Deviations of Stationary Curves in the Bundle $Osc^{(2)}(M)$

R.Miron, V.Balan, P.C.Stavrinos and Gr.Tsagas

Abstract

Within the framework of higher-order geometry, the paper describes the osculator bundle of order two (the bundle of accelerations) and develops the equations of geodesics and of their deviations; important particular cases are evidentiated.

Mathematics Subject Classification: 53C22, 53B40, 58E15.
Key words: osculator bundle, N-linear connections, curvatures, torsions, stationary curves.

1 Introduction

Lagrange geometry of higher order was recently defined and studied by R.Miron and Gh.Atanasiu [15,16], and represents the geometry of Lagrange spaces $L^{(k)n} = (M, L)$, where $L : J_0^k(M) \to R$ is a regular Lagrangian of order $k \geq 1$ and $J_0^k(M)$ is the total space of the bundle of velocities of k-order.
The replacement of the bundle of k-jets $(J_0^k(M), \pi, M)$ by the osculator bundle $(Osc^k(M), \pi, M)$ infers profound changes, the last bundle having a deeper geometrical meaning and being more suitable for applications; any point of the total space is in fact a contact element of k-th order for curves in M, and has as representative an arc of curve on the base manifold M.
This concept is extremely useful in the geometrization of higher-order Lagrangians [13,15] and for prolongation of Riemannian, Finslerian and Lagrangian structures [1,13,16].
In the present paper, we develop the basics for the study of stationary curves for the case $k = 2$, which provides the so-called "bundle of accelerations" - a particular, but suggestive example of the general case.

2 The bundle of accelerations

Let M be a $\mathcal{C}^\infty$ real differentiable, n-dimensional manifold. Two curves

Balkan Journal of Geometry and Its Applications, Vol.2, No.1, 1997, pp. 51-60

$$\rho, \sigma : I \subset \mathbf{R} \to M, 0 \in I,$$

such that $\rho(0) = \sigma(0) = x_0 \in M$ have *a contact of order* 2 at x_0 if $\forall f \in \mathcal{F}(U)$ - with $x_0 \in U \subset M, U$ open set, we have

$$\begin{cases} (f \circ \rho)'(0) = (f \circ \sigma)'(0) \\ (f \circ \rho)''(0) = (f \circ \sigma)''(0) \end{cases}$$

The relation above is an equivalence one. Denoting it with $\sim$, and its classes like $[\rho]_{x_0} = \{\sigma \mid \rho \sim \sigma\}$ will form *the* 2*-osculator space at* x_0; then *the osculator space of* M will be the set $Osc^{(2)}(M) = \bigcup\limits_{x_0 \in M} Osc^{(2)}_{x_0}$.

Therefore, representing locally the curve like $\rho(t) = (x^1(t), \ldots, x^n(t))$, we remark that $\rho \sim \overline{\rho}$, where $\bar\rho$ is given by $\bar\rho^i(t) = x^i(0) + \frac{t}{1!}x^{i\prime}(0) + \frac{t^2}{2!}x^{i\prime\prime}(0)$. Thus, $[\rho]_{x_0}$ is uniquely determined by the triple of n-uples

$$(x^i(0), x^{i\prime}(0), x^{i\prime\prime}(0)),$$

and therefore a convenient local chart on $Osc^{(2)}(M)$ in the neighborhood of $[\rho]_{x_0}$ will be described by a mapping of the type

$$[\rho]_{x_0} \to (x^i, \overset{(1)}{y}{}^i, \overset{(2)}{y}{}^i) \equiv (x^i(0), x^{i\prime}(0), x^{i\prime\prime}(0)/2);$$

the last $3n$-ulple provides *the coordinates* of the point $[\rho]_{x_0} \in Osc^{(2)}(M)$.

The change of coordinates on $Osc^{(2)}(M)$ is given by

$$\text{(1)} \qquad \begin{cases} \tilde{x}^i = \tilde{x}^i(x) \\ \overset{(1)}{\tilde{y}}{}^i = \dfrac{\partial \tilde{x}^i}{\partial x^j}\overset{(1)}{y}{}^j \\ \overset{(2)}{\tilde{y}}{}^i = \dfrac{1}{2}\dfrac{\partial \overset{(1)}{\tilde{y}}{}^i}{\partial x^j}\overset{(1)}{y}{}^j + \dfrac{\partial \overset{(1)}{\tilde{y}}{}^i}{\partial \overset{(1)}{y}{}^j}\overset{(2)}{y}{}^j, det\left(\dfrac{\partial \tilde{x}^i}{\partial x^j}\right) \neq 0 \end{cases}$$

The mapping $\pi : Osc^{(2)}(M) \to M, \pi([\rho]_x) = x, \forall [\rho]_x \in Osc^{(2)}(M)$, is a surjective submersion, and thus, $\xi = (Osc^{(2)}(M), \pi, M)$ becomes a fiber bundle, called *the osculator bundle of second order*. We shall denote hereafter $E = Osc^{(2)}(M)$.

Considering on $\xi = (E, \pi, M)$ a non-linear connection [13,14], given by its coefficients $\{\underset{(1)}{N}{}^i_j, \underset{(2)}{N}{}^i_j\}$, $\mathcal{X}(E)$ is locally spanned by the adapted basis

$$\mathcal{B} = \left\{ \delta_i \overset{not}{=} \frac{\delta}{\delta x^i} \overset{not}{=} \frac{\delta}{\delta \overset{(0)}{y}{}^i}, \delta_i{}' \overset{not}{=} \frac{\delta}{\delta \overset{(1)}{y}{}^i}, \delta_i{}'' \overset{not}{=} \frac{\delta}{\delta \overset{(2)}{y}{}^i} \right\} \overset{not}{=} \{\delta_i, \delta_a, \delta_p\} = \{\delta_\alpha\},$$

with the indices running as follows: $i, j, k, \ldots = \overline{1, n}$; $a, b, c, \ldots = \overline{n+1, 2n}$; $p, q, r, \ldots = \overline{2n+1, 3n}$; $\alpha, \beta, \gamma, \ldots = \overline{1, 3n}$. While computing, the first three types of indices carry information on the tensor-type (h, v_1, v_2) (horizontal, velocity and acceleration-type, respectively) and are determined from the first cathegory by adding the corresponding multiple of n. The fields of the adapted basis are

$$\left\{\begin{array}{l} \dfrac{\delta}{\delta \overset{(0)}{y}{}^{i}} = \dfrac{\partial}{\partial x^{i}} - \underset{(1)}{N}{}_{i}^{j}\dfrac{\partial}{\partial \overset{(1)}{y}{}^{j}} - \underset{(2)}{N}{}_{i}^{j}\dfrac{\partial}{\partial \overset{(2)}{y}{}^{j}} \\ \dfrac{\delta}{\delta \overset{(1)}{y}{}^{i}} = \dfrac{\partial}{\partial \overset{(1)}{y}{}^{i}} - \underset{(1)}{N}{}_{i}^{j}\dfrac{\partial}{\partial \overset{(2)}{y}{}^{j}} \\ \dfrac{\delta}{\delta \overset{(2)}{y}{}^{i}} = \dfrac{\partial}{\partial \overset{(2)}{y}{}^{i}}. \end{array}\right. \tag{2}$$

The dual adapted basis relative to $\mathcal{B}$ will be denoted by

$$\mathcal{B}^{*} = \{\delta \overset{(0)}{y}{}^{i} \overset{not}{=} dx^{i}, \delta \overset{(1)}{y}{}^{i}, \delta \overset{(2)}{y}{}^{i}\} \overset{not}{=} \{dx^{i}, \delta y^{a}, \delta y^{p}\} = \{\delta^{\alpha}\}$$

given explicitly by

$$\left\{\begin{array}{l} \delta \overset{(0)}{y}{}^{i} \overset{not}{=} dx^{i} \\ \delta \overset{(1)}{y}{}^{i} = d\overset{(1)}{y}{}^{i} + \overset{(1)}{M}{}_{j}^{i} dx^{j} \\ \delta \overset{(2)}{y}{}^{i} = d\overset{(2)}{y}{}^{i} + \overset{(1)}{M}{}_{j}^{i} d\overset{(1)}{y}{}^{j} + \overset{(2)}{M}{}_{j}^{i} dx^{j}, \end{array}\right. \tag{3}$$

with $\overset{(1)}{M}{}_{j}^{i} = \underset{(1)}{N}{}_{j}^{i}; \overset{(2)}{M}{}_{j}^{i} = \underset{(2)}{N}{}_{j}^{i} + \underset{(1)}{N}{}_{s}^{i}\underset{(1)}{N}{}_{j}^{s}$.

These bases correspond to the direct-sum decompositions

$$T_{u}E = N_{u} \oplus N_{1_{u}} \oplus V_{2_{u}}, \qquad T_{u}^{*}E = N_{u}^{*} \oplus N_{1_{u}}^{*} \oplus V_{2_{u}}^{*}. \tag{4}$$

Then any vector field $X \in \mathcal{X}(E)$ can be locally decomposed like

$$X = \overset{(0)}{X}{}^{i}\delta_{i} + \overset{(1)}{X}{}^{a}\delta_{a} + \overset{(2)}{X}{}^{p}\delta_{p},$$

with the three right terms belonging to the distributions N, N_1, V_2 provided by (4); also, any 1-form $\omega \in \Lambda^{1}(E)$ can be decomposed as

$$\omega = \overset{(0)}{\omega}{}_{i}\delta^{i} + \overset{(1)}{\omega}{}_{a}\delta^{a} + \overset{(2)}{\omega}{}_{p}\delta^{p}.$$

Similarly, any tensor-field $T \in \mathcal{T}_{s}^{r}(E)$ can be split into components, which will be called *d-tensor fields*. As particular cases of d-tensor fields, the h-tensor fields form the algebra generated by $\{1, \delta_{i}, \delta^{i}\}$, and the v_1 and v_2-tensor fields, the algebras generated by $\{1, \delta_{a}, \delta^{a}\}$ and $\{1, \delta_{p}, \delta^{p}\}$ respectively.

3 N-linear connections

On the total space $E = Osc^{(2)}(M)$, there exist linear connections which are compatible with the decomposition (4). Their coefficients are in the adapted basis (2) geometrical objects which are much simpler than for a general connection. A linear connection D on E is said to be an *$N-$connection* if D preserves by parallelism the distributions of the decomposition (4).

A linear N-connection D is given locally by its coefficients

$$D = \{L^i_{jk}(x, \overset{(1)}{y}, \overset{(2)}{y}), \underset{(1)}{C}{}^i_{jk}(x, \overset{(1)}{y}, \overset{(2)}{y}), \underset{(2)}{C}{}^i_{jk}(x, \overset{(1)}{y}, \overset{(2)}{y})\}$$

determined by the relations

$$(5)\qquad \begin{cases} D_{\frac{\delta}{\delta x^j}} \dfrac{\delta}{\delta \overset{(\alpha)}{y}{}^i} = L^m_{ij} \dfrac{\delta}{\delta \overset{(\alpha)}{y}{}^m}, \alpha = \overline{0,2} \\ D_{\frac{\delta}{\delta \overset{(\beta)}{y}{}^j}} \dfrac{\delta}{\delta \overset{(\alpha)}{y}{}^i} = \underset{(\beta)}{C}{}^m_{ij} \dfrac{\delta}{\delta \overset{(\alpha)}{y}{}^m}, \alpha = \overline{0,2}, \beta = \overline{1,2} \end{cases}$$

or, briefly, $D_{\delta_\alpha}\delta_\beta = \underset{(\bar\beta)}{C}{}^\gamma_{\beta\alpha} \cdot \delta_\gamma, \bar\beta = \bar\gamma$, where we denoted $L^i_{jk} \overset{not}{=} \underset{(0)}{C}{}^i_{jk}$ and $\bar\beta = \left[\dfrac{\beta - 1}{n}\right]$ is the type of the index $\beta, \bar\beta = \overline{0,2}$. One can prove [13,14,15] that the coefficients $\underset{(\alpha)}{C}{}^i_{jk} \alpha = \overline{1,2}$ provide d-tensor fields; also, if M is paracompact, then there exist an non-linear connection on $E, N = \{\underset{(1)}{N}{}^i_j, \underset{(2)}{N}{}^i_j\}$, and hence the Berwald N-linear connection

$$B\Gamma(N) = \{L^i_{jk} = \delta'_k \underset{(1)}{N}{}^i_j, \underset{(1)}{C}{}^i_{jk} = 0, \underset{(2)}{C}{}^i_{jk} = 0\}.$$

Corresponding to the type of the leading d-vector field in (1), the N-linear connection produces the rules of covariant derivation (h-, v_1- and v_2-covariant derivations) for d-tensor fields, described by

$$\overset{(\bar\alpha)}{D}_X T = X^\alpha T^{i_1 \ldots i_r}_{j_1 \ldots j_s} \overset{(\bar\alpha)}{|}_\alpha \frac{\delta}{\delta x^{i_1}} \otimes \ldots \otimes \frac{\delta}{\delta \overset{(2)}{y}{}^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes \delta \overset{(2)}{y}{}^{j_s},$$

with

$$T^{i_1 \ldots i_r}_{j_1 \ldots j_s} \overset{(\bar\alpha)}{|}_\alpha = \delta_\alpha T^{i_1 \ldots i_r}_{j_1 \ldots j_s} + \underset{(\bar\alpha)}{C}{}^{i_1}_{h\alpha} T^{h i_2 \ldots i_r}_{j_1 \ldots j_s} + \ldots + \underset{(\bar\alpha)}{C}{}^{i_r}_{h\alpha} T^{i_1 \ldots j_{r-1} h}_{j_1 \ldots j_s} -$$
$$- \underset{(\bar\alpha)}{C}{}^h_{j_1\alpha} T^{i_1 \ldots i_r}_{h j_2 \ldots j_s} - \ldots - \underset{(\bar\alpha)}{C}{}^h_{j_s\alpha} T^{i_1 \ldots i_r}_{j_1 \ldots j_{s-1} h},$$

and $X = X^\alpha \delta_\alpha, \bar\alpha = \overline{0,2}$ fixed, $L \equiv \underset{(0)}{C}$. For $\bar\alpha = 0,1,2$, the h-, v_1- and v_2-derivations are respectively defined by the relations above.

Theorem. *Any N-linear connection is compatible with the following natural structures, whose matrices with respect to the adapted bases are given:*

a) The almost product structure $\mathcal{P}, [\mathcal{P}] = \begin{pmatrix} I_n & 0 & 0 \\ 0 & -I_n & 0 \\ 0 & 0 & -I_n \end{pmatrix}, \mathcal{P}^2 = Id,$

b) The n-almost contact structure $\mathcal{F}, [\mathcal{F}] = \begin{pmatrix} 0 & 0 & I_n \\ 0 & 0 & 0 \\ -I_n & 0 & 0 \end{pmatrix}, \mathcal{F}^3 + \mathcal{F} = 0,$

c) The 2-tangent structure $\mathcal{J}, [\mathcal{J}] = \begin{pmatrix} 0 & 0 & 0 \\ I_n & 0 & 0 \\ 0 & I_n & 0 \end{pmatrix}, \mathcal{J}^3 = 0.$

4 Torsions and curvatures

The torsion tensor $\mathcal{T} \in \mathcal{T}_2^1(E)$ of the N-linear connection D has the coefficients given by the relation

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = \underset{(\bar{\alpha}\bar{\beta})}{\overset{(\bar{\gamma})}{T}}{}^{\gamma}_{\alpha\beta} Y^\alpha X^\beta \delta_\gamma, \forall X,Y \in \mathcal{X}(E),$$
$$\bar{\alpha},\bar{\beta},\bar{\gamma} = \overline{0,2}.$$

Also, we can introduce the non-holonomy coefficients $\underset{(\bar{\alpha}\bar{\beta})}{\overset{(\bar{\gamma})}{B}}{}^{\gamma}_{\alpha\beta}$ given by

$$[\delta_\alpha, \delta_\beta] = \underset{(\bar{\alpha}\bar{\beta})}{\overset{(\bar{\gamma})}{B}}{}^{\gamma}_{\alpha\beta}\delta_\gamma, \bar{\alpha},\bar{\beta},\bar{\gamma} = \overline{0,2}$$

The detailed expressions and the inter-relations of the coefficients of non-holonomy and torsion can be found in [16].

The curvature tensor field $\mathcal{R} \in \mathcal{T}_3^1(E)$ has the coefficients given by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$
$$= \underset{(\bar{\alpha}\bar{\beta}\bar{\gamma})}{\overset{(\bar{\kappa})}{R}}{}^{\kappa}_{\alpha\beta\gamma} X^\gamma Y^\beta Z^\alpha \delta_\kappa, \forall X,Y,Z \in \mathcal{X}(E),$$

where

$$R(\delta_\alpha,\delta_\beta)\delta_\gamma = \underset{(\bar{\alpha}\bar{\beta}\bar{\gamma})}{\overset{(\bar{\kappa})}{R}}{}^{\kappa}_{\gamma\beta\alpha}\delta_\kappa,\ \bar{\alpha},\bar{\beta},\bar{\gamma} = \overline{0,2}.$$

5 Equations of stationary curves on $Osc^{(2)}(M)$

Let $E = Osc^{(2)}(M)$ be endowed with a 3-block *metric tensor field*

$$G = \overset{(0)}{g}_{ij} dx^i \otimes dx^j + \overset{(1)}{g}_{ij} \delta\overset{(1)}{y}{}^i \otimes \delta\overset{(1)}{y}{}^j + \overset{(2)}{g}_{ij} \delta\overset{(2)}{y}{}^i \otimes \delta\overset{(2)}{y}{}^j$$

where we set $G \equiv \{\overset{(0)}{g}_{ij}, \overset{(1)}{g}_{ij}, \overset{(2)}{g}_{ij}\} = \{G_{\alpha\beta}\}, \alpha,\beta = \overline{1,3n}$,

Let $c : I = [a,b] \subset \mathbf{R} \to E$ be a smooth curve in the osculator bundle, such that its image lies in a chart $(\tilde{U},\tilde{h})$ of E,

$$c(t) = (x^i(t), \overset{(1)}{y}{}^i(t), \overset{(2)}{y}{}^i(t)), t \in [a,b].$$

Let be *the Lagrangian* given by $L(t) = \left(G_{\alpha\beta}\mathcal{V}^\alpha\mathcal{V}^\beta\right)^{1/2}$ where $\mathcal{V} \equiv (\mathcal{V}^\alpha) = (\mathcal{V}^i, \mathcal{V}^a, \mathcal{V}^p)$ is *the covariant velocity* tensor-field, given by

$$\begin{cases} \mathcal{V}^i = \dfrac{dx^i}{dt} \\ \mathcal{V}^a = \dfrac{d\overset{(1)}{y}{}^a}{dt} + \overset{(1)}{M}{}^a_j(c(t))\dfrac{dx^j}{dt} \\ \mathcal{V}^p = \dfrac{d\overset{(2)}{y}{}^p}{dt} + \overset{(1)}{M}{}^p_a(c(t))\dfrac{d\overset{(1)}{y}{}^a}{dt} + \overset{(2)}{M}{}^p_j(c(t))\dfrac{dx^j}{dt} \end{cases}$$

Let $\mathcal{F} = \frac{\nabla\mathcal{V}}{dt} \equiv (\mathcal{F}^\alpha) = (\mathcal{F}^i, \mathcal{F}^a, \mathcal{F}^p)$ be the *force* moving the test-body along the curve c,

$$\mathcal{F}^\alpha = \frac{\nabla \mathcal{V}^\alpha}{dt} = \frac{d\mathcal{V}^\alpha}{dt} + \underset{(\bar\kappa)}{C}{}^{\alpha}_{\beta\kappa}\mathcal{V}^\beta \mathcal{V}^\kappa, \alpha = \overline{1,3n},$$

where κ involves summation after $\bar\kappa = \overline{0,2}$ and we denoted $L \equiv \underset{(0)}{C}$. Applying the variational principle to the Lagrangian L [2-3,5-9,12,20], we infer the following

Theorem. *The expressions of the force d-tensor fields with respect to the connection* $D \equiv \{\underset{(0)}{C}, \underset{(1)}{C}, \underset{(2)}{C}\}$ *are*

$$\begin{cases} \mathcal{F}^i = \mathcal{V}^i \dfrac{d(lnL)}{dt} + \mathcal{V}^\alpha \mathcal{V}^\beta \overset{(0)}{g}{}^{im}\left(\dfrac{1}{2}\overset{(\bar\alpha)}{g}{}_{\alpha\beta}\overset{(0)}{|}_m - \mathcal{T}_{m\alpha\beta}\right) - \mathcal{V}^j\mathcal{V}^\beta \overset{(0)}{g}{}^{im}\overset{(0)}{g}{}_{jm}\overset{\bar\beta}{|}_\beta \\ \mathcal{F}^a = \mathcal{V}^a \dfrac{d(lnL)}{dt} + \mathcal{V}^\alpha \mathcal{V}^\beta \overset{(1)}{g}{}^{ab}\left(\dfrac{1}{2}\overset{(\bar\alpha)}{g}{}_{\alpha\beta}\overset{(1)}{|}_b - \mathcal{T}_{b\alpha\beta}\right) - \mathcal{V}^c\mathcal{V}^\beta \overset{(1)}{g}{}^{ab}\overset{(1)}{g}{}_{cb}\overset{\bar\beta}{|}_\beta \\ \mathcal{F}^p = \mathcal{V}^p \dfrac{d(lnL)}{dt} + \mathcal{V}^\alpha \mathcal{V}^\beta \overset{(2)}{g}{}^{pq}\left(\dfrac{1}{2}\overset{(\bar\alpha)}{g}{}_{\alpha\beta}\overset{(2)}{|}_q - \mathcal{T}_{q\alpha\beta}\right) - \mathcal{V}^r\mathcal{V}^\beta \overset{(2)}{g}{}^{pq}\overset{(2)}{g}{}_{rq}\overset{\bar\beta}{|}_\beta \end{cases}$$

where $\bar\beta = \overline{0,2}$ *and* $\overset{(\bar\alpha)}{g}{}^{\alpha\beta}$ *is the reciprocal d-tensor field of* $\overset{(\bar\alpha)}{g}{}_{\alpha\beta}, \bar\alpha = \overline{0,2}$.

We have denoted $\mathcal{T}_{\alpha\beta\gamma} = \overset{(\bar\kappa)}{g}{}_{\beta\kappa}\mathcal{T}_{\alpha\cdot\gamma}^{\ \kappa}$, i.e. the lowering/raising of indices is performed using the metric G. For a stationary curve, we have $\mathcal{F} \equiv 0$, and the three relations above become *the equations of stationary curves* on the fibration ξ, relative to the connection D.

6 Equations of deviations of stationary curves

Let the N-connection D be metrical, i.e. the following relations take place

$$G_{\alpha\beta}\overset{\bar\gamma}{|}_\gamma = 0; \alpha, \beta, \gamma = \overline{1,3n};$$

but still having in general nonvanishing torsion. Let $c(t,u)$ be a family of stationary curves, having t as arc-parameter, and u the deviation parameter. Then let $\mathcal{Z} \equiv (\mathcal{Z}^\alpha) = (\mathcal{Z}^i, \mathcal{Z}^a, \mathcal{Z}^p)$ be the *deviation vector field*, given by

$$\begin{cases} \mathcal{Z}^i = \partial_u x^i \\ \mathcal{Z}^a = \partial_u \overset{(1)}{y}{}^a + \overset{(1)}{M}{}^a_j \partial_u x^i \\ \mathcal{Z}^p = \partial_u \overset{(2)}{y}{}^p + \overset{(1)}{M}{}^p_a \partial_u \overset{(1)}{y}{}^a + \overset{(2)}{M}{}^p_i \partial_u x^i \end{cases}$$

and let $\mathcal{V} \equiv (\mathcal{V}^\alpha) = (\mathcal{V}^i, \mathcal{V}^a, \mathcal{V}^p)$ be the overall velocity vector field

$$\begin{cases} \mathcal{V}^i = \partial_t x^i \\ \mathcal{V}^a = \partial_t \overset{(1)}{y}{}^a + \overset{(1)}{M}{}^a_j \partial_t x^i \\ \mathcal{V}^p = \partial_t \overset{(2)}{y}{}^p + \overset{(1)}{M}{}^p_a \partial_t \overset{(1)}{y}{}^a + \overset{(2)}{M}{}^p_i \partial_t x^i \end{cases}$$

where we assumed that $c : I_1 \times I_2 \subset \mathbf{R}^2 \to U \subset E = Osc^{(2)}(M)$ with U an open chart-domain for E, with

$$c(t,u) = (x^i(t,u), \overset{(1)}{y}{}^i(t,u), \overset{(2)}{y}{}^i(t,u)) \in U, \forall (t,u) \in I_1 \times I_2.$$

Considering the partial covariant derivatives

$$\begin{cases} \delta_t \mathcal{W}^\alpha = \partial_t \mathcal{W}^\alpha + \underset{(\bar\gamma)}{C}{}^\alpha_{\beta\gamma} \mathcal{W}^\beta \mathcal{V}^\gamma \\ \delta_u \mathcal{W}^\alpha = \partial_u \mathcal{W}^\alpha + \underset{(\bar\gamma)}{C}{}^\alpha_{\beta\gamma} \mathcal{W}^\beta \mathcal{Z}^\gamma \end{cases}$$

for any overall vector field on $Im(c)$, $\mathcal{W} \equiv (\mathcal{W}^\alpha) = (\mathcal{W}^i, \mathcal{W}^a, \mathcal{W}^p)$, we obtain, using the second variation of the Lagrangian L the following

Theorem. *The equations of deviations of the sheaf of curves (which determine the deviation force $\mathcal{Z}$), are*

$$\begin{cases} \delta_t^2 \mathcal{Z}^i + \delta_t \mathcal{T}^i = \rho^i_j \mathcal{V}^j + \delta_u \mathcal{F}^i \\ \delta_t^2 \mathcal{Z}^a + \delta_t \mathcal{T}^a = \rho^a_b \mathcal{V}^b + \delta_u \mathcal{F}^a \\ \delta_t^2 \mathcal{Z}^p + \delta_t \mathcal{T}^p = \rho^p_q \mathcal{V}^q + \delta_u \mathcal{F}^p, \end{cases}$$

where we denoted

$$\mathcal{T}^i = \underset{\bar\alpha\bar\beta}{\overset{(0)}{T}}{}^i_{\alpha\beta} \mathcal{V}^\alpha \mathcal{Z}^\beta, \mathcal{T}^a = \underset{\bar\alpha\bar\beta}{\overset{(1)}{T}}{}^a_{\alpha\beta} \mathcal{V}^\alpha \mathcal{Z}^\beta, \mathcal{T}^p = \underset{\bar\alpha\bar\beta}{\overset{(2)}{T}}{}^p_{\alpha\beta} \mathcal{V}^\alpha \mathcal{Z}^\beta,$$

and $\rho^\alpha_\beta = R^\alpha_{\beta\gamma\kappa} \mathcal{Z}^\gamma \mathcal{V}^\kappa$.

We remark that for stationary curves, the last term of each equation identically vanishes.

7 Applications. Particular cases

Definitions. Let $c : I = [a,b] \subset \mathbf{R} \to Osc^{(2)}(M)$ be a curve, $D = B\Gamma(N)$, and N the canonical non-linear connection induced by a Riemannian metric of M in $Osc^{(2)}(M)$[13].

a) The curve c is called *h-curve* (horizontal curve) if $\dfrac{\delta \overset{(1)}{y}(t)}{dt} = \dfrac{\delta \overset{(2)}{y}(t)}{dt} = 0.$

b) The curve c is called *v-curve* (vertical curve) if

$$\frac{dx(t)}{dt} = 0 \; (x(t) = x_0 \in U, \forall t \in I).$$

c) The curve c is called *v_2-curve* if

$$\frac{dx(t)}{dt} = \frac{\delta \overset{(1)}{y}(t)}{dt} = 0.$$

If the curves in a), b), c) satisfy also the condition $D_{\dot c}\dot c = 0$, $(\mathcal{F}^\alpha \equiv 0)$, then they are called *h-path, v-path* and *v_2-path*, respectively.

In terms of velocity and force, these notions can be condensed as follows:

a) h-curve: $\mathcal{V}^a = \mathcal{V}^p = 0$; h-path: $\mathcal{V}^a = \mathcal{V}^p = \mathcal{F}^i = 0$;

b) v-curve: $\mathcal{V}^i = 0$; v-path: $\mathcal{V}^i = \mathcal{F}^a = \mathcal{F}^p = 0$;

c) v_2-curve: $\mathcal{V}^i = \mathcal{V}^a = 0$; v_2-path: $\mathcal{V}^i = \mathcal{V}^a = \mathcal{F}^p = 0$;
We can easily remark that all the three kinds of paths are particular cases of stationary curves.

Also, their equations of deviations are considerably simplified, as can be easily seen from the following example

Proposition. *If c is an h-path and D is a metrical N-linear connection, then the equations of deviations become*

$$\begin{cases} \delta_t^2 \mathcal{Z}^i + \delta_t \mathcal{T}^i = \rho^i_j \mathcal{V}^j \\ \delta_t^2 \mathcal{Z}^a + \delta_t \mathcal{T}^a = 0 \\ \delta_t^2 \mathcal{Z}^p + \delta_t \mathcal{T}^p = 0. \end{cases}$$

It must be noticed, that if dropping the v_2 terms in the previous equations one obtains the $Osc^{(1)}(M) \equiv TM$ case, already investigated [9,21]; if dropping all the vertical (v_1 and v_2) terms, the equations of geodesics and of deviations of geodesics for the classical Riemannian case are obtained [10,12]. Hence, the present framework extends both the classical framework and the tangent bundle case.

8 Conclusions

The equations of stationary curves and of their deviations play a major role in finding the curves that extremize the first and second variations of the length Lagrangian [2,5,12]. In the present paper, these equations were obtained for the bundle of accelerations, a particular case of the k-osculator bundle investigated within the framework of higher-order geometry; on the other side, the obtained equations generalize the Riemannian and tangent bundle cases. Also, the equations of deviations of paths have a considerably simpler form, and hence, are easier to integrate.

Acknowledgements. A version of this paper was presented at the First Conference of Balkan Society of Geometers, Politehnica University of Bucharest, September 23-27, 1996.

References

[1] M.Anastasiei, I.Bucătaru, *A Notable Submersion in Higher Order Geometry*, Workshop, 1994, Thessaloniki; in Ed.Gr.Tsagas, Global Analysis, Differential Geometry, Lie Algebras, BSG Proceedings I, Geometry Balkan Press, 1997, 1-10.

[2] G.S.Asanov, *Finsler Geometry, Relativity and Gauge Theories*, D.Reidel, 1985.

[3] G.S.Asanov, S.F.Ponomarenko, *Finsler Fibrations over Space-Time. Associated Gauge Fields and Connections*, Nauka Eds., Chishinew, 1989, (in Russian).

[4] G.S.Asanov, P.C.Stavrinos, *Finslerian Deviations of Geodesics over Tangent Bundle*, Rep. Math. Phys.,30, (1991), 63-69.

[5] V.Balan, *Deviation of Geodesics in Vector Bundles*, Proc. of the Nat. Conf. of Geometry and Topology, sept.1993, Cluj-Napoca, 6-13.

[6] V.Balan, *On Gedesics and Deviations of Geodesics in Fibered Finslerian Approach*, Mathematical Reports, Romanian Academy Eds., 4.6, 4(1994), 415-422.

[7] V.Balan, P.C.Stavrinos, *Stationary Curves in Fibered Finslerian Approach*, Proc. of the Int. Conf. of Diff. Geom., 1994, Braşov.

[8] V.Balan, K.Trencevski, *Deviations of Stationary Curves in the Extended Fibered Finslerian Approach. Characterization and Analytical Solutions for E.D.G.*, Rep.on Math.Phys., Torun, 1995.

[9] V.Balan, K.Trencevski, *Equations of Deviations of Geodesics. Special Cases in the Fibered Finslerian Case*, Tensor N.S., vol. 56, (1995), 184-192.

[10] S.Ianuş, *Differential Geometry with Applications to General Relativity*, Rom. Academy Ed., Bucharest, 1983, (in Romanian).

[11] M.Matsumoto, *Foundations of Finsler Geometry and Special Finsler Spaces*, Kaisheisha Press, Kyoto, 1986.

[12] J.Milnor, *Morse Theory*, Princeton Univ. Press, Princeton, 1963.

[13] R.Miron, *The Geometry of Higher Order Lagrange Spaces. Applications in Mathematics and Physics*, FTPH 82, Kluwer Acad. Publ. , 1996.

[14] R.Miron, M.Anastasiei, *The Geometry of Vector Bundles. Theory and Applications*, FTPH 59, Kluwer, Dordrecht 1994.

[15] R.Miron, Gh.Atanasiu, *Compendium sur les espaces Lagrange d'ordre superieur*, Sem. Mec. 40, Timişoara, 1994.

[16] R.Miron, Gh.Atanasiu, *Differential Geometry of k-Osculator Bundles*, Rev. Roum.Math.Pures et Appl., vol. 41, 3-4(1996), 205-236.

[17] Miron R., Gh.Atanasiu, *Lagrange Geometry of Second Order*, Math. Comput. Modelling, vol 20, 4-5(1994), 41-56.

[18] D.Opriş, *Geometrical Aspects of the Covariant Dynamics of Higher Order*, Sem.Mec.49, Timişoara, 1996.

[19] G.Sardanashvili, O.Zacharov, *Gauge Gravitation Theory*, World Scientific, 1992.

[20] M.F.Shirokov, *On One New Effect of the Einsteinian Theory of Gravitation, General Relativity and Gravitation*, v.4, 2(1973), 131-136.

[21] P.C.Stavrinos, H.Kawaguchi, *Deviation of Geodesics in the Gravitational Field of Finslerian Space-Time*, Memoirs of Shonan Inst. of Technology, 27, 1(1993), 35-40.

[22] K.Yano, S.Ishihara, *Tangent and Cotangent Bundles*, Marcel Dekker, N.Y., 1973.

Faculty of Mathematics, University of Iassy, Romania,
email: rmiron@uaic.ro

Dept. Mathematics I, Politehnica University of Bucharest, Romania,
email: vbalan@mathem.pub.ro

Department of Mathematics, University of Athens, Greece,
email: pstavrin@atlas.uoa.gr

Department of Mathematics, University of Thessaloniki, Greece,
email: tsagas@vergina.auth.gr