# Recent Progress in the Integration of Poisson Systems via the Mid–Point Rule and Runge–Kutta Algorithm

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**Abstract**

Some recent progress in the integration of Poisson systems via the mid–point rule and Runge–Kutta algorithm are discussed and some of their properties are pointed out.

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**Key words:** Poisson manifold, mid–point–rule, Runge–Kutta–approximation

## 1 Introduction

In the last time many dynamical systems have been found to be Hamilton–Poisson systems. These include the Euler equations for the free rigid body, the Maxwell–Vlasov equations from plasma physics, the Maxwell–Bloch equations from laser–matter dynamics and others, see for details [**4**] and [**6**]. It is an interesting and very tempting problem to try to numerically integrate these Hamilton–Poisson systems such that the corresponding algorithms to preserve as much as possible from their Poisson pictures.

The goal of our paper is to present some recent progress in the integration of Poisson systems via the mid–point rule and Runge–Kutta algorithm and to point out some of their properties.

## 2 Hamilton–Poisson systems

Let $P$ be a smooth $n$–dimensional manifold and $C^\infty(P, \mathbf{R})$ the space of smooth ($= C^\infty$) real valued functions defined on $P$. Consider a given bracket operation denoted

$$\{\cdot,\cdot\}: C^\infty(P,\mathbf{R}) \times C^\infty(P,\mathbf{R}) \to C^\infty(P,\mathbf{R}).$$

**Definition 2.1.** The pair $(P, \{\cdot,\cdot\})$ is called a *Poisson manifold* if $\{\cdot,\cdot\}$ satisfies:

(PB1) bilinearity, i.e. $\{\cdot,\cdot\}$ is $\mathbf{R}$–bilinear.

(PB2) anticommutativity, i.e. $\{f, g\} = -\{g, f\}$ for every $f, g \in C^\infty(P, \mathbf{R})$.

(PB3) Jacobi's identity, i.e. $\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$ for every $f, g, h \in C^\infty(P, \mathbf{R})$.

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(PB4) Leibniz' rule, i.e. $\{fg, h\} = f\{g, h\} + g\{f, h\}$ for every $f, g, h \in C^\infty(P, \mathbf{R})$. Conditions (PB1)–(PB3) make $(C^\infty(P, \mathbf{R}), \{\cdot, \cdot\})$ into a Lie algebra and moreover it is not hard to see that every Poisson manifold is essentially a union of symplectic manifolds which fit together in a smooth way.

If $(P, \{\cdot, \cdot\})$ is a Poisson manifold, then because of (PB1) and (PB4), there is a tensor field $B$ on $P$, assigning to each $x \in P$ a linear map

$$B(x): T_x^*P \to T_xP$$

such that

$$\{f, g\}(x) = \langle B(x) \cdot df(x), dg(x) \rangle.$$

Here $\langle \cdot, \cdot \rangle$ denotes the natural pairing between vectors and covectors. Because of (PB2), $B(x)$ is antisymmetric. Letting $x^i$, $i = 1, 2, \ldots, n$, denote local coordinates on $P$ we have:

$$\{f, g\} = B^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}.$$

**Definition 2.2.** Let $(P_1, \{\cdot, \cdot\}_1)$ and $(P_2, \{\cdot, \cdot\}_2)$ be Poisson manifolds. A mapping $\phi: P_1 \to P_2$ is called *Poisson* if for all $f, g \in C^\infty(P, \mathbf{R})$ we have

$$\{f, g\}_2 \circ \phi = \{f \circ \phi, g \circ \phi\}_1,$$

or equivalently

$$JB_1J^T = B_2,$$

where $J$ is the Jacobian matrix associated to $\phi$. Locally the above relation can be written in the following form:

$$B_1^{ij}(x) \frac{\partial y^k}{\partial x^i} \cdot \frac{\partial y^l}{\partial x^j} = B_2^{kl}(x).$$

**Definition 2.3.** An *Hamilton–Poisson system* is a triple $(P, \{\cdot, \cdot\}, H)$, where $(P, \{\cdot, \cdot\})$ is a Poisson manifold and $H$ is a smooth real valued function defined on $P$ called the *Hamiltonian* or *the energy.* Its dynamics is described by the integral curves of the Hamiltonian vector field $X_H$ defined by

$$X_H(f) = \{H, f\},$$

or locally

$$\dot{x}_i = \{x_i, H\}, \;\; i = 1, 2, \ldots, n.$$

Moreover a *Casimir* of our configuration is a smooth function $C \in C^\infty(P, \mathbf{R})$ such that

$$\{C, f\} = 0$$

for each $f \in C^\infty(P, \mathbf{R})$.

**Example 2.1.** (the free rigid body) The Euler angular momentum equations of the free rigid body are written in the following form

$$\text{(2.1)} \qquad \begin{cases} \dot{m}_1 = a_1 m_2 m_3 \\ \dot{m}_2 = a_2 m_1 m_3 \\ \dot{m}_3 = a_3 m_1 m_2, \end{cases}$$

where

$$a_1 = \frac{1}{I_3} - \frac{1}{I_2},\ \ a_2 = \frac{1}{I_1} - \frac{1}{I_3},\ \ a_3 = \frac{1}{I_2} - \frac{1}{I_1},$$

$I_1$, $I_2$, $I_3$ being the components of the inertia tensor and we suppose as usually that $I_1 > I_2 > I_3$.

The free rigid body is a Hamilton–Poisson system with the phase space $P = \mathbf{R}^3$, the Poisson bracket given by

$$\{f, g\}_{RB}(m) = -m \cdot (\nabla f \times \nabla g) \tag{2.2}$$

and the Hamiltonian $H$ defined by

$$H(m_1, m_2, m_3) = \frac{1}{2}\left(\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^2}{I_3}\right). \tag{2.3}$$

Moreover, a Casimir of our configuration $(\mathbf{R}^3, \{\cdot, \cdot\}_{RB})$ is given by the function

$$C(m_1, m_2, m_3) = \frac{1}{2}(m_1^2 + m_2^2 + m_3^2). \tag{2.4}$$

It follows that the trajectories of motion are intersections of the ellipsoids

$$H = constant$$

with the spheres

$$C = constant.$$

There is also a very nice interpretation of the rigid body bracket (2.2) namely, the rigid body bracket (2.2) is the minus–Lie–Poisson structure on $so(3)^*$. Indeed, let $SO(3)$ be the Lie group of all linear orientation preserving orthogonal transformations of $\mathbf{R}^3$ to itself. Its Lie algebra $so(3)$ is the set of all $3 \times 3$ skew–symmetric matrices. It can be canonically identified with $\mathbf{R}^3$ via the map.

$$\wedge\colon v = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \in \mathbf{R}^3 \mapsto \widehat{v} = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \in so(3).$$

Then the Lie bracket on $so(3)$ corresponds to the cross product on $\mathbf{R}^3$ in the sense that

$$[\widehat{v}, \widehat{w}] = \widehat{v \times w}.$$

The dual of $so(3)$, i.e., $so(3)^*$ can be also identified with $\mathbf{R}^3$ and then the minus–Lie–Poisson structure on $so(3)^*$ is given by the matrix

$$\Pi_{RB} = \begin{bmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{bmatrix},$$

which is nothing else than the Rigid–Body bracket (2.2) as easily can be verified.

**Example 2.2.** (the Maxwell–Bloch equations). The 3–dimensional real valued Maxwell–Bloch equations from laser matter dynamics are usually written as

(2.5)
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1 x_3 \\ \dot{x}_3 = -x_1 x_2. \end{cases}$$

They have a Hamilton–Poisson realization with the phase space $P = (\mathbf{R}^3_{[\cdot,\cdot]})^*$, where $\mathbf{R}^3_{[\cdot,\cdot]}$ is the Lie algebra $\mathbf{R}^3$ with the bracket operation given by

$$[e_1, e_2] = e_3, \ \ [e_1, e_3] = e_2, \ \ [e_2, e_3] = 0,$$

$\{e_1, e_2, e_3\}$ being the canonical basis of $\mathbf{R}^3$, the minus Lie–Poisson structure given by the matrix

(2.6)
$$\Pi_{MB} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & 0 \\ -x_2 & 0 & 0 \end{bmatrix}$$

and the Hamiltonian $H$ defined by:

(2.7)
$$H(x_1, x_2, x_3) = \frac{1}{2}x_1^2 + x_3.$$

Moreover, a Casimir of our configuration is given by

(2.8)
$$C(x_1, x_2, x_3) = \frac{1}{2}(x_2^2 + x_3^2)$$

**Theorem 2.1.** ([4], [6]). *Let $(P, \{\cdot,\cdot\}, H)$ be a Hamilton–Poisson system and $\phi_t$ the flow for $X_H$. Then we have:*

(i) *Each $\phi_t$ is a Poisson map.*

(ii) *$\phi_t$ preserves $H$ for each $t \in \mathbf{R}$ .*

(iii) *Each $\phi_t$ preserves the symplectic leaves of the Poisson manifold $(P, \{\cdot,\cdot\})$ .*

# 3 Mid–point rule

Let us start with a Hamilton–Poisson system with the phase space $P = \mathbf{R}^n$, the Poisson structure given by the matrix $\Pi$ and the Hamiltonian $H$. Its dynamics can be written in the following form

(3.1)
$$\dot{x} = \Pi(x) \cdot \nabla H(x).$$

We are interested in the numerical integration of this system via the mid–point rule. It is an implicit recurrence given in our case by

(3.2)
$$\frac{x^{k+1} - x^k}{h} = \Pi\left(\frac{x^k + x^{k+1}}{2}\right) \cdot \nabla H\left(\frac{x^k + x^{k+1}}{2}\right),$$

where $h$ is the size step (or time step). If $h$ is small enough, then (3.2) defines a diffeomorphism $\phi_H^h$ via

(3.3)
$$x^{k+1} = \phi_H^h(x^k).$$

We compute the Fréchet derivative $D\phi_H^h(x)$ as follows. By definition, $y = \phi_H^h(x)$ is the unique solution of the implicit equation

$$F(x,y) \stackrel{\text{def}}{=} y - x - h\Pi\left(\frac{x+y}{2}\right) \cdot \nabla H\left(\frac{x+y}{2}\right) = 0. \tag{3.4}$$

Differentiating $F(x, \phi_H^h(x)) = 0$ gives:

$$D_1 F + D_2 F \circ D\phi_H^h = 0, \tag{3.5}$$

where $D_i F$, $i = 1, 2$, denote the partial Fréchet derivatives. For $h$ small enough, $D_2 F$ has an inverse, and (3.5) may be written as

$$D\phi_H^h = -(D_2 F)^{-1} \circ (D_1 F). \tag{3.6}$$

For the particular case $\Pi(x) = \Pi =$ *constant* it is easy to see that

$$D_1 F = -I_n - \frac{h}{2}\Pi H_{xx}\left(\frac{x + \phi_H^h(x)}{2}\right)$$

and

$$D_2 F = I_n - \frac{h}{2}\Pi H_{xx}\left(\frac{x + \phi_H^h(x)}{2}\right).$$

Letting

$$Q(x) \stackrel{\text{def}}{=} H_{xx}\left(\frac{x + \phi_H^h(x)}{2}\right)$$

denote the symmetric Hessian matrix, we obtain

$$D\phi_H^h(x) = \left[I_n - \frac{h}{2}\Pi Q(x)\right]^{-1}\left[I_n + \frac{h}{2}\Pi Q(x)\right]. \tag{3.7}$$

Now, following Austin, Krishnaprasad and Wang [**1**], we can prove:

**Theorem 3.1.** (Wang [**11**]). *If $\Pi(x) = \Pi =$ constant, then the mid–point integrator (3.1) is a Poisson one.*

**Proof.** We need to show that

$$D\phi_H^h(x)\Pi[D\phi_H^h(x)]^T = \Pi.$$

Based on the above calculations, this reduces to showing that

$$[(I_n - \frac{h}{2}\Pi Q(x))^{-1}(I_n + \frac{h}{2}\Pi Q(x))]\Pi[(I_n - \frac{h}{2}\Pi Q(x))^{-1}(I_n + \frac{h}{2}\Pi Q(x))]^T = \Pi$$

or equivalently

$$[I_n + \frac{h}{2}\Pi Q(x)]\Pi[I_n + \frac{h}{2}\Pi Q(x)]^T = [I_n - \frac{h}{2}\Pi Q(x)]\Pi[I_n - \frac{h}{2}\Pi Q(x)]^T.$$

This follows from the fact that $\Pi^T = -\Pi$ and $Q(x)^T = Q(x)$. Indeed,

$$[I_n + \frac{h}{2}\Pi Q(x)]\Pi[I_n + \frac{h}{2}\Pi Q(x)]^T = [I_n + \frac{h}{2}\Pi Q(x)][I_n - \frac{h}{2}\Pi Q(x)] = \Pi - \frac{h}{2}\Pi Q(x)\Pi +$$

$$+\frac{h}{2}\Pi Q(x)\Pi - \frac{h^2}{4}\Pi Q(x)\Pi Q(x)\Pi = \Pi - \frac{h^2}{4}\Pi Q(x)\Pi Q(x)\Pi Q(x)\Pi,$$

and similarly

$$[I_n - \frac{h}{2}\Pi Q(x)]\Pi[I_n - \frac{h}{2}\Pi Q(x)]^T = [I_n - \frac{h}{2}\Pi Q(x)][\Pi + \frac{h}{2}\Pi Q(x)\Pi] =$$

$$= \Pi + \frac{h}{2}\Pi Q(x)\Pi - \frac{h}{2}\Pi Q(x)\Pi - \frac{h^2}{4}\Pi Q(x)\Pi Q(x)\Pi = \Pi - \frac{h^2}{4}\Pi Q(x)\Pi Q(x)\Pi,$$

as required.

In the particular case when

$$\Pi = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix}$$

we again come upon Feng's theorem, namely

**Theorem 3.2.** (Feng [**2**]). *The mid–point integrator is a symplectic one. When* $\Pi(x)$ *is not a constant the mid-point rule is not in general a Poisson integrator.*

**Example 3.1.** (Puta and Birtea [**8**]). Let us take the case of symmetric rigid body, i.e. $I_2 = I_3$. Its dynamics is described by the equations:

$$\left\{ \begin{array}{l} \dot{m}_1 = 0 \\ \dot{m}_2 = a_2 m_1 m_3 \\ \dot{m}_3 = -a_2 m_1 m_2. \end{array} \right. \tag{3.8}$$

Then the mid–point rule takes the following form

$$\left\{ \begin{array}{l} m_1^{k+1} = m_1^k \\ \\ m_2^{k+1} = \dfrac{4m_2^k + 4ha_2 m_1^k m_3^k - h^2 a_2^2 (m_1^k)^2 m_2^k}{4 + h^2 a_2^2 (m_1^k)^2} \\ \\ m_3^{k+1} = \dfrac{4m_3^k - 4ha_2 m_1^k m_2^k - h^2 a_2^2 (m_1^k)^2 m_3^k}{4 + h^2 a_2^2 (m_1^k)^2}. \end{array} \right. \tag{3.9}$$

A straightforward computation shows us that the algorithm (3.9) is not of Poisson type. More precisely, it is of Poisson type if and only if $h = 0$.

Let $F$ be a first integral of (3.1), i.e.

$$\dot{F} = 0,$$

or equivalently

$$(\nabla F)^T \Pi \nabla H = 0. \tag{3.10}$$

Assuming $F$ is also three times differentiable, then by Taylor's formula we can expand $F$ around $x^k$ as:

$$F(x^{k+1}) = F(x^k) + (\nabla F(x^k))^T (x^{k+1} - x^k) + \frac{1}{2} D^2 F(x^k)(x^{k+1} - x^k)(x^{k+1} - x^k)$$

$$+\frac{1}{6}D^3F(x^k)(x^{k+1}-x^k)(x^{k+1}-x^k)(x^{k+1}-x^k)+O(\|x^{k+1}-x^k\|^4). \tag{3.11}$$

It can be checked that when the mid–point rule (3.2) is plugged into (3.11) we get

$$F(x^{k+1}-F(x^k)=\frac{1}{24}D^3F(x^k)u\cdot u\cdot u+O(h^4), \tag{3.12}$$

where

$$u=\Pi\left(\frac{x^{k+1}+x^k}{2}\right)\nabla H\left(\frac{x^{k+1}+x^k}{2}\right).$$

Equation (3.12) is an error formula dues to Austin, Krishnaprasad and Wang [**1**] for conserved quantities of (3.2), which contains only third or higher order terms. It follows that the mid–point rule (3.2) preserves exactly and conserved quantity having only linear and quadratic terms, including Casimir functions and the Hamiltonian of (3.1). So we have proved:

**Theorem 3.3.** (Austin, Krishnaprasad and Wang [**1**]). *The mid–point integrator (3.2) conserves all Casimir functions and the Hamiltonian $H$ of (3.1) if they contain only linear and quadratic terms.*

**Example 3.2.** In the case of the free rigid body (see Example 2.1), the mid–point rule can be written in the following form

$$\begin{cases}\dfrac{m_1^{k+1}-m_1^k}{h}=a_1\cdot\dfrac{m_2^{k+1}+m_2^k}{2}\cdot\dfrac{m_3^{k+1}+m_3^k}{2}\\[2ex]\dfrac{m_2^{k+1}-m_2^k}{h}=a_2\cdot\dfrac{m_1^{k+1}+m_1^k}{2}\cdot\dfrac{m_3^{k+1}+m_3^k}{2}\\[2ex]\dfrac{m_3^{k+1}-m_3^k}{h}=a_3\cdot\dfrac{m_1^{k+1}+m_1^k}{2}\cdot\dfrac{m_2^{k+1}+m_2^k}{2}.\end{cases} \tag{3.13}$$

Given $m_1^k,m_2^k,m_3^k$, equations (3.13) are solved for $m_1^{k+1},m_2^{k+1},m_3^{k+1}$. It follows via the above theorem that this integrator preserves both $H$ and $C$ given respectively by (2.3) and (2.4), but doesn't preserve the Poisson structure (2.2).

**Example 3.3.** In the case of 3–dimensional real valued Maxwell–Bloch equations (see Example 2.2), the mid–point rule can be written in the following form

$$\begin{cases}\dfrac{x_1^{k+1}-x_1^k}{h}=\dfrac{x_2^{k+1}+x_2^k}{2}\\[2ex]\dfrac{x_2^{k+1}-x_2^k}{h}=\dfrac{x_1^{k+1}+x_1^k}{2}\cdot\dfrac{x_3^{k+1}+x_3^k}{2}\\[2ex]\dfrac{x_3^{k+1}-x_3^k}{h}=-\dfrac{x_1^{k+1}+x_1^k}{2}\cdot\dfrac{x_2^{k+1}+x_2^k}{2}.\end{cases} \tag{3.14}$$

Given $x_1^k,x_2^k,x_3^k$, equations (3.14) are solved for $x_1^{k+1},x_2^{k+1},x_3^{k+1}$. It follows via Theorem 3.3 that this integrator preserves both $H$ and $C$, given respectively by (2.7) and (2.8).

# 4 Runge–Kutta algorithm

Let us start with the system of differential equations

$$\dot{x} = f(x), \qquad x \in \mathbf{R}^n. \tag{4.1}$$

Then the $s$–stage Runge–Kutta algorithm can be written in the following form

$$\begin{cases} x^{k+1} = x^k + h \cdot \sum_{i=1}^{s} b_i f(y_i) \\ y_i = x^k + h \cdot \sum_{j=1}^{s} a_{ij} f(y_j), \end{cases} \tag{4.2}$$

where $1 \leq i \leq s$.

If our system (4.1) is Hamiltonian, i.e., it can be put in the equivalent form

$$\begin{cases} \dot{q}_i = \frac{\partial H}{\partial p_i} \\ \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, 2, \ldots, n, \end{cases} \tag{4.3}$$

then we have the following result proved independently by Lasagni [**3**], Sanz–Serna [**9**] and Suris [**10**].

**Theorem 4.1.** (Lasagni, Sanz–Serna and Suris) *Assume that the coefficients of the s–stage Runge–Kutta algorithm satisfy the relations*

$$b_i a_{ij} + b_j a_{ji} - b_i b_j = 0, \qquad 1 \leq i, j \leq s.$$

*Then the integrator (4.1) is a symplectic (so a Poisson) one.*

**Proof.** We shall sketch the proof following Sanz–Serna [**9**]. For beginning let us write the relations (4.2) for our particular system (4.3). We get

$$\begin{cases} p^{k+1} = p^k + h \cdot \sum_{i=1}^{s} b_i f(P_i, Q_i), \\ q^{k+1} = q^k + h \cdot \sum_{i=1}^{s} b_i g(P_i, Q_i), \end{cases} \tag{4.4}$$

$$\begin{cases} P_i = p^k + h \cdot \sum_{j=1}^{s} a_{ij} f(P_j, Q_j), \\ Q_i = q^k + h \cdot \sum_{j=1}^{s} a_{ij} g(P_j, Q_j), \end{cases} \tag{4.5}$$

where $f$ and $g$ respectively denote the vectors with components $-\partial H/\partial q_i$ and $\partial H/\partial p_i$. We employ also the notation

$$r_i = f(P_i, Q_i) \text{ and } l_i = g(P_i, Q_i)$$

for the slope of the stages. Differentiate (4.4) and form the exterior product to arrive at

$$dp^{k+1} \wedge dq^{k+1} = dp^k \wedge dq^k + h \sum_{i=1}^{s} b_i dr_i \wedge dq^k + h \sum_{j=1}^{s} b_j dp^k \wedge dl_j + h^2 \sum_{i,j=1}^{s} b_i b_j dr_i \wedge dl_j.$$

Our next step is to eliminate $dr_i \wedge dq^k$ and $dp^k \wedge dl_j$ from this expression. This is easily achieved by differentiating (4.5) and taking the exterior product of the result with $dr_i$, $dl_j$. The outcome of the elimination is

$$dp^{k+1} \wedge dq^{k+1} - dp^k \wedge dq^k = h \sum_{i=1}^{s} b_i [dr_i \wedge dQ_i + dP_i \wedge dl_i] -$$

$$-h^2 \sum_{i,j=1}^{s} (b_i a_{ij} + b_j a_{ji} - b_i b_j) dr_i \wedge dl_j.$$

The second term in the right hand side vanishes by hypothesis. To finish the proof is then sufficient to show that, for each $i$,

$$dr_i \wedge dQ_i + dP_i \wedge dl_i = 0.$$

In fact, dropping the subscript $i$ that numbers the stages, we can write

$$dr \wedge dQ + dP \wedge dl = \sum_{\mu=1}^{n} (dr_\mu \wedge dQ_\mu + dP_\mu \wedge dl_\mu) =$$

$$= \sum_{\mu,\nu=1}^{n} \left( \frac{\partial f_\mu}{\partial p_\nu} dP_\nu \wedge dQ_\nu + \frac{\partial f_\mu}{\partial q_\nu} dQ_\nu \wedge dQ_\mu + \frac{\partial g_\mu}{\partial p_\nu} dP_\mu \wedge dP_\nu + \frac{\partial g_\mu}{\partial q_\nu} dP_\mu \wedge dQ_\nu \right).$$

To see that this expression vanishes, express $f_\mu$ and $g_\nu$ as derivatives of $H$ and recall the skew–symmetry of the exterior product.

q.e.d.

If the system (4.1) is of Poisson type, i.e. it is equivalent to

$$\dot{x} = \Pi \cdot \nabla H,$$

then we have:

**Theorem 4.2.** (McLachlan [**5**]) *If* $\Pi$ *is constant, then the* $s$*–stage Runge–Kutta integrator is of Poisson type.*

**Proof.** It is known that $s$–stage Runge–Kutta algorithm is invariant under linear maps, that is, changing variables in the map or in the vector field results in the same Runge–Kutta map. $s$–stage Runge–Kutta for the Poisson system is, therefore, equivalent to $s$–stage Runge–Kutta for the system in canonical form with Poisson tensor given by the matrix:

$$\begin{bmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & 0 & 0_n \end{bmatrix}.$$

For this system, $s$–stage Runge–Kutta algorithm leaves the last $n$ variables fixed, so it is equivalent to $s$–stage Runge–Kutta for a Hamiltonian system in the first $m$ variables, for which it is a symplectic map and so a Poisson map. Thus, $s$–stage Runge–Kutta for the original system preserves the symplectic leaves and is symplectic on them as required.

q.e.d.

If the matrix $\Pi$ is not constant, then $s$–stage Runge–Kutta algorithm is not in general a Poisson one.

**Example 4.1.** (Puta [**7**]) Let us consider the Hamilton–Poisson system

$$\left(\mathbf{R}^2, \Pi = \begin{bmatrix} 0 & x_2 \\ -x_2 & 0 \end{bmatrix}, H(x_1,x_2) = Ax_1 + Bx_2 + C, \ \ A \neq 0\right).$$

Then the 1–stage Runge–Kutta algorithm is not of Poisson type. Indeed, the dynamics of our system is given by

$$\begin{cases} \dot{x}_1 = Bx_2 \\ \dot{x}_2 = -Ax_2. \end{cases}$$

Then the 1–stage Runge–Kutta algorithm with size step $h$ is given by

$$\begin{cases} x_1^{k+1} = x_1^k + \dfrac{hbB}{1+haA} x_2^k \\ \\ x_2^{k+1} = \left(1 - \dfrac{hbA}{1+haA}\right). \end{cases} \tag{4.6}$$

Now, an easy computation shows us that it doesn't preserve the Poisson tensor $\Pi$. Let us mention also that the algorithm (4.6) is also not energy preserving. Moreover the following assertions are equivalent:

(i) (4.6) is a Poisson integrator;
(ii) (4.6) is an energy integrator;
(iii) $A = 0$.

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