Notes on a New Finsler Metric Function

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Abstract

The Weyl metric is a Finsler metric of the (α, β) type and is closely related to the Randers and Beil (or Kaluza–Klein) metrics. Some properties of this metric are computed. The suitability of the metric for physical applications is discussed.

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A new Finsler metric has been introduced which is closely related to the *Randers and Beil metrics* [1]. The metric is of the (α, β) type [2] and appears as a special case of these metrics as classified by Park and Choi [3]. The metric has been labeled the *Weyl metric* [1] since it has a similar structure to metrics used in the old unified field theory of Weyl. The new metric can produce the Lorentz equation as an equation of motion.

The purpose of these notes is to develop a few of the mathematical and physical properties of the Weyl metric and to indicate whether or not it might be useful in the unification of gravitation and electromagnetism or other physical applications.

The basic form of the Finsler metric function for this metric is

$$F_W^2 = 2\alpha\beta, \quad \alpha^2 = \eta_{\mu\nu} v^\mu v^\nu, \quad \beta = k^{\frac{1}{2}} (B_\mu v^\mu), \tag{1}$$

where $v^\mu = dx^\mu / d\tau$ is the tangent vector component at a point in the Finsler space and $B_\mu = \eta_{\mu\nu} B^\nu$ is related to the electromagnetic potential vector A_μ by a gauge

$$B_\mu = A_\mu + \frac{\partial \Lambda}{\partial x^\mu} \tag{2}$$

The v^μ will be taken to be independent of x while B_μ is x–dependent. The constant k will be determined. For simplicity, the Lorentz metric $\eta_{\mu\nu}$ is used here as the metric for the original (background) Riemannian space. A more general gravitational metric $g_{\mu\nu}$ would give similar results, but with additional purely gravitational terms in the connections and curvatures.

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The metric function (1) can be compared with the metric functions of the Randers, F_R, and Kaluza-Klein, or Beil, metric, F_K:

$$F_R^2 = (\alpha+\beta)^2, \quad F_K^2 = \alpha^2+\beta^2 \tag{3}$$

so that

$$F_R^2 = F_W^2 + F_K^2 \tag{4}$$

which shows the close relationship of these metrics [1]. A similar way of classifying these metrics has been given in more recent work [3]. A general discussion of (α, β) metrics with a useful list of references appeas in Antonelli et al [4]. See also [5].

Each of the metric functions (1), (3) leads to a version of the Lorentz equation for charged particles. The necessary treatment of the constants in the Lorentz equations, however, leads to different physical interpretations [1, 6]. There appears to be some physical reason to prefer F_K over F_R [6]. For another approach to particle dynamics see a new paper in this journal [7].

The interesting discussion in Ingarden et al [8] on this question suggests that F_R is better adapted to an "optical" description of a trajectory whereas F_K is more suitable for a "microscopic" or dynamical description of particle motion in time.

For purposes of further discussion, the metric function, F_W, is modified slightly by introducing a more general constant factor w,

$$F_W^2 = w\alpha\beta. \tag{5}$$

One reason for this, as will be seen, is to preserve physical dimensionality.

Various consequences of the metric (5) are now examined. The metric itself is

$$f_{\mu\nu} = \frac{1}{2}\frac{\partial^2 F_W^2}{\partial v^\mu \partial v^\nu} = \frac{w\beta}{2\alpha}\left[\eta_{\mu\nu} - \frac{1}{\alpha^2}v_\mu v_\nu + \frac{k^{\frac{1}{2}}}{\beta}\left(B_\mu v_\nu + v_\mu B_\nu\right)\right] \tag{6}$$

with $v_\nu = \eta_{\mu\nu}v^\nu$.

For future reference, the determinant of the metric is,

$$f = |f_{\mu\nu}| = \frac{w^4\beta^2}{16\alpha^2}\left[kB_\mu B^\mu - 3\frac{\beta^2}{\alpha^2}\right]. \tag{7}$$

The contravariant form of the metric is

$$f^{\mu\nu} = \frac{2\alpha}{w\beta}\left\{\eta^{\mu\nu} - \frac{w^4\beta^2}{16\alpha^2[(-f)^{\frac{1}{2}}]^2}w^{\mu\nu}\right\} \tag{8}$$

$$w^{\mu\nu} = \frac{2\beta k^{\frac{1}{2}}}{\alpha^2}\left(B^\mu v^\nu + v^\mu B^\nu\right) - Qv^\mu v^\nu - kB^\mu B^\nu$$

$$Q = \frac{\beta^2}{\alpha^4} + \frac{kB_\lambda B^\lambda}{\alpha^2}$$

as can be verified by computing $f^{\mu\nu}f_{\nu\lambda} = \delta^\mu_\lambda$.

The equation of motion is most easily computed from

$$\frac{d}{d\tau}\left(\frac{\partial F_W^2}{\partial v^\mu}\right) - \frac{\partial F_W^2}{\partial x^\mu} = 0. \tag{9}$$

This gives

$$\frac{w\beta}{\alpha}\eta_{\mu\nu}\frac{dv^\nu}{d\tau} + w\alpha k^{\frac{1}{2}}\left(\frac{dB_\mu}{d\tau} - \frac{\partial B_\nu}{\partial x^\mu}v^\nu\right) = 0 \tag{10}$$

which leads to

$$\frac{dv_\mu}{d\tau} + \frac{\alpha^2 k^{\frac{1}{2}}}{\beta}\left(\frac{\partial B_\mu}{\partial x^\nu} - \frac{\partial B_\nu}{\partial x^\mu}\right)v^\nu = 0. \tag{11}$$

If the condition

$$\frac{\alpha^2}{\beta} = \frac{e}{mck^{\frac{1}{2}}} \tag{12}$$

holds, then (11) is the Lorentz equation,

$$\frac{dv_\mu}{d\tau} + \frac{e}{mc}F_{\nu\mu}y^\nu = 0. \tag{13}$$

The most convenient choice of values for α and β is

$$\alpha^2 = c^2 \tag{14}$$

which, from (12), immediately gives

$$\beta = \frac{mc^3 k^{\frac{1}{2}}}{e}. \tag{15}$$

Other choices are possible, but (14) and (15) give satisfactory results. Another convenient choice is

$$w = \frac{2e}{mc^2 k^{\frac{1}{2}}} \tag{16}$$

which produces the value

$$F_W^2 = f_{\mu\nu}v^\mu v^\nu = 2c^2 \tag{17}$$

with reasonable physical dimensionality.

The metric (6) then, can be evaluated as

$$f_{\mu\nu} = \eta_{\mu\nu} - \frac{v_\mu v_\nu}{c^2} + \frac{e}{mc^3}\left(B_\mu v_\nu + v_\mu B_\nu\right). \tag{18}$$

This shows that the Weyl metric, like the Randers metric, explicitly contains the charge to mass ratio of the particle. This property is an apparent disadvantage of the metric since it would imply that the structure of the space is dependent on the parameters of the test particle. This has previously been noted for the Randers metric by several authors [8, 9, 10]. The F_K metric avoids this problem, however.

The Riemann-Christoffel connection

$$\gamma_{\lambda\mu\nu} = \frac{1}{2}\left(\frac{\partial f_{\mu\lambda}}{\partial x^{\nu}} + \frac{\partial f_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial f_{\mu\nu}}{\partial x^{\lambda}}\right) \tag{19}$$

is easily computed (assuming that α and β are independent of x), using(6),

$$\gamma_{\lambda\mu\nu} = \frac{wk^{\frac{1}{2}}}{4\alpha}\left[v_\mu F_{\nu\lambda} + v_\nu F_{\mu\lambda} + v_\lambda\left(\frac{\partial B_\mu}{\partial x^\nu} + \frac{\partial B_\nu}{\partial x^\mu}\right)\right]. \tag{20}$$

An alternate expression to (9) for the equation of motion is

$$f_{\kappa\lambda}\frac{dv^\kappa}{d\tau} + \gamma_{\lambda\mu\nu}v^\mu v^\nu = 0. \tag{21}$$

It is easy to show that (13) follows from (20) and (21).

The vertical connection, also called the Cartan torsion tensor, is easily obtained

$$\begin{aligned} C_{\lambda\mu\nu} &= \tfrac{1}{2}\tfrac{\partial f_{\mu\nu}}{\partial v^\lambda} = \tfrac{w}{4\alpha}\left(k^{\frac{1}{2}}a_{\mu\nu\lambda} + \tfrac{3\beta}{\alpha^4}v_\lambda v_\mu v_\nu - \tfrac{\beta}{\alpha^2}b_{\mu\nu\lambda} - \tfrac{k^{\frac{1}{2}}}{\alpha^2}c_{\mu\nu\lambda}\right)\\ a_{\mu\nu\lambda} &= \eta_{\mu\nu}B_\lambda + \eta_{\mu\lambda}B_\nu + \eta_{\nu\lambda}B_\mu\\ b_{\mu\nu\lambda} &= \eta_{\mu\nu}v_\lambda + \eta_{\mu\lambda}v_\nu + \eta_{\nu\lambda}v_\mu\\ c_{\mu\nu\lambda} &= v_\mu v_\nu B_\lambda + v_\lambda v_\mu B_\nu + v_\lambda v_\nu B_\mu. \end{aligned} \tag{22}$$

This can be written in a more compact form,

$$\begin{aligned} C_{\lambda\mu\nu} &= \tfrac{w}{4\alpha}\left(u_{\mu\nu}b_\lambda + u_{\mu\lambda}b_\nu + u_{\nu\lambda}b_\mu\right)\\ u_{\mu\nu} &= \eta_{\mu\nu} - \tfrac{v_\mu v_\nu}{\alpha^2}\\ b_\lambda &= k^{\frac{1}{2}}B_\lambda - \tfrac{\beta}{\alpha^2}v_\lambda. \end{aligned} \tag{23}$$

A Finsler metric which is C-reducible [11] can be written in the form

$$C_{\lambda\mu\nu} = h_{\mu\nu}M_\lambda + h_{\mu\lambda}M_\nu + h_{\nu\lambda}M_\mu, \tag{24}$$

where M is an arbitrary vector and

$$h_{\mu\nu} = f_{\mu\nu} - \frac{\partial F}{\partial v^\mu}\frac{\partial F}{\partial v^\nu} \tag{25}$$

is called the *angular metric tensor.*

For the Weyl metric we obtain

$$h_{\mu\nu} = \frac{w\beta}{2\alpha}\left[\eta_{\mu\nu} - \frac{3v_\mu v_\nu}{2\alpha^2} + \frac{k^{\frac{1}{2}}}{2\beta}\left(B_\mu v_\nu + v_\mu B_\nu\right) - \frac{k\alpha^2}{2\beta^2}B_\mu B_\nu\right]. \tag{26}$$

It is obvious that the Weyl metric is not C-reducible. This is in agreement with well known results that the only (α, β) metrics which are C-reducible are of the Randers or Kropina type [4, 11].

At this point calculations of the various Finsler torsions and curvatures could be undertaken using (20) and (23). These will be postponed to future efforts.

An easier calculation would be that of the Ricci tensor associated with (20). The details will not be given here, but it is not difficult to see that one would obtain terms in an Einstein tensor which would have the same form as an electromagnetic energy–momentum tensor. In this way the Weyl metric is similar to the metric derived from F_K^2 [12]. However, for the Weyl metric there is no favorable comparison of constants as in [12]. The magnitudes of the energy-momentum tensors do not match those of gravitational theory. This leaves the status of the Weyl metric as a candidate for a unified field theory somewhat in doubt.

There is a similar magnitude problem for the Randers metric, in addition to the explicit presence of e/m in the metric. This means that F_K^2 becomes a clear choice for a unified theory.

The Randers and Weyl metrics are, however, very suitable for electron optics and other applications since they produce the Lorentz equation. Applications of the Randers metric are discussed extensively in [4].

Finally, another possible application of the Weyl metric is obtained by inserting the auxiliary condition

$$B_\mu = \frac{mc}{e}\left(1 - \frac{1}{2n^2}\right) v_\mu \tag{27}$$

into Equation (18). This sort of condition can be introduced after differential computations are completed. The parameter n can be interpreted as an index of refraction.

The resulting form of the metric is

$$f_{\mu\nu} = \eta_{\mu\nu} + \left(1 - \frac{1}{n^2}\right) \frac{v_\mu v_\nu}{c^2} \tag{28}$$

which is the geometrical optics metric of Miron and Kawaguchi [13].

The condition (27) thus effectively transforms the problem from the case of a theory of charged particles to the case of geometrical optics. This might show a useful correspondence between the two theories and allow computations in one theory to be applied to the other.

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