Caracterizations of the Nonlinear Connection in the Higher Order Geometry

Ioan Bucataru

Abstract

In the paper [8] a nonlinear connection on k-osculator bundle is characterised by a system of functions defined on each domain of local chart, which verify a special formula.

Starting with this result, to a nonlinear connection on the k-osculator bundle was associated a special map, called connection map [2].

The aim of this paper is to present the notion of connection map, independently of the choice of a nonlinear connection. The kernel of the connection map is a nonlinear connection. In this way we obtain a characterization for the nonlinear connection using only the k-tangent structure on the k-osculator bundle.

In the last part of this paper we present the notion of horizontal lift, independently of the choice of a nonlinear connection and connection map. Using this map we obtain a characterization for a connection map and a nonlinear connection. The nonlinear connection appears as the image of the horizontal lift. The connection map is defined using the inverse map of the vertical lift.

AMS subject classification: 53C60
Key words: nonlinear connection, connection map, horizontal lift

1 Introduction

Let M be a real, smooth manifold of dimension n and $(Osc^k M, \pi^k, M)$ its k-osculator bundle. Then $Osc^k M$ is a real, smooth manifold of dimension $n(k+1)$. We set $E = Osc^k M$.

Let (x^i) be the local coordinates in a local chart $U \subset M$. The local coordinates on $(\pi^k)^{-1}(U) \subset Osc^k M$ will be denoted by $(x^i, y^{(1)i}, \ldots, y^{(k)i})$.

A change of coordinates $(x, y^{(1)}, ..., y^{(k)}) \to (\widetilde{x}, \widetilde{y}^{(1)}, \ldots, \widetilde{y}^{(k)})$ on E is given by:

Balkan Journal of Geometry and Its Applications, Vol.2, No.2, 1997, pp. 13-22

$$
(1.1)\quad \begin{cases}
\widetilde{x}^i = \widetilde{x}^i(x^1, x^2, \ldots, x^n);\ \text{rang} \left\| \dfrac{\partial \widetilde{x}^i}{\partial x^j} \right\| = n \\
\widetilde{y}^{(1)i} = \dfrac{\partial \widetilde{x}^i}{\partial x^j} y^{(1)j} \\
2\widetilde{y}^{(2)i} = \dfrac{\partial \widetilde{y}^{(1)i}}{\partial x^j} y^{(1)j} + 2\dfrac{\partial \widetilde{y}^{(1)i}}{\partial y^{(1)j}} y^{(2)j} \\
\ldots \\
k\widetilde{y}^{(k)i} = \dfrac{\partial \widetilde{y}^{(k-1)i}}{\partial x^j} y^{(1)j} + 2\dfrac{\partial \widetilde{y}^{(k-1)i}}{\partial y^{(1)j}} y^{(2)j} + \ldots + k\dfrac{\partial \widetilde{y}^{(k-1)i}}{\partial y^{(k-1)j}} y^{(k)j}.
\end{cases}
$$

Under a change of coordinates (1.1) on E, for each $u \in E$, the natural basis $\left\{ \dfrac{\partial}{\partial x^i} \mid_u, \dfrac{\partial}{\partial y^{(1)i}} \mid_u, \ldots, \dfrac{\partial}{\partial y^{(k)i}} \mid_u \right\}$ changes as follows

$$
(1.2)\quad \begin{cases}
\dfrac{\partial}{\partial \widetilde{x}^i} \mid_u = \dfrac{\partial x^j}{\partial \widetilde{x}^i}(u) \dfrac{\partial}{\partial x^j} \mid_u + \dfrac{\partial y^{(1)j}}{\partial \widetilde{x}^i}(u) \dfrac{\partial}{\partial y^{(1)j}} \mid_u + \ldots + \dfrac{\partial y^{(k)j}}{\partial \widetilde{x}^i}(u) \dfrac{\partial}{\partial y^{(k)j}} \mid_u \\
\dfrac{\partial}{\partial \widetilde{y}^{(1)i}} \mid_u = \dfrac{\partial y^{(1)j}}{\partial \widetilde{y}^{(1)i}}(u) \dfrac{\partial}{\partial y^{(1)j}} \mid_u + \ldots + \dfrac{\partial y^{(k)j}}{\partial \widetilde{y}^{(1)i}}(u) \dfrac{\partial}{\partial y^{(k)j}} \mid_u \\
\ldots \\
\dfrac{\partial}{\partial \widetilde{y}^{(k)i}} \mid_u = \dfrac{\partial y^{(k)j}}{\partial \widetilde{y}^{(k)i}}(u) \dfrac{\partial}{\partial y^{(k)j}} \mid_u .
\end{cases}
$$

For $\alpha \in \{1, 2, \ldots, k-1\}$ we denote $\pi^k_\alpha : E \to Osc^\alpha M$ the canonical submersion which is expressed in the local chart as follows

$$
\pi^k_\alpha : (x, y^{(1)}, \ldots, y^{(k)}) \mapsto (x, y^{(1)}, \ldots, y^{(\alpha)}).
$$

As $(\pi^k)_* : (TE, \tau_E, E) \to (TM, \tau, M)$ is a π^k epimorphism of vector bundles, it results that its kernel is a vector subbundle of the bundle (TE, τ_E, E). This will be denoted by V_1E and will be called the *vertical subbundle* of the TE. The fibres of V_1E determine an integrable distribution $V_1 : u \in E \mapsto V_1(u) \subset T_uE$ which has the dimension kn, called *vertical distribution.*

In the same manner, for each submersion $\pi^k_\alpha : E \to Osc^\alpha M$ we obtain a vector subbundle of TE denoted by $V_{\alpha+1}E = Ker(\pi^k_\alpha)_*$. The fibres of $V_{\alpha+1}E$ determine an integrable distribution $V_{\alpha+1} : u \in E \mapsto V_{\alpha+1}(u) \subset T_uE$, of dimension $(k-\alpha)n$.

In this way we obtain k-vertical distributions $V_1, V_2, \ldots, V_k$ of dimensions $kn, (k-1)n, \ldots, n$ respectively, such that $\forall u \in E, \quad V_k(u) \subset V_{k-1}(u) \subset \cdots \subset V_1(u)$.

For each $u \in E$ we consider the linear map $J_u : T_uE \to T_uE$ defined on natural basis by

$$
(1.3)\quad J_u\left(\frac{\partial}{\partial x^i} \mid_u\right) = \frac{\partial}{\partial y^{(1)i}} \mid_u, \ldots, J_u\left(\frac{\partial}{\partial y^{(k-1)i}} \mid_u\right) = \frac{\partial}{\partial y^{(k)i}} \mid_u, J_u\left(\frac{\partial}{\partial y^{(k)i}} \mid_u\right) = 0
$$

and extended by linearity.

Proposition 1.1.

1. For each $u \in E$ the map J_u is well defined.

2. For $\alpha \in \{1, 2, \ldots, k-1\}$ we have

$$(1.4)\qquad \begin{cases} J_u^{k-\alpha} V_{k-\alpha}(u) = V_\alpha(u), \quad \forall u \in E \\ J_u^{k+1} = 0 \end{cases} .$$

The maps $J_u, \quad u \in E$ determine a morphism of vector bundles $J : (TE, \tau_E, E) \to (TE, \tau_E, E)$ and an $\mathcal{F}$(E) linear map $J : \chi(E) \to \chi(E)$. The map J is called the *k–tangent structure* of the *k–osculator bundle.*

2 The nonlinear connection associated to a connection map

We denote by $(TM^{(k)}, \tau^{(k)}, M)$ the Whitney sum of the tangent bundle (TM, τ, M) on itself of k times.

Definition 2.1. We call *connection map* on the k-osculator bundle E a π^k morphism of vector bundles

$$K = (\overset{(1)}{K}, \overset{(2)}{K}, \dots, \overset{(k)}{K}) : (TE, \tau_E, E) \to (TM^{(k)}, \tau^{(k)}, M)$$

which satisfies

$$(2.1)\qquad \begin{cases} \overset{(k)}{K} \circ J^\alpha = \overset{(k-\alpha)}{K}, \quad \forall \alpha = \overline{1, k-1} \\ \overset{(k)}{K} \circ J^k = (\pi^k)_* \end{cases} .$$

Proposition 2.1. For a connection map $K = (\overset{(1)}{K}, \overset{(2)}{K}, \dots, \overset{(k)}{K})$ we have the following relations

$$(2.2)\qquad \begin{cases} \overset{(\alpha)}{K} = \overset{(\alpha+1)}{K} \circ J, \quad \forall \alpha = \overline{1, k-1} \\ (\pi^k)_* = \overset{(\alpha)}{K} \circ J^\alpha \quad \forall \alpha = \overline{1, k} \end{cases} .$$

Proof.

$$\overset{(\alpha)}{K} = \overset{(k)}{K} \circ J^{k-\alpha} = \overset{(k)}{K} \circ J^{k-\alpha-1} \circ J = \overset{(\alpha+1)}{K} \circ J$$

$$(\pi^k)_* = \overset{(k)}{K} \circ J^k = \overset{(k)}{K} \circ J^{k-1} \circ J = \overset{(1)}{K} \circ J$$

Remark 2.1. The kernel of the connection map $N = \mathrm{Ker}K$ is a vector subbundle of the bundle (TE, τ_E, E). Its fibres determine a distribution $N : u \in E \mapsto N(u) \subset E_u$ of dimension n, called the *horizontal distribution* associated to the connection map K.

Next, we shall prove that a connection map determines a nonlinear connection on k–osculator bundle.

For each $u \in E$, the map $K_u : T_u E \to \underbrace{T_{\pi^k(u)} M \times \dots \times T_{\pi^k(u)} M}_{k\ times}$ is linear. We denote by $\underset{(1)}{M}{}^i_j(u), \dots, \underset{(k)}{M}{}^i_j(u)$ the coordinate functions, defined on every domain of local charts, for the vectors $\overset{(1)}{K}_u \frac{\partial}{\partial x^j} |_u, \ldots,$ and $\overset{(k)}{K}_u \frac{\partial}{\partial x^j} |_u$ respectively, in the natural basis $\frac{\partial}{\partial x^i} |_{\pi^k(u)}$ of $T_{\pi^k(u)}M$. Therefore

$$K_u \frac{\partial}{\partial x^j} |_u = (\underset{(1)}{M}{}^i_j(u) \frac{\partial}{\partial x^i} |_{\pi^k(u)}, \ldots, \underset{(k)}{M}{}^i_j(u) \frac{\partial}{\partial x^i} |_{\pi^k(u)}). \tag{2.3}$$

Taking account of Proposition 2.1 and formula (2.3), we obtain the following formulae

$$\begin{cases} K_u \frac{\partial}{\partial y^{(1)j}} |_u = (\frac{\partial}{\partial x^i} |_{\pi^k(u)}, \underset{(1)}{M}{}^i_j(u) \frac{\partial}{\partial x^i} |_{\pi^k(u)}, \ldots, \underset{(k-1)}{M}{}^i_j(u) \frac{\partial}{\partial x^i} |_{\pi^k(u)}) \\ K_u \frac{\partial}{\partial y^{(2)j}} |_u = (0, \frac{\partial}{\partial x^i} |_{\pi^k(u)}, \underset{(1)}{M}{}^i_j(u) \frac{\partial}{\partial x^i} |_{\pi^k(u)}, \ldots, \underset{(k-2)}{M}{}^i_j(u) \frac{\partial}{\partial x^i} |_{\pi^k(u)}) \\ \ldots \\ K_u \frac{\partial}{\partial y^{(k-1)j}} |_u = (0, \ldots, 0, \frac{\partial}{\partial x^i} |_{\pi^k(u)}, \underset{(1)}{M}{}^i_j(u) \frac{\partial}{\partial x^i} |_{\pi^k(u)}) \\ K_u \frac{\partial}{\partial y^{(k)j}} |_u = (0, \ldots, 0, \frac{\partial}{\partial x^i} |_{\pi^k(u)}). \end{cases} \tag{2.4}$$

Theorem 2.1. *Under a change of coordinates (1.1) on E, the set of functions* $(\underset{(\alpha)}{M}{}^i_j)_{\alpha=\overline{1,k}}$ *is changing according to the following rules*

$$\begin{cases} \underset{(1)}{M}{}^m_j \frac{\partial \widetilde{x}^i}{\partial x^m} = \widetilde{\underset{(1)}{M}{}^i_m} \frac{\partial \widetilde{x}^m}{\partial x^j} + \frac{\partial \widetilde{y}^{(1)i}}{\partial x^j} \\ \underset{(2)}{M}{}^m_j \frac{\partial \widetilde{x}^i}{\partial x^m} = \widetilde{\underset{(2)}{M}{}^i_m} \frac{\partial \widetilde{x}^m}{\partial x^j} + \widetilde{\underset{(1)}{M}{}^i_m} \frac{\partial \widetilde{y}^{(1)m}}{\partial x^j} + \frac{\partial \widetilde{y}^{(2)i}}{\partial x^j} \\ \ldots \\ \underset{(k)}{M}{}^m_j \frac{\partial \widetilde{x}^i}{\partial x^m} = \widetilde{\underset{(k)}{M}{}^i_m} \frac{\partial \widetilde{x}^m}{\partial x^j} + \widetilde{\underset{(k-1)}{M}{}^i_m} \frac{\partial \widetilde{y}^{(1)m}}{\partial x^j} + \ldots + \widetilde{\underset{(1)}{M}{}^i_m} \frac{\partial \widetilde{y}^{(k-1)m}}{\partial x^j} + \frac{\partial \widetilde{y}^{(k)i}}{\partial x^j}. \end{cases} \tag{2.5}$$

Proof. For $(U, \phi = (x^i, y^{(1)i}, \ldots, y^{(k)i}))$ and $(V, \psi = (\widetilde{x}^i, \widetilde{y}^{(1)i}, \ldots, \widetilde{y}^{(k)i}))$ two local charts in $u \in U \cap V$, we have

$$K_u \frac{\partial}{\partial \widetilde{x}^j} |_u = \left(\widetilde{\underset{(1)}{M}{}^i_j}(u) \frac{\partial}{\partial \widetilde{x}^i} |_{\pi^k(u)}, \ldots, \widetilde{\underset{(k)}{M}{}^i_j}(u) \frac{\partial}{\partial \widetilde{x}^i} |_{\pi^k(u)} \right) = K_u \frac{\partial}{\partial x^j} |_u .$$

According to (1.2),

$$K_u\frac{\partial}{\partial\widetilde{x}^i}\mid_u = K_u(\frac{\partial x^m}{\partial\widetilde{x}^i}(u)\frac{\partial}{\partial x^m}\mid_u + \frac{\partial y^{(1)m}}{\partial\widetilde{x}^i}(u)\frac{\partial}{\partial y^{(1)m}}\mid_u + \ldots +$$
$$+\frac{\partial y^{(k)m}}{\partial\widetilde{x}^i}(u)\frac{\partial}{\partial y^{(k)m}}\mid_u) = \frac{\partial x^m}{\partial\widetilde{x}^i}(u)K_u\frac{\partial}{\partial x^m}\mid_u + \frac{\partial y^{(1)m}}{\partial\widetilde{x}^i}(u)K_u\frac{\partial}{\partial y^{(1)m}}\mid_u + \ldots +$$
$$+\frac{\partial y^{(k)m}}{\partial\widetilde{x}^i}(u)K_u\frac{\partial}{\partial y^{(k)m}}\mid_u = \frac{\partial x^m}{\partial\widetilde{x}^i}(\underset{(1)}{M}{}^j_m(u)\frac{\partial}{\partial x^j}\mid_{\pi^k(u)}, \ldots, \underset{(k)}{M}{}^j_m(u)\frac{\partial}{\partial x^j}\mid_{\pi^k(u)}) +$$
$$+\frac{\partial y^{(1)m}}{\partial\widetilde{x}^i}(\frac{\partial}{\partial x^m}\mid_{\pi^k(u)}, \underset{(1)}{M}{}^j_m(u)\frac{\partial}{\partial x^j}\mid_{\pi^k(u)}, \ldots, \underset{(k-1)}{M}{}^j_m(u)\frac{\partial}{\partial x^j}\mid_{\pi^k(u)}) + \ldots +$$
$$+\frac{\partial y^{(k-1)m}}{\partial\widetilde{x}^i}(0,\ldots,0,\frac{\partial}{\partial x^m}\mid_{\pi^k(u)}, \underset{(1)}{M}{}^j_m(u)\frac{\partial}{\partial x^j}\mid_{\pi^k(u)}) + \frac{\partial y^{(k)m}}{\partial\widetilde{x}^i}(0,\ldots,0,\frac{\partial}{\partial x^m}\mid_{\pi^k(u)}).$$

On this way we obtain

$$\begin{cases} \widetilde{\underset{(1)}{M}{}^j_i}(u)\dfrac{\partial}{\partial\widetilde{x}^j}\mid_{\pi^k(u)} = \dfrac{\partial x^m}{\partial\widetilde{x}^i}\underset{(1)}{M}{}^j_m(u)\dfrac{\partial}{\partial x^j}\mid_{\pi^k(u)} + \dfrac{\partial y^{(1)j}}{\partial\widetilde{x}^i}\dfrac{\partial}{\partial x^j}\mid_{(\pi^k(u)} \\ \ldots \\ \widetilde{\underset{(k)}{M}{}^j_i}(u)\dfrac{\partial}{\partial\widetilde{x}^j}\mid_{\pi^k(u)} = \dfrac{\partial x^m}{\partial\widetilde{x}^i}\underset{(k)}{M}{}^j_m(u)\dfrac{\partial}{\partial x^j}\mid_{\pi^k(u)} + \dfrac{\partial y^{(1)m}}{\partial\widetilde{x}^i}\underset{(k-1)}{M}{}^j_m(u)\dfrac{\partial}{\partial x^j}\mid_{\pi^k(u)} + \\ \qquad + \ldots + \dfrac{\partial y^{(k)j}}{\partial\widetilde{x}^i}\dfrac{\partial}{\partial\widetilde{x}^j}\mid_{\pi^k(u)}. \end{cases}$$

and using $\dfrac{\partial}{\partial\widetilde{x}^j}\mid_{\pi^k(u)} = \dfrac{\partial x^s}{\partial\widetilde{x}^j}\dfrac{\partial}{\partial x^s}\mid_{\pi^k(u)}$ it results (2.5).

Next, on every domain of local chart on E we consider the set of functions $(\underset{(1)}{N}{}^i_j, \ldots, \underset{(k)}{N}{}^i_j)$ defined by

$$(2.6)\qquad \begin{cases} \underset{(1)}{N}{}^i_j = \underset{(1)}{M}{}^i_j \\ \underset{(2)}{N}{}^i_j = \underset{(2)}{M}{}^i_j - \underset{(1)}{N}{}^i_m\underset{(1)}{M}{}^m_j \\ \ldots \\ \underset{(k)}{N}{}^i_j = \underset{(k)}{M}{}^i_j - \underset{(k-1)}{N}{}^i_m\underset{(1)}{M}{}^m_j - \ldots - \underset{(1)}{N}{}^i_m\underset{(k-1)}{M}{}^m_j. \end{cases}$$

We use the following notations

$$(2.7)\qquad \begin{cases} \dfrac{\delta}{\delta x^i}\mid_u = \dfrac{\partial}{\partial x^i}\mid_u - \underset{(1)}{N}{}^j_i(u)\dfrac{\partial}{\partial y^{(1)j}}\mid_u - \ldots - \underset{(k)}{N}{}^j_i(u)\dfrac{\partial}{\partial y^{(k)j}}\mid_u \\ \dfrac{\delta}{\delta y^{(1)i}}\mid_u = J_u\left(\dfrac{\delta}{\delta x^i}\mid_u\right), \ldots, \dfrac{\delta}{\delta y^{(k-1)i}}\mid_u = J_u\left(\dfrac{\delta}{\delta y^{(k-2)i}}\mid_u\right). \end{cases}$$

Theorem 2.2. *The vector fields*

$$\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \ldots, \frac{\delta}{\delta y^{(k-1)i}}, \frac{\partial}{\partial y^{(k)i}}$$

are d-vector fields, that is under a change of coordinates (1.1) these are changing by the following rules

$$(2.8)\qquad \begin{cases} \dfrac{\delta}{\delta\tilde{x}^i} = \dfrac{\partial x^j}{\partial\tilde{x}^i}\dfrac{\delta}{\delta x^j}, \\ \dfrac{\delta}{\delta\tilde{y}^{(\alpha)i}} = \dfrac{\partial x^j}{\partial\tilde{x}^i}\dfrac{\delta}{\delta y^{(\alpha)j}}\ (\alpha = \overline{1,k-1}), \\ \dfrac{\partial}{\partial\tilde{y}^{(k)i}} = \dfrac{\partial x^j}{\partial\tilde{x}^i}\dfrac{\partial}{\partial y^{(k)j}}. \end{cases}$$

Proof. Under a change of coordinates on E, the set of functions $(\underset{(1)}{N}{}^i_j, \dots, \underset{(k)}{N}{}^i_j)$ changes as follows

$$(2.9)\qquad \begin{cases} \underset{(1)}{\widetilde{N}}{}^i_m\dfrac{\partial\widetilde{x}^m}{\partial x^j} = \underset{(1)}{N}{}^m_j\dfrac{\partial\widetilde{x}^i}{\partial x^m} - \dfrac{\partial\widetilde{y}^{(1)i}}{\partial x^j} \\ \underset{(2)}{\widetilde{N}}{}^i_m\dfrac{\partial\widetilde{x}^m}{\partial x^j} = \underset{(2)}{N}{}^m_j\dfrac{\partial\widetilde{x}^i}{\partial x^m} + \underset{(1)}{N}{}^m_j\dfrac{\partial\widetilde{y}^{(1)i}}{\partial x^m} - \dfrac{\partial\widetilde{y}^{(2)i}}{\partial x^j} \\ \dots \\ \underset{(k)}{\widetilde{N}}{}^i_m\dfrac{\partial\widetilde{x}^m}{\partial x^j} = \underset{(k)}{N}{}^m_j\dfrac{\partial\widetilde{x}^i}{\partial x^m} + \underset{(k-1)}{N}{}^m_j\dfrac{\partial\widetilde{y}^{(1)i}}{\partial x^m} + \dots + \underset{(1)}{N}{}^m_j\dfrac{\partial\widetilde{y}^{(k-1)i}}{\partial x^m} - \dfrac{\partial\widetilde{y}^{(k)i}}{\partial x^j}. \end{cases}$$

Using (2.9) and (1.2) it results that under a change of coordinates (1.1) on E, (2.8) is true. The vectors $\dfrac{\delta}{\delta x^i}\mid_u, \dfrac{\delta}{\delta y^{(1)i}}\mid_u, \dots, \dfrac{\delta}{\delta y^{(k-1)i}}\mid_u, \dfrac{\partial}{\partial y^{(k)i}}\mid_u$ are linearly independent on T_uE, $\forall u \in E$.

Proposition 2.2. *We have the following formulae*

$$(2.10)\qquad \begin{cases} (\pi^k)_{*,u}\left(\dfrac{\delta}{\delta x^i}\mid_u\right) = \dfrac{\partial}{\partial x^i}\mid_{\pi^k(u)} \\ \overset{(\alpha)}{K}{}_u\left(\dfrac{\delta}{\delta y^{(\alpha)i}}\mid_u\right) = \dfrac{\partial}{\partial x^i}\mid_{\pi^k(u)}\ \ \forall\alpha = \overline{1,k-1} \\ \overset{(k)}{K}{}_u\left(\dfrac{\partial}{\partial y^{(k)i}}\mid_u\right) = \dfrac{\partial}{\partial x^i}\mid_{\pi^k(u)}. \end{cases}$$

Proof.

$$(\pi^k)_{*,u}\left(\frac{\delta}{\delta x^i}\mid_u\right) = (\pi^k)_{*,u}\left(\frac{\partial}{\partial x^i}\mid_u - \underset{(1)}{N}{}^j_i(u)\frac{\partial}{\partial y^{(1)j}}\mid_u - \dots - \underset{(k)}{N}{}^j_i(u)\frac{\partial}{\partial y^{(k)j}}\mid_u\right) =$$

$$= (\pi^k)_{*,u}\left(\frac{\partial}{\partial x^i}\mid_u\right) = \frac{\partial}{\partial x^i}\mid_{\pi^k(u)}.$$

The other formulae are proved by using (2.2) and (2.7).

Theorem 2.3. *For a connection map* $K = (\overset{(1)}{K}, \overset{(2)}{K}, \dots, \overset{(k)}{K}) : (TE, \tau_E, E) \to (TM^{(k)}, \tau^{(k)}, M)$, *its kernel* $N = KerK$ *is a nonlinear connection on* k*-osculator bundle.*

Proof. To prove this theorem it is sufficient to show the following decomposition in the Whitney sum

$$(2.11)\qquad TE = N \oplus V_1.$$

Firstly, we prove that $\left\{\frac{\delta}{\delta x^i}\mid_u\right\}_{i=\overline{1,n}}$ is a basis for $KerK_u = N(u)$. Since $\left\{\frac{\delta}{\delta x^i}\mid_u\right\}_{i=\overline{1,n}}$ are n linearly independent vectors and the dimension of $N(u)$ is n, we need to prove that $K_u\left(\frac{\delta}{\delta x^i}\mid_u\right) = 0$. We have the following sequence of equalities

$$K_u(\frac{\delta}{\delta x^i}\mid_u) = K_u(\frac{\partial}{\partial x^i}\mid_u - \underset{(1)}{N}{}^j_i(u)\frac{\partial}{\partial y^{(1)j}}\mid_u - \ldots - \underset{(1)}{N}{}^j_i(u)\frac{\partial}{\partial y^{(k)j}}\mid_u) =$$

$$= K_u(\frac{\partial}{\partial x^i}\mid_u) - \underset{(1)}{N}{}^j_i(u)K_u(\frac{\partial}{\partial y^{(1)j}}\mid_u) - \ldots - \underset{(k)}{N}{}^j_i(u)K_u(\frac{\partial}{\partial y^{(k)j}}\mid_u) =$$

$$(\underset{(1)}{M}{}^i_j(u)\frac{\partial}{\partial x^i}\mid_{\pi^k(u)}, \ldots, \underset{(k)}{M}{}^i_j(u)\frac{\partial}{\partial x^i}\mid_{\pi^k(u)}) - \underset{(1)}{N}{}^i_j(u)(\frac{\partial}{\partial x^i}\mid_{\pi^k(u)}, \underset{(1)}{M}{}^m_j(u)\frac{\partial}{\partial x^m}\mid_{\pi^k(u)},$$

$$, \ldots, \underset{(k-1)}{M}{}^m_j(u)\frac{\partial}{\partial x^m}\mid_{\pi^k(u)}) - \underset{(2)}{N}{}^i_j(u)(0, \frac{\partial}{\partial x^i}\mid_{\pi^k(u)}, \underset{(1)}{M}{}^m_j(u)\frac{\partial}{\partial x^m}\mid_{\pi^k(u)}, \ldots,$$

$$\underset{(k-2)}{M}{}^i_j(u)\frac{\partial}{\partial x^i}\mid_{\pi^k(u)}) - \cdots - \underset{(k-1)}{N}{}^i_j(u)(0, \ldots, 0, \frac{\partial}{\partial x^i}\mid_{\pi^k(u)}, \underset{(1)}{M}{}^m_j(u)\frac{\partial}{\partial x^m}\mid_{\pi^k(u)}) -$$

$$-\underset{(k)}{N}{}^i_j(u)(0, \ldots, 0, \frac{\partial}{\partial x^i}\mid_{\pi^k(u)}).$$

Using (2.6) one obtains $K_u\left(\frac{\delta}{\delta x^i}\mid_u\right) = 0$. Finally, we have to prove that $\forall u \in E$, $N(u) \cap V_1(u) = \{0\}$. Let $X_u \in N(u) \cap V_1(u)$. Because $X_u \in N(u)$ it results $K_u X_u = 0$ and because $X_u \in V_1(u)$ we have $(\pi^k)_{*,u}X_u = 0$. If X_u is expressed in the basis (2.7) by

$$X_u = \overset{(0)}{X}{}^i\frac{\delta}{\delta x^i} + \overset{(1)}{X}{}^i\frac{\delta}{\delta y^{(1)i}} + \ldots + \overset{(k)}{X}{}^i\frac{\partial}{\partial y^{(k)i}}$$

it follows that

$$K_u X_u = (\overset{(1)}{X}{}^i\frac{\partial}{\partial x^i}, \overset{(2)}{X}{}^i\frac{\partial}{\partial x^i}, \ldots, \overset{(k)}{X}{}^i\frac{\partial}{\partial x^i}) = 0$$

and

$$(\pi^k)_{*,u}X_u = \overset{(0)}{X}{}^i\frac{\partial}{\partial x^i} = 0 \Longrightarrow \overset{(0)}{X}{}^i = \overset{(1)}{X}{}^i = \cdots = \overset{(k)}{X}{}^i = 0.$$

We call the functions $(\underset{(1)}{N}{}^i_j, \ldots, \underset{(k)}{N}{}^i_j)$ the *coefficients* of the nonlinear connection N.

Next, we prove the existence of connection map.

Theorem 2.4. *If the manifold M is smooth, then there exists a connection map on the k-osculator bundle of the manifold M.*

Proof. In the paper [2] we have proved that every nonlinear connection N on the k–osculator bundle determines a connection map. Because on $Osc^k M$ there exist nonlinear connections, the proof is finished.

We denote $N_0 = N, N_1 = J(N_0), \cdots, N_{k-1} = J(N_{k-2})$. The following decomposition in the Whitney sum is true

$$TE = N_0 \oplus N_1 \oplus \cdots \oplus N_{k-1} \oplus V_k. \tag{2.12}$$

The distributions $N_\alpha : u \in E \mapsto N_\alpha(u) \subset T_uE$ are of dimensions n. Generally these are not integrable.

Concluding, the existence of the nonlinear connection N on the k-osculator bundle E is characterized by a special π^k morphism of vector bundles K such that $N = Ker\ K$.

3 Characterizations of the connection map with the lifts.

Let $(V_kE, \tau_E\,|_{V_kE}, E)$ be a vector bundle over E, whose fibres are $V_k(u)$, $u \in E$.

For each $u \in E$, we denote by $(l_{v_k})_{\pi^k(u),u} : T_{\pi^k(u)}M \to V_k(u)$ the linear map defined by: $(l_{v_k})_{\pi^k(u),u}\left(\frac{\partial}{\partial x^i}\,|_{\pi^k(u)}\right) = \frac{\partial}{\partial y^{(k)i}}\,|_u$ and extended by linearity.

The map $(l_{v_k})_{\pi^k(u),u}$ is a linear isomorphism for $\forall u \in E$. It is called the *vertical lift.*

Definition 3.1. The *horizontal lift* is defined as a linear map $(l_h)_{\pi^k(u),u} : T_{\pi^k(u)}M \to T_uE$ which satisfies

$$(l_{v_k})_{\pi^k(u),u} = J_u^k \circ (l_h)_{\pi^k(u),u}. \tag{3.1}$$

We use the notation $N(u) = (l_h)_{\pi^k(u),u}(T_{\pi^k(u)}M)$.

Remark. The map $(l_h)_{\pi^k(u),u} : T_{\pi^k(u)}M \to N(u)$ is a linear isomorphism of vector spaces.

Proposition 3.1. *Every connection map K on the k–osculator bundle E determines a horizontal lift.*

Proof. The map $\pi^k_{*,u}\,|_{Ker\ K_u}: Ker\ K_u \to T_{\pi^k(u)}M$ is an isomorphism of vector spaces. We denote by $(l_h)_{\pi^k(u),u} : T_{\pi^k(u)}M \to Ker\ K_u$ its inverse map. We have $\pi^k_{*,u} \circ (l_h)_{\pi^k(u),u} = id_{T_{\pi^k(u)}M}$. Firstly, we prove $\overset{(k)}{K}_u \circ (l_{v_k})_{\pi^k(u),u} = id_{T_{\pi^k(u)}M}$. Using that $(l_{v_k})_{\pi^k(u),u}\left(\frac{\partial}{\partial x^i}\,|_{\pi^k(u)}\right) = \frac{\partial}{\partial y^{(k)i}}\,|_u$ and $\overset{(k)}{K}_u\left(\frac{\partial}{\partial y^{(k)i}}\,|_u\right) = \frac{\partial}{\partial x^i}\,|_{\pi^k(u)}$ we obtain the previous formula. The proof of proposition is finished by the following sequence of implications: $\pi^k_{*,u} = \overset{(k)}{K}_u \circ J_u^k \Rightarrow \overset{(k)}{K}_u \circ J_u^k \circ (l_h)_{\pi^k(u),u} = \overset{(k)}{K}_u \circ (l_{v_k})_{\pi^k(u),u} \Rightarrow$ the map l_h verify (3.1).

Proposition 3.2. *Every horizontal lift determines a connection map on k-osculator bundle.*

Proof. Let $(l_h)_{\pi^k(u),u} : T_{\pi^k(u)}M \to T_uE$ be a map which satisfy (3.1).

Let $\overset{(k)}{K}_u$: $V_k(u) \to T_{\pi^k(u)}$ be the inverse map of the vertical lift and $K_u = (\overset{(k)}{K}_u \circ J_u^{k-1}, \ldots \overset{(k)}{K}_u \circ J_u, \overset{(k)}{K}_u)$. For proving that the map K is a connection map it is sufficient to show that $\overset{(k)}{K}_u \circ J_u^k = \pi^k_{*,u}$. From (3.1), compound at left by $\overset{(k)}{K}_u$ we have $\overset{(k)}{K}_u \circ J_u^k \circ (l_h)_{\pi^k(u),u} = \overset{(k)}{K}_u \circ (l_{v_k})_{\pi^k(u),u} = id_{T_{\pi^k(u)}M}$. Since $(l_h)_{\pi^k(u),u}$ is a linear isomorphism it results that $\overset{(k)}{K}_u \circ J_u^k$ is its inverse and $\overset{(k)}{K}_u \circ J_u^k = \pi^k_{*,u}$.

For $\alpha \in \{1,\ldots,k-1\}$ and $u \in E$ we denote $(l_{v_\alpha})_{\pi^k(u),u} : T_{\pi^k(u)}M \to T_uE$ the map defined by $(l_{v_\alpha})_{\pi^k(u),u} = J_u^\alpha \circ (l_h)_{\pi^k(u),u}$. If we use the notations $N_\alpha(u) = (l_{v_\alpha})_{\pi^k(u),u}(T_{\pi^k(u)}M)$ we obtain that $(l_{v_\alpha})_{\pi^k(u),u}$ are linear isomorphisms.

Corollary 3.1. *The following formulae are true*

$$\overset{(\alpha)}{K}_u \circ (l_{v_\alpha})_{\pi^k(u),u} = id_{T_{\pi^k(u)}M} \quad \forall\alpha \in \{1,2,\ldots\} \text{ and } u \in E.$$

Proof. We have $(l_{v_k})_{\pi^k(u),u} = J_u^k \circ (l_h)_{\pi^k(u),u} = J^{k-\alpha} \circ J_u^\alpha \circ (l_h)_{\pi^k(u),u} = J_u^{k-\alpha} \circ (l_{v_\alpha})_{\pi^k(u),u}$. Compound at left to $\overset{(k)}{K}_u$ in the formula $(l_{v_k})_{\pi^k(u),u} = J_u^{k-\alpha} \circ (l_{v_\alpha})_{\pi^k(u),u}$ and using $\overset{(k)}{K}_u \circ (l_{v_k})_{\pi^k(u),u} = id_{T_{\pi^k(u)}M}$ and from (2.1) it results $\overset{(\alpha)}{K}_u \circ (l_{v_\alpha})_{\pi^k(u),u} = id_{T_{\pi^k(u)}M} \quad \forall\alpha \in \{1,2,\ldots\}$.

Acknowledgements. A version of this paper was presented at the First Conference of Balkan Society of Geometers, Politehnica University of Bucharest, September 23-27, 1996.

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"Octav Mayer" Institute of Matematics
Iaşi Branch of the Romanian Academy
6600, Iaşi, Romania