \star \phi_{\alpha} : (f \star \mathcal{P}) \mid \mathcal{V}_{\alpha} \equiv \mathcal{V}_{\alpha} \times_{\mathcal{X}} (\mathcal{P} \mid \mathcal{U}_{\alpha}) \longrightarrow \mathcal{V}_{\alpha} \times_{\mathcal{X}} (\mathcal{G} \mid \mathcal{U}_{\alpha}) \equiv (\{ \star \mathcal{G}) \mid \mathcal{V}_{\alpha},$$

with $V_{\alpha} = f^{-1}(U_{\alpha})$. Finally, working as in the proof of Lemma 3.11, we establish the last assertion of the statement.

4 Mappings of vector sheaf connections

We consider now two vector sheaves $\mathcal{E} \equiv (\mathcal{E}, \mathcal{X}, \pi)$ and $\mathcal{E}' \equiv (\mathcal{E}', \mathcal{X}, \pi')$ over the same base X, the latter being equipped with the same differential triad $(\mathcal{A}, \lceil, \otimes^{\infty})$. We assume that $\mathcal{E}$ and $\mathcal{E}'$ are of respective ranks m, n.

Definition 4.1. Let $F : \mathcal{E} \to \mathcal{E}'$ be a morphism of vector sheaves. Two connections ∇ and ∇', on $\mathcal{E}$ and $\mathcal{E}'$ respectively, are said to be F-*conjugate* (or F-related) if

(4.1) $$\nabla' \circ F = (F \otimes 1) \circ \nabla \quad (1 := id \mid \Omega^1).$$

In a more illustrating way, the following diagram is commutative:

For the local description of the previous situation we fix a gauge $(U_{\alpha}, \Phi_{\alpha})$ as in (2.8) and we denote by (e_i^{α}) and $(e_i^{-\alpha})$ the corresponding natural bases of $\mathcal{E}(\mathcal{U}_{\alpha})$ and $\mathcal{E}'(\mathcal{U}_{\alpha})$ respectively, defined by (2.14). The local connection matrix (2.12) of ∇ is determined by (2.13). Analogously, the local matrix of ∇'

$$\bar{\omega}^{\alpha} = (\omega_{kl}^{-\alpha}) \in M_n(\Omega^1(U_{\alpha})); \quad 1 \leq k, \quad 1 \leq n,$$

is determined by

$$\nabla'(\bar{e}_l^\alpha) = \sum_{k=1}^{n} \bar{e}_k^\alpha \otimes \bar{\omega}_{kl}^\alpha, \quad 1 \leq 1 \leq n. \tag{4.2}$$

Similarly, the $\mathcal{A}$-linearity of F implies that, over U_α, F is determined by an $n \times m$ matrix

$$F^\alpha = (F_{kj}^\alpha) \in M_{n\times m}(\mathcal{A})(\mathcal{U}_\alpha))$$

such that

$$F\,|_{U_\alpha}\,(e_i^\alpha) = \sum_{k=1}^{n} F_{ki}^\alpha \cdot \bar{e}_k^\alpha; \quad 1 \leq i \leq m. \tag{4.3}$$

Evaluating now both sides of (4.1) at $e_j := e_j^\alpha$ (for convenience, in the subsequence calculations we omit the superscript α from all entries) and taking into account (2.13), (2.14), (4.2) and (4.3) we check that

$$(\nabla' \circ F)(e_j) = \nabla'\left(\sum_{k=1}^{n} F_{kj} \cdot \bar{e}_k\right) =$$

$$= \sum_{k=1}^{n} (F_{kj} \cdot \nabla'(\bar{e}_k) + \bar{e}_k \otimes dF_{kj}) =$$

$$= \sum_{k=1}^{n} F_{kj}\left(\sum_{l=1}^{n} \bar{e}_l \otimes \bar{\omega}_{lk}\right) + \sum_{k=1}^{n} \bar{e}_k \otimes dF_{kj} =$$

$$= \sum_{k=1}^{n}\sum_{l=1}^{n} \bar{e}_l \otimes \bar{\omega}_{lk} \cdot F_{kj} + \sum_{k=1}^{n} \bar{e}_k \otimes dF_{kj} =$$

$$= \sum_{l=1}^{n} \bar{e}_l \otimes \left(\sum_{k=1}^{n} \bar{\omega}_{lk} \cdot F_{kj}\right) + \sum_{l=1}^{n} \bar{e}_l \otimes dF_{lj} =$$

$$= \sum_{l=1}^{n} \bar{e}_l \otimes \left[\left(\sum_{k=1}^{n} \bar{\omega}_{lk} \cdot F_{kj}\right) + dF_{lj}\right].$$

Similarly,

$$[(F \otimes 1) \circ \nabla](e_j) = \sum_{i=1}^{m} F(e_i) \otimes \omega_{ij} =$$

$$= \sum_{i=1}^{m}\left(\sum_{k=1}^{n} F_{ki} \cdot \bar{e}_k\right) \otimes \omega_{ij} = \sum_{l=1}^{n} \bar{e}_l \otimes \left(\sum_{i=1}^{m} F_{li} \cdot \omega_{ij}\right).$$

Therefore, the previous expressions and (4.1) imply that

$$\sum_{i=1}^{m} F_{li} \cdot \omega_{ij} = \left(\sum_{k=1}^{n} \bar{\omega}_{lk} \cdot F_{kj}\right) + dF_{lj}; \quad 1 \leq j \leq m, \quad 1 \leq l \leq n,$$

from which we obtain (in matrix notation),

$$F^{\alpha} \cdot \omega^{\alpha} = \bar{\omega}^{\alpha} \cdot F^{\alpha} + dF^{\alpha}, \quad \alpha \in I. \tag{4.4}$$

Thus, we have proved in fact the following main

Theorem 4.2. *Two connections ∇ and ∇' on $\mathcal{E}$ and $\mathcal{E}'$, respectively, are F-conjugate if and only if there exists a 0-cochain of local matrices*

$$(F^{\alpha}) \in C^{0}(\mathcal{C}, \mathcal{M}_{\backslash\times\Updownarrow}(\mathcal{A}))$$

such that (4.4) is satisfied, for every $\alpha \in I$.

As we have seen in Theorem 2.4, each connection ∇ on $\mathcal{E}$ corresponds bijectively to a connection D on the principal sheaf of frames $\mathcal{P}(\mathcal{E})$, both having the same local connection matrices/forms $\omega^{\alpha}(\alpha \in I)$ over $\mathcal{C}$. Since a morphism F does not necessarily induce a morphism between the principal sheaves $\mathcal{P}(\mathcal{E})$ and $\mathcal{P}(\mathcal{E}')$. condition (4.1) does not imply that D and D' are necessarily conjugate by means of a principal sheaf morphism.

However, if F is an *isomorphism* (thus $\mathcal{E}$ and $\mathcal{E}'$ have the same rank, say, n), its local matrices F^{α} are invertible, i.e.

$$F^{\alpha} \in GL(n, \mathcal{A}(\mathcal{U}_{\alpha})) \cong \mathcal{GL}(\backslash, \mathcal{A})(\mathcal{U}_{\alpha});$$

hence, (4.1) takes the equivalent form

$$\omega^{\alpha} = \mathcal{A}\lceil((\mathcal{F}^{\alpha})^{-\infty}) \cdot \omega^{\alpha} + \partial\mathcal{F}^{\alpha}, \quad \alpha \in \mathcal{I}. \tag{4.5}$$

On the other hand, F induces an obvious $\mathcal{GL}(\mathcal{A})$-X-isomorphism f between $\mathcal{P}(\mathcal{E})$ and $\mathcal{P}(\mathcal{E}')$. Moreover, the construction of f from F implies the analogue of (3.4) for $h_{\alpha} = F^{\alpha}$ $(\alpha \in I)$. Consequently, Theorems 3.4 and 4.2, along with equality (4.5), yield

Theorem 4.3 *With the previous notations, the following assertions are equivalent:*

i) ∇ and ∇' are F-conjugate.

ii) D and D' are f-conjugate.

Both isomorphisms F and f are fully determined by the same zero cochain $(F^{\alpha}) \in C^{0}(\mathcal{C}, \mathcal{M}_{\backslash}(\mathcal{A}))$, in virtue of equalities (4.3) and (3.4) respectively.

Remarks. **1)** Completing the discussion after Theorem 4.1, in an opposite direction, one may ask the following question:

Given a morphism $(f, \phi, \bar{\phi}, id_X)$ between the principal sheaves of frames $(\mathcal{P}(\mathcal{E}), \mathcal{GL}(\Updownarrow, \mathcal{A}), \mathcal{X}, \pi)$ and $(\mathcal{P}(\mathcal{E}'), \mathcal{GL}(\backslash, \mathcal{A}), \mathcal{X}, \pi')$ with $\bar{\phi} : \mathcal{M}_{\Updownarrow}(\mathcal{A}) \to \mathcal{M}_{\Updownarrow}(\mathcal{A})$, under what conditions is it possible to induce a morphism F between $\mathcal{E}$ and $\mathcal{E}'$ so that the $(f, \phi, \bar{\phi}, id_X)$-conjugation (in the sense of Theorem 3.6) is equivalent with the F-conjugation (defined at the beginning of this section)?

The bundle analogue of this has been answered in [13] (cf. Lemma 4.2 and Theorem 4.3 therein).

2) Obviously, using Lemma 3.10 and 3.11 (with $\mathcal{L} = \mathcal{M}_{\backslash}(\mathcal{A})$), we can prove the vector sheaf analogue of Theorem 3.12.

3) It is clear that, for all the results refering to transformations of connections, both on principal and vector sheaves, there are analogous statements for the corresponding curvatures.

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