Some Theorems on Austere Submanifolds

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Abstract

In §1 there are described the introductory notions of the geometry of submanifolds, as they are required in the paper. There are introduced the austere submanifold and are given same examples. The shape discriminant, depending upon the principal curvatures of the submanifold, is introduced in §2, repeating the idea from [8]. Here are proved completely all its properties necessary to prepare the main results, which are stated in §3, where the geometry of the austere submanifolds is presented related on some local techniques involving the principal curvatures.

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1 Preliminaries

Let M be an n-dimensional Riemannian manifold isometrically imersed in an almost complex Hermitian $n+m$-dimensional manifold $\bar{M}$ with almost complex structure J and almost Hermitian metric g. For any vector X tangent to M we write

$$JX = PX + FX, \tag{1.1}$$

where PX and FX are the tangent and the normal components of JX, respectively. For any point $p \in M$ we have the Gauss and Weingarten formulas

$$(\bar{\nabla}_X Y)_p = (\nabla_X Y)_p + h_p(X,Y), \ \ \forall X, Y \in TM \tag{1.2}$$

$$(\bar{\nabla}_X \xi)_p = -(A_\xi X)_p + (D_X \xi)_p, \ \ \forall X \in TM, \forall \xi \in T^\perp M, \tag{1.3}$$

where $\bar{\nabla}$ and ∇ are the Levi-Civita connections in the Riemannian manifolds $\bar{M}$ and M, h is the second fundamental form of the submanifold M of $\bar{M}$, A_ξ the shape operator for the normal field ξ, and D the normal connection in $T^\perp M$.

We will denote $(A_\xi)_p$ not only the shape operator in p, but also the matrix associated to the linear mapping from T_pM to T_pM.

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Definition 1.1 The principal curvatures of the submanifold M associated to the field $\xi \in T^{\perp}M$ are the roots of the characteristic polynomial

$$\det((A_{\xi})_p - \rho(p)I_n) = 0. \tag{1.4}$$

Notation 1.1. Let $\xi_1, ..., \xi_m$ be an orthonormal base for a neighbourhood U of the point $p \in M$, where $\xi_i (i = \overline{1,m})$ are sections in the normal bundle $T^{\perp}M$. For these normal fields $\xi_1, ...\xi_m$ we have the principal curvatures, respectively:

$$\begin{array}{c} \rho_1^1(p), ..., \rho_n^1(p) \\ \dots\dots\dots\dots \\ \rho_1^m(p), ..., \rho_n^m(p) \end{array}$$

Remark 1.1. Some properties of the principal curvatures are presented in [1], pg.67.
Definition 1.2. Let M be a submanifold of the Riemannian manifold $(\bar{M}, g)$. M is called *minimal* if trace $H = 0$. M is called *austere* if for every $\xi \in T^{\perp}M$ the family of the eigenvalues of A_{ξ} - repeated as many times as it is their multiplicity - is invariable by its multiplication with (-1).
Remark 1.2. Obviously, an austere submanifold is minimal.
Example 1.1.. The trivial case of austere submanifolds are the totally geodesic submanifolds. For example, for any hyperplane $(H) \subseteq \mathbf{E}^n = (\mathbf{R}^n, \text{can})$, all the eigenvalues are 0, and (H) is austere since (H) is totally geodesic.
Example 1.2. The Clifford torus is an austere submanifold. Let $f : \mathbf{R}^2 \to \mathbf{R}^4$ given by

$$f(\theta, \varphi) = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi), \ \ (\theta, \varphi) \in \mathbf{R}^2.$$

The mapping f is an imersion of $\mathbf{R}^2$ into the unit sphere $S^3(1) \subset \mathbf{R}^4$, where image $f(\mathbf{R}^2)$ is the flat torus. The vectors

$$e_1 = (-\sin\theta, \cos\theta, 0, 0)$$

$$e_2 = (0, 0, -\sin\varphi, \cos\varphi)$$

form an orthonormal basis of the tangent space $T_{f(\theta,\varphi)}$ *Imf*, and the normal vectors

$$\eta_1 = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi)$$

$$\eta_2 = \frac{1}{\sqrt{2}}(-\cos\theta, -\sin\theta, \cos\varphi, \sin\varphi)$$

form an orthonormal basis of the normal space. The matrices A_{η_1} and A_{η_2} with respect to the basis $\{e_1, e_2\}$ are

$$A_{\eta_1} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix},$$

$$A_{\eta_2} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

As we can see, $T_{f(\theta,\varphi)}Imf \oplus \mathbf{R}\eta_2 = T_{f(\theta,\varphi)}S^3(1)$ and $T_{f(\theta,\varphi)}S^3(1) \oplus \mathbf{R}\eta_1 = T_{f(\theta,\varphi)}\mathbf{E}^4$. The eigenvalues $\frac{1}{\sqrt{2}}$ and $-\frac{1}{\sqrt{2}}$ show us that the Clifford torus is an austere submanifold of the unit sphere S^3.

2 The shape discriminant

We follow now the construction from [8]. We begin with the following
Lemma 2.1. *Let* $g \in \mathbf{R}[X]$ *be the polynomial*

$$g(X) = a_0X^n + a_1X^{n-1} + a_2X^{n-2} + ... + a_n$$

with $a_0 \neq 0$ *and* $n > 1$. *If all the roots of* g *are real, then*

$$\Delta(g,n) = \frac{2(n-1)}{n}a_1^2 - 4a_0a_2 \geq 0.$$

Proof. If g has all the roots real, then g' has also all the roots real. After $(n-2)$ steps, $g^{(n-2)}$ has all the roots real. Actually, $g^{(n-2)}$ is a polynomial of degree 2 and the condition $x_1, x_2 \in \mathbf{R}$ means $\Delta(g,n) \geq 0$.
Remark 2.1. The converse of this lemma is not true.
Lemma 2.2. *Let* $x_i \in \mathbf{R}$, $i = \overline{1,n}$. *If* $\sum_{i=1}^{n} x_i = 0$, *then we have*

$$(2.1.) \qquad (\max_{i=\overline{1,n}} x_i - \min_{i=\overline{1,n}} x_i)^2 \leq -4 \sum_{1\leq i<j\leq n} x_ix_j.$$

Proof. If we suppose $x_1 \leq x_2 \leq ... \leq x_n$ we have:

$$(\max_{i=\overline{1,n}} x_i - \min_{i=\overline{1,n}} x_i)^2 = (x_n - x_1)^2.$$

The condition $\sum_{i=1}^{n} x_i = 0$ gives

$$\sum_{i=1}^{n} x_i^2 + 2\sum_{1\leq i<j\leq n} x_ix_j = 0,$$

and to prove the claim we have to show that

$$(x_n - x_1)^2 \leq 2\sum_{i=1}^{n} x_i^2$$

or $0 \leq (x_1 + x_n)^2 + 2(x_2^2 + x_3^2 + ... + x_{n-1}^2)$, q.e.d.
Definition 2.1. Let M be a submanifold of the Riemannian manifold ($\bar{M}$,g). The mapping

$$(2.2) \qquad \delta_\xi : M \to R, \quad \delta_\xi(p) = \Delta(det((A_\xi)_p - \rho(p)I_n), n)$$

is called *the shape discriminant* of the submanifold M associated to the normal vector field $\xi \in T^{\perp}M$.

Remark 2.2. Because the matrix $(A_{\xi})_p$ is symmetric and its eigenvalues are all real, the lemma 2.1 gives

$$\delta_{\xi}(p) \geq 0, \ \ \forall p \in M, \forall \xi \in T^{\perp}M. \tag{2.3}$$

Notation 2.1. Let $\xi_1, ...\xi_m$ be m normal sections in the normal bundle $T^{\perp}M$. We denote

$$\delta_i = \delta_{\xi_i} \ ; \ A_i = A_{\xi_i}.$$

For every section ξ_i we have the set of eigenvalues $\rho_1^i, ..., \rho_n^i$. Let $L_i : M \to \mathbf{R}$ be the mapping which measures the length of the interval where lies the eigenvalues

$$L_i(p) = \max_{j=\overline{1,n}} \rho_j^i(p) - \min_{j=\overline{1,n}} \rho_j^i(p). \tag{2.4}$$

Remark 2.3. With the notation

$$||A_i||^2 = (\rho_1^i)^2 + ... + (\rho_n^i)^2$$

the shape discriminant is

$$\delta_i = 2||A_i||^2 - \frac{2}{n}(trA_i)^2. \tag{2.5}$$

Theorem 2.1. *With the notations above, for any $p \in M$ and any basis $\xi_1, ..., \xi_m$ in $T^{\perp}M$, we have the double inequality*

$$\sqrt{\binom{n}{2}^{-1} \delta_i(p)} \leq L_i(p) \leq \sqrt{\delta_i(p)}, \quad \forall i = \overline{1,n}. \tag{2.6}$$

Proof. Let $f \in \mathbf{R}[X]$ the polynomial corresponding to the characteristic equation of the shape operator A_i in $p \in M$,

$$f(X) = a_0 X^n + a_1 X^{n-1} + ... + a_n.$$

We have seen that $f^{(k)}(X) = 0$ is an equation with all the roots real,
$\forall k = \overline{1, n-1}$, and for every $k = \overline{1, n-1}$, we have

$$\min_{i=\overline{1,n}} x_i \leq \min_{i=\overline{1,n}} x_i^{(k)} \leq \max_{j=\overline{1,n}} x_j^{(k)} \leq \max_{j=\overline{1,n}} x_j.$$

For $k = n-2$, we find

$$|x_1^{n-2)} - x_2^{(n-2)}| = \frac{1}{|a_0|\sqrt{\frac{n(n-1)}{2}}}\sqrt{\Delta}_i,$$

and $|a_0| = 1$ proves the first inequality. To prove the second one, we use the lemma 2.2. We denote $y_i = x_i + \frac{a_1}{na_0}$, and $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{a_1}{na_0} = 0$, from the Viète's formula.

For y_i chosen as above, we can apply the lemma 2.2:

$$(\max_{i=\overline{1,n}} y_i - \min_{i=\overline{1,n}} y_i)^2 \leq -4 \sum_{1\leq i<j\leq n} y_i y_j =$$

$$= -4\left[\frac{a_2}{a_0} + \frac{a_1}{na_1}(n-1)\sum_{i=1}^{n} x_i + \frac{a_1^2}{n^2a_0^2}\cdot\frac{n(n-1)}{2}\right] = \frac{1}{a_0^2}\Delta_i.$$

3 Austere Submanifolds

Remark 3.1. Obviously, not any minimal submanifold is austere. An austere submanifold with $L_i = 0$, $\forall i = \overline{1,m}$, for every base $\xi_1, ..., \xi_m$ in $T^{\perp}M$ is totally geodesic. For the submanifolds of a Riemannian manifold we have the following inclusions:

totally geodesic $\subset$ austere $\subset$ minimal.

In [3], with the notations established as above, is presented

Theorem 3.1. *If M is a proper slant submanifold of a Kaehlerian manifold $(\bar{M}, J, g)$, and if $\nabla F = 0$, then M is austere.*

Remark 3.2. For an austere submanifold we have

$$L_i(p) = 2 \max_{i=\overline{1,n}} \rho_j^i(p) = 2\rho_{\max}^i(p), \tag{3.1}$$

and for every point p of an austere submanifold $M \subset \bar{M}$ we have

$$\sqrt{\binom{n}{2}^{-1}\delta_i(p)} \leq 2\rho_{\max}^i(p) \leq \sqrt{\delta_i(p)}, \quad \forall i = \overline{1,p}, \tag{3.2}$$

(see Theorem 2.1).

Theorem 3.2. *Let M be a n-dimensional austere submanifold in the n+m dimensional Riemannian manifold $(\bar{M}, g)$. If for any orthonormal basis $\xi_1, ..., \xi_m$ in $T^{\perp}M$ we have*

$$\delta_i(p) = 0, \quad \forall i = \overline{1,m},\ \forall p \in M,$$

then M is totally geodesic.

Proof. The relation (2.6) means $L_i(p) = 0$, $\forall i = \overline{1,m}$ and $\forall p \in M$, and from (3.2) we have $\rho_{max}^i(p) = 0$, $\forall p \in M$, q.e.d. Following the theorem 3.1 we find

Corollary 3.1. *Let M be a n-dimensional proper slant submanifold of the n+m-dimensional Kaehlerian manifold $(\bar{M}, J, g)$. If $\nabla F = 0$ and for every basis $\xi_1, ..., \xi_m$ in $T^{\perp}M$ we have $\delta_i \equiv 0$, $\forall i = \overline{1,m}$, then M is totally geodesic in $\bar{M}$.*

Proof. The theorem 3.1 shows that M is an austere submanifold, and the theorem 3.2 yields that M is totally geodesic.

Definition 3.1. Let $\xi_1, ..., \xi_m$ be as above an orthonormal basis in $T^{\perp}M$, and the m shape discriminants, respectively $\delta_1, ...\delta_m$. We consider the mapping

$$\delta : M \to \mathbf{R}^m, \quad \delta(p) = (\delta_1(p), ..., \delta_m(p)).$$

We call δ the *principal discriminant* of the submanifold M, corresponding to the basis $\xi_1, ..., \xi_m$.

Remark 3.3. If M is a totally geodesic submanifold, then $\delta = 0_{\mathbf{R}^p}$. The principal discriminant measures how far from the property of being totally geodesic is the submanifold $M \subset \bar{M}$.
Theorem 3.3. *Let M be a submanifold of the Riemannian manifold $\bar{M}$. If there exist $i \in \{1, ..., m\}$ and $p \in M$ such that $\delta_i(p) > 0$, then M cannot be totally geodesic in $\bar{M}$.*
Remark 3.4. The condition $||\delta|| > 0$ is an obstruction for the submanifold M to be totally geodesic.

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