Discrete-Continuous Dynamical Systems with Symmetry

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Abstract

This work develops the geometry and dynamics for discrete-continuous systems with nonholonomic constraints and symmetry from the perspective of Lagrangian mechanics.

The basic methodology is that of geometric mechanics of discrete-continuous type, applied to the formulation of Lagrange - d'Alembert for these systems, generalizing the momentum maps associated with a given symmetry group to this case.

One of the purposes of this paper is to derive the evolution equation for the momentum, a discrete-continuous momentum equation and to distinguish geometrically and mechanically the cases where it is conserved.

We give detailed exemples to ilustrate this theory.

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1 Introduction

Problems of nonholonomic mechanics, including many problems in robotics are of considerable interest. Greenspan (1973,1974) and more recently Labudde, T.D.Lee (1987), Yuhuawu (1990) and Shibberu (1992) developed a discrete mechanics. Discrete mechanics schemes are distinguished from numerical conventional schemes in that they are based on fundamental principles, as opposed to approximations of differential equations derived from continuum mechanics.

The purpose of this work is to bring these topics closer together with nonholonomic systems with symmetry.

We begin by formulating a discrete-continuous variational principle and deduce the discrete-continuous Lagrange d'Alembert principle. Following this, we add the hypothesis of symmetry, derive a discrete-continuous version of the Noether theorem and develop an evolution equation for the momentum, of course a discrete-continuous one, that generalize the usual conservation laws associated to a symmetry group.

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In the final part of the paper we extend this theory for discrete-continuous systems with special constraints (homogeneous and affine).

In this paper we give some examples to illustrate our approach including a discrete version of a nonholonomically constrained particle.

2 The discrete-continuous variations

Let $\{\tau_k\}_{k\in[0,N]}$, $[0,N]=\{0,1,\ldots,N]\}$ be a division of the interval $[\tau_0,\tau_N]\subset\mathbf{R}$, where

$$(1)\qquad \tau_k=\tau_0+k\Delta\tau,\qquad \Delta\tau=\frac{\tau_N-\tau_0}{N},$$

and let $\mathcal{R}=[\tau_0,\tau_N]\times[a,b]\subset\mathbf{R}^2$ be the two-dimensional network whose arbitrary element $(\tau_k,t)\in\mathcal{R}$ is denoted by $(k,t), k\in[0,N], t\in[a,b]$. For a function $q:\mathcal{R}\to\mathbf{R}^n$, C^1-differentiable with respsct to $t\in[a,b]$, we denote

$$(2)\qquad \begin{aligned} q(k,t)&=(q^i(k,t))\\ q^1(k,t)&=\frac{1}{\Delta\tau}(q(k+1,t)-q(k,t))\\ \dot q(k,t)&=\frac{dq(k,t)}{dt}. \end{aligned}$$

The set

$$(3)\qquad \mathcal{L}^2(\mathcal{R})=\left\{q:\mathcal{R}\to\mathbf{R}^n|\int_a^b\sum_{k=0}^N\delta_{ij}q^i(k,t)q^j(k,t)\Delta\tau dt<\infty\right\}$$

endowed with the scalar product

$$(4)\qquad (q,\bar q)=\int_a^b\left(\sum_{k=0}^N\delta_{ij}q^i(k,t)\bar q^j(k,t)\Delta\tau\right)dt$$

is a Hilbert space. The tangent space of the manifold

$$(5)\qquad \begin{aligned} \Omega=\{q\in\mathcal{L}^2(\mathcal{R}),\ q(k,a)&=q_1(k),\ q(k,b)=q_2(k), k\in[1,N-1]\\ q(0,t)&=q_1(t), q(N,t)=q_N(t), t\in[a,b]\} \end{aligned}$$

at a point $q\in\Omega$ is

$$(6)\qquad \begin{aligned} T_q\Omega=\{\eta:\mathcal{R}\to\mathbf{R}^n,\ \eta(k,a)&=0,\ \eta(k,b)=0,\ k\in[1,N-1]\\ \eta(0,t)&=0, \eta(N,t)=0, t\in[a,b]\}, \end{aligned}$$

where

$$(7)\qquad \eta(k,t)=\left.\frac{dq(k,t,\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0},\ q_{(\varepsilon)}\in\Omega,\ \varepsilon\in I\subset\mathbf{R},\ 0\in I, q_{(0)}=q_0.$$

For a C^1-differentiable function $F : \Omega \to \mathbf{R}$ we define the variation of the function F [1],

$$\delta F : T_q\Omega \to \mathbf{R},$$

(8) $$(\delta F)(\eta) = \left.\frac{dF(q_{(\varepsilon)})}{d\varepsilon}\right|_{\varepsilon=0} = \sum_{k=0}^{N} \frac{\partial F(q(k,t))}{\partial q^i(k,t)}\eta^i(k,t).$$

The point $q \in \Omega$ is *critical* (or *stationary*) for F, if at this point $\delta F = 0$.
Let be the sets

(9) $$\Omega^1 = \left\{q^1(k,t), (k,s) \in \mathcal{R}\right\}, \qquad \dot{\Omega} = \{\dot{q}(k,t), (k,s) \in \mathcal{R}\}$$

and $L : \mathcal{R} \times \Omega \times \Omega^1 \times \dot{\Omega} \to \mathcal{R}$ be a C^2-function with respect to the variables from Ω, Ω^1, $\dot{\Omega}$ and $t \in [a,b]$. We denote this function by

(10) $$L(k,t) = L(k,t,q(k,t),q^1(k,t),\dot{q}(k,t)), \qquad (k,t) \in \mathcal{R}.$$

For $L(k,t)$ we obtain

$$\frac{dL(k,t)}{dq^i(k,t)} = \frac{\partial L(k,t)}{\partial q^i(k,t)} - \frac{1}{\Delta\tau}\frac{\partial L(k,t)}{\partial q^{1i}(k,t)}$$

(11) $$\frac{dL(k,t)}{dq^i(k+1,t)} = \frac{1}{\Delta\tau}\frac{\partial L(k,t)}{\partial q^{1i}(k,t)}, \qquad i = \overline{1,n}, (k,t) \in \mathcal{R}.$$

The functional

(12) $$\mathcal{A}(q) = \int_a^b \left(\sum_{k=0}^{N-1} L(k,t)\Delta\tau\right) dt$$

is called the *action* of L with respect to $q \in \Omega$.
Proposition 1 (first variation formula) [6]. The variation of the functional $\mathcal{A}$ is

(13) $$\delta\mathcal{A}(\eta,\eta^1,\dot{\eta}) = \int_a^b \left[\sum_{k=0}^{N-1}\left(\frac{\partial L(k,t)}{\partial q^i(k,t)}\eta^i(k,t) + \frac{\partial L(k,t)}{\partial q^{1i}(k,t)}\eta^{1i}(k,t) + \right.\right.$$
$$\left.\left. + \frac{\partial L(k,t)}{\partial \dot{q}^i(k,t)}\dot{\eta}^i(k,t)\right)\right]\Delta\tau dt,$$

where

$$\eta^i(k,t) = \left.\frac{dq_{(\varepsilon)}(k,t)}{d\varepsilon}\right|_{\varepsilon=0}, \qquad \eta^{1i}(k,t) = \left.\frac{dq^1_{(\varepsilon)}(k,t)}{d\varepsilon}\right|_{\varepsilon=0}$$

(14) $$\dot{\eta}^i(k,t) = \left.\frac{d\dot{q}_{(\varepsilon)}(k,t)}{d\varepsilon}\right|_{\varepsilon=0},$$

$$q_{(\varepsilon)} \in \Omega, q^1_{(\varepsilon)} \in \Omega^1, \dot{q}_{(\varepsilon)} \in \dot{\Omega},$$

with

$$\varepsilon \in I \subset \mathbf{R}, 0 \in I, q_{(0)} = q, q^1_{(0)} = q^1, \dot{q}_{(0)} = \dot{q}.$$

For

$$\eta^1(k,t) = \frac{1}{\Delta\tau}(\eta(k+1,t) - \eta(k,t)), \qquad \dot{\eta}(k,t) = \frac{d\eta(k,t)}{dt},$$

from (12) we obtain

$$\delta\mathcal{A}(q) = \int_a^b \sum_{k=1}^{N-1} [E]_i(k,t)\eta^i(k,t)\Delta\tau dt, \tag{15}$$

where

$$[E]_i(k,t) = \frac{d(L(k,t) + L(k-1,t))}{dq^i(k,t)} - \frac{d}{dt}\left(\frac{\partial L(k,t)}{\partial \dot{q}^i(k,t)}\right). \tag{16}$$

3 The discrete-continuous Lagrange-d′Alembert principle

The path $q(k,t) \in \Omega$ is called a discrete-continuous motion of the lagrangian system (Ω, L), if it is a critical point of the action functional $\mathcal{A}(q)$. From (13) we obtain
Proposition 1 *(discrete-continuous variation principle). The function $q \in \Omega$ is a critical point for $\mathcal{A}(q)$ iff*

$$[E]_i(k,t) = \frac{d(L(k,t) + L(k-1,t))}{dq^i(k,t)} - \frac{d}{dt}\left(\frac{\partial L(k,t)}{\partial \dot{q}^i(k,t)}\right) = 0 \quad i = \overline{1,n}, (k,t) \in \mathcal{R}. \tag{17}$$

For $L(k,t) = L(t, q(t), \dot{q}(t))$, $t \in [a,b]$, we obtain from (16), the continuous Euler-Lagrange equation. For $L(k,t) = L(k, q(k), q^1(k))$, $k \in [0, N-1]$, we obtain from (16) the discrete Euler-Lagrange equations

$$\frac{d(L(k) + L(k-1))}{dq^i(k)} = 0, \qquad i = \overline{1,n}. \tag{18}$$

For $\Delta\tau = 1, \tau_0 = 1, \tau_N = N$, from (17) we deduce the discrete Euler equations given in [5]

$$\frac{\partial L(k)}{\partial q^i(k)} - \frac{\partial L(k)}{\partial q^{1i}(k)} + \frac{\partial L(k-1)}{\partial q^{1i}(k-1)} = 0, \qquad i = \overline{1,n}, k \in \mathcal{R}. \tag{19}$$

Let $f_a : \mathcal{R} \times \Omega \times \Omega^1 \times \dot{\Omega} \to \mathbf{R}$, $a = \overline{1,m}$, a C^1-function with respect to the variables of Ω, Ω^1, $\dot{\Omega}$ and $t \in [a,b]$, given by

$$f_a(k,t) = f_a(k, t, q(k,t), q^1(k,t)), \qquad a = \overline{1,m}, (k,t) \in \mathcal{R} \tag{20}$$

with

$$\text{rang} \left\| \frac{\partial f_a(k,t)}{\partial q^{1i}(k,t)} \right\| = m < n, \qquad (k,t) \in \mathcal{R}. \tag{21}$$

The elements $\eta \in T_q\Omega$ satisfying the conditions

$$\frac{\partial f_a(k,t)}{\partial q^{1i}(k,t)} \cdot \eta^i(k,t) = 0, \qquad a = \overline{1,m}, (k,t) \in \mathcal{R} \tag{22}$$

are called *virtual variations* of the constrained discrete lagrangian system (Ω, L, S), where L is given by (10) and

$$S = \Big\{ (q(k,t), q^1(k,t), \dot{q}(k,t)) \in \Omega \times \Omega^1 \times \dot{\Omega}, f_a(k,t) = 0,$$

$$a = \overline{1,m}, (k,t) \in \mathcal{R} \Big\} \tag{23}$$

The discrete Lagrange-d′Alembert definition: The admissible element $q \in \Omega$ is called a *discrete motion* of the discrete constrained lagrangian system (Ω, L, S) if for $(k,t) \in \mathcal{R}$, $[E]_i(k,t)\eta^i(k,t) = 0$ for all virtual variation η.

Proposition 2. *The discrete motion of the discrete constrained lagrangian system* (Ω, L, S) *is given by*

$$[E]_i(k,t) = \mu^a \frac{\partial f_a(k,t)}{\partial q^{1i}(k,t)} \tag{24}$$

$$f_a(k,t) = 0, \qquad a = \overline{1,m}, i = \overline{1,n}, (k,t) \in \mathcal{R}.$$

Let $f_a(k,t)$ given by (20) with

$$\text{rang} \left\| \frac{\partial f_a(k,t)}{\partial \dot{q}^i(k,t)} \right\| = m < n, \qquad (k,t) \in \mathcal{R}. \tag{25}$$

The elements $\eta \in T_q\Omega$ satisfying the condition

$$\frac{\partial f_a(k,t)}{\partial \dot{q}^i(k,t)} \cdot \eta^i(k,t) = 0, \qquad a = \overline{1,m}, (k,t) \in \mathcal{R} \tag{26}$$

are called *virtual variations* of the constrained continuous lagrangian system (Ω, L, S), where L is given by (10) and S is given by (23).

The continuous Lagrange-d′Alembert principle is: The admisible element $q \in \Omega$ is called a continuous motion of the continuous lagrangian system (Ω, L, S) if for $(k,t) \in \mathcal{R}$, $[E]_i(k,t)\eta^i(k,t) = 0$, for all virtual variation η.

Proposition 3. *The continuous motion of the continuous lagrangian system* (Ω, L, S) *is given by*

$$[E]_i(k,t) = \bar{\mu}^a \frac{\partial f_a(k,t)}{\partial \dot{q}^i(k,t)} \tag{27}$$

$$f_a(k,t) = 0, \qquad a = \overline{1,m}, i = \overline{1,n}, (k,t) \in \mathcal{R}.$$

4 The discrete-continuous Noether theorem

Let be the Lie group G acts (from the left) on Ω, by $(g,q)\in G\times\Omega\to g\cdot q\in\Omega$, which $(gq)(k,t)=\bar{q}(k,t,g)$. Let $\mathcal{G}$ be the Lie algebra of G and $\mathcal{G}^*$ the linear dual of $\mathcal{G}$. To each vector $\xi\in\mathcal{G}$ there correspond the one-parameter group $exp(\varepsilon\xi),\varepsilon\in I\subset\mathbf{R}$ of G whose action on Ω determines a discrete-continuous vector field ξ_Ω given by

$$(28)\qquad \xi_\Omega(k,t)=\frac{d}{d\varepsilon}[exp(\varepsilon\xi)\cdot q(k,t)]|_{\varepsilon=0}\qquad (k,t)\in\mathcal{R}.$$

From (28) we obtain

$$(29)\qquad \xi^i_\Omega(k,t)=K^i_\alpha(k,t)\xi^\alpha,\qquad i=\overline{1,n},\ \alpha=\overline{1,\dim G},\dim G=r,$$

where

$$(30)\qquad \xi=\xi^\alpha e_\alpha\in\mathcal{G},\qquad K^i_\alpha(k,t)=\left.\frac{d\bar{q}^i(k,t,\exp(\varepsilon e_\alpha))}{d\varepsilon}\right|_{\varepsilon=0}.$$

Let $\bar{q}^p$ be the canonical prolongation of G on $\Omega\times\Omega^1\times\dot{\Omega}$. The Lie group G is called a *symmetry group* of the discrete–continuous lagrangian system (Ω,L) if

$$(31)\qquad L\circ\bar{q}^p(k,t,g)=L(k,t),\qquad \forall(k,t)\in\mathcal{R},\forall g\in G.$$

The function L is called *G–invariant lagrangian*. The corresponding discrete momentum map is the mapping $J^d:T_q\Omega^1\to\mathcal{G}^*$ [2] given by

$$(32)\qquad J^d_\alpha(k,t)=\frac{\partial L(k-1,t)}{\partial q^{1i}(k,t)}K^i_\alpha(k,t),\qquad \alpha=\overline{1,n},(k,t)\in\mathcal{R}$$

and the corresponding continuous momentum map is $J^c:T_q\dot{\Omega}\to\mathcal{G}^*$ given by

$$(33)\qquad J^c_\alpha(k,t)=\frac{\partial L(k,t)}{\partial\dot{q}^i(k,t)}K^i_\alpha(k,t),\qquad \alpha=\overline{1,n},(k,t)\in\mathcal{R}.$$

Proposition 1 *(discrete-continuous Noether theorem). For a solution of the discrete–continuous Euler–Lagrange equations, we obtain*

$$(34)\qquad J^{d1}_\alpha(k,t)+\frac{dJ^c_\alpha(k,t)}{dt}=0,\qquad \alpha=\overline{1,r},(k,t)\in\mathcal{R},$$

where

$$(35)\qquad J^{d1}_\alpha(k,t)=J^d_\alpha(k+1,t)-J^d_\alpha(k,t).$$

If $L(k,t)=L(k,q(k),q^1(k))$, for a solution of the discrete Euler–Lagrange equations, we obtain [5]

$$(36)\qquad J^{1d}_\alpha(k)=0,\qquad \alpha=\overline{1,r}.$$

If $L(k,t)=L(t,q(t),\dot{q}(t))$, for a solution of the continuous Euler–Lgrange equations, we obtain [2], [4]

$$\frac{dJ^c_\alpha(t)}{dt} = 0, \qquad \alpha = \overline{1,r}. \tag{37}$$

Suppose that the lagrangian L does not depend on q^j, j fixed. The system (Ω, L) admit locally the symmetry group G_j. The action on Ω is given by

$$\bar{q}^i(k,t) = q^i(k,t), \qquad \bar{q}^j(k,t) = q^j(k,t) + \alpha^j, \qquad i \neq j, \alpha^j \in \mathcal{R}. \tag{38}$$

The discrete–continuous generator is given by $\xi^i_\Omega(k,t) = 0$, $\xi^j_\Omega(k,t) = 1$, $i \neq j$. From (40) we find

$$\frac{\partial L(k,t)}{\partial q^{1j}(k,t)} - \frac{\partial L(k-1,t)}{\partial q^{1j}(k-1,t)} + \frac{d}{dt}\left(\frac{\partial L(k,t)}{\partial \dot{q}^j(k,t)}\right) = 0, \qquad (k,t) \in \mathcal{R}. \tag{39}$$

The coordinate q^j is called a *cyclic coordinate.*

For discrete–continuous lagrangian system (Ω, L), the conservation energy hold; along a solution, the energy function

$$E(k,t) = \frac{\partial(L(k,t) + L(k-1,t))}{\partial \dot{q}^i(k,t)}\dot{q}^i(k,t) - (L(k,t) + L(k-1,t)) \tag{40}$$

is constant in time.

5 The discrete–continuous momentum equation

In this section we use the discrete-continuous Lagrange-d'Alembert principle to derive an equation for a generalized discrete-continuous momentum as a consequence of the symmetries. We make the assumption that the action of G on Ω is free and proper. Let $\mathcal{S} \subset \Omega \times \Omega^1 \times \dot{\Omega}$. We say that $\mathcal{S}$ is *invariant* if $\mathcal{S}_{gq} = \mathcal{S}_q, \forall q \in \Omega, \forall g \in G$. The group orbit through a point $q \in \Omega$ is denoted by $orb(q) = \{gq, q \in G\}$. Let $\bar{q}^p$ be the canonical prolongation of G on $\Omega \times \Omega^1 \times \dot{\Omega}$, $T_{\bar{q}^p}S$ the virtual variation and $T_{\bar{q}}(orb(\bar{q}^p))$ the tangent space to the group orbit through a point $q^p = (q, q^1, \dot{q})$. In general, $\mathcal{S}_{\bar{q}^E} = T_{\bar{q}^p}S \cap T_{\bar{q}}(orb(q^p)) \neq \emptyset$. We define for each point $q \in \Omega \times \Omega^1 \times \dot{\Omega}$, the vector subspace $\mathcal{G}(q)$ to be the set of Lie algebra elements $\mathcal{G}$ whose infinitesimal discrete-continuous generators evaluated at $\bar{q}$ are in $\mathcal{S}_{\bar{q}^E}$ so $\mathcal{G}(q)\,\{\xi \in \mathcal{G} : \xi_\Omega(\bar{q}) \in \mathcal{S}_{\bar{q}}\}$. Since the action is free, the vector field $\xi_\Omega(\bar{q})$ defines the elements $\xi(q) \in \mathcal{G}(q)$.

The discrete nonholonomic momentum map J^{nhd} is defined by

$$J^{nhd}(k,t) = \frac{\partial L(k-1,t)}{\partial q^{1i}(k-1,t)}\xi^i_\Omega(k,t), \qquad (k,t) \in \mathcal{R}, \tag{41}$$

where

$$\xi^i_\Omega(k,t) = K^i_\alpha(k,t)\xi^\alpha(k,t), \qquad \xi(q(k,t)) = \xi(k,t) \in \mathcal{G}(q(k,t)). \tag{42}$$

The continuous nonholonomic momentum map J^{nhc} is defined by

$$J^{nhc}(k,t) = \frac{\partial L(k,t)}{\partial \dot{q}^i(k,t)}\xi^i_\Omega(k,t), \qquad (k,t) \in \mathcal{R}. \tag{43}$$

Proposition 1. *Assume that the lagrangian L is invariant under the group action of G and $\xi(q) \in \mathcal{G}(q)$. Then any solution of the Lagrange-d'Alembert equations for a $\mathcal{S}$ (discrete-continuous nonholonomic system) must satisfy the discrete-continuous momentum equation*

$$J^{nhd}(k+1,t) - J^{nhd}(k,t) + \frac{dJ^{nhc}(k,t)}{dt} =$$

$$= \frac{\partial L(k,t)}{\partial q^{1i}(k,t)} K^i_\alpha \xi^{1\alpha}(k,t) + \frac{\partial L(k,t)}{\partial \dot{q}^i(k,t)} \left[\frac{d}{dt}\xi(k,t)\right]^i_\Omega . \quad (44)$$

If $L(k,t) = L(k, q(k), q^1(k))$, for a solution of the discrete Lagrange- d'Alembert equation, we obtain

$$J^{nhd1}_\alpha(k) = \frac{\partial L(k)}{\partial q^{1i}(k)} K^i_\alpha(k)\xi^{1\alpha}(k). \quad (45)$$

If $L(k,t) = L(t, q(t), \dot{q}(t))$, for a solution of the continuous Lagrange- d'Alembert equations, we obtain [4]

$$\frac{dJ^{nhc}_\alpha(t)}{dt} = \frac{\partial L(t)}{\partial \dot{q}^i(t)} K^i_\alpha(t) \frac{\partial \xi^\alpha(t)}{\partial q^e(t)} \dot{q}^e(t). \quad (46)$$

6 The discrete-continuous momentum equation in a moving basis

At the point $\bar{q} \in \Omega$ introduce a basis $\{e_1, \ldots, e_m, e_{m+1}, \ldots, e_r\}$ of the Lie algebra such that the first m elements form a basis of $\mathcal{G}(\bar{q})$. Thus $r = dim\mathcal{G}$ and $m = dim\mathcal{G}(\bar{q})$ which by assumption is locally constant. We can introduce a similar basis $\{e_1(q), \ldots, e_m(q), e_{m+1}(q), \ldots, e_r(q)\}$ for $q \in \Omega$. We consider a change of basis matrix by writing

$$e_\alpha(q(k,t)) = e_\alpha(k,t) = \psi^\beta_\alpha(q(k,t)) e_\beta(\bar{q}(k,t)) = \psi^\beta_\alpha(k,t) e_\beta, \qquad \alpha, \beta = \overline{1,r}. \quad (47)$$

Here, the change of basis matrix ψ^β_α is an $r \times r$ inversable matrix. By definition

$$J^{nhd}_a(k,t) = \frac{\partial L(k,t)}{\partial \dot{q}^{1i}(k,t)} [e_a(k,t)]^i_\Omega, \qquad a = \overline{1,m}, (k,t) \in \mathcal{R},$$

$$J^{nhc}_a(k,t) = \frac{\partial L(k,t)}{\partial \dot{q}^i(k,t)} [e_a(k,t)]^i_\Omega, \qquad a = \overline{1,m}, (k,t) \in \mathcal{R}. \quad (48)$$

Using this notation, the momentum equations, with the choice given by $\xi(q(k,t)) = \xi(k,t) = e_a(k,t)$ $a = \overline{1,m}$, reads as follows:

$$J^{nhd}_a(k+1,t) - J^{nhd}_a(k,t) + \frac{dJ^{nhc}(k,t)}{dt} = \quad (49)$$

$$= \frac{\partial L(k,t)}{\partial q^{1i}(k,t)}[e_a(k+1,t) - e_a(k,t)]^i_\Omega + \frac{\partial L(k,t)}{\partial \dot{q}^i(k,t)}\left[\frac{d}{dt}e_a(k,t)\right]^i_\Omega .$$

We define Christoffel like symbols by

$$\Gamma^\lambda_{ai}(k,t) = \frac{\partial \psi^\beta_a(k,t)}{\partial q^i(k,t)}\tilde{\psi}^\lambda_\beta(k,t) \tag{50}$$
$$\Lambda^\lambda_a(k,t) = [\psi^\beta_a(k+1,t) - \psi^\beta_a(k,t)]\tilde{\psi}^\lambda_\beta(k,t),$$

where $\tilde{\psi}^\lambda_\beta(k,t)$ denotes the inverse of the matrix $\psi^\alpha_\beta(k,t)$.

Observe that

$$\frac{d}{dt}e_a(k,t) = \frac{\partial \psi^\beta_a(k,t)}{\partial q^i}\dot{q}^i(k,t)e_\beta = \Gamma^\lambda_{ai}(k,t)\dot{q}^i(k,t)e_\lambda(k,t) \tag{51}$$

$$e_a(k+1,t) - e_a(k,t) = [\psi^\beta_a(k+1,t) - \psi^\beta_a(k,t)]\tilde{\psi}^\lambda_\beta(k,t)e_\lambda(k,t) = \Lambda^\lambda_a(k,t)e_\lambda(k,t)$$

which implies

$$\left[\frac{d}{dt}e_a(k,t)\right]^i_\Omega = \Gamma^\lambda_{aj}(k,t)\dot{q}^j(k,t)[e_\lambda(k,t)]^i_\Omega$$

$$[e_a(k+1,t) - e_a(k,t)]^i_\Omega = \Lambda^\lambda_a(k,t)[e_\lambda(k,t)]^i_\Omega. \tag{52}$$

Thus we can write the momentum equation as

$$J^{nhd}_a(k+1,t) - J^{nhd}_a(k,t) + \frac{dJ^{nhc}(k,t)}{dt} = \tag{53}$$

$$= \left[\frac{\partial L(k,t)}{\partial q^{1i}(k,t)}\Lambda^\lambda_a(k,t) + \frac{\partial L(k,t)}{\partial \dot{q}^j(k,t)}\Gamma^\lambda_{ai}(k,t)\dot{q}^j(k,t)\right][e_\lambda(k,t)]^i_\Omega.$$

7 Discrete-continuous systems with special constraints

Consider the set $S = \Big\{(q(k,t), q^1(k,t), \dot{q}(k,t) \in \Omega \times \Omega^1 \times \dot{\Omega}, f^a(k,t) = 0,$ $a = \overline{1,p}\Big\}$. An element $\eta(k,t) \in T_{q(k,t)}\Omega$ is called a *discrete virtual variation* of the constrained lagrangian system (Ω, S) if

$$\frac{\partial f^a(k,t)}{\partial q^{1i}(k,t)}\eta^i_d(k,t) = 0, \quad a = \overline{1,p}, \qquad \text{rang}\left(\frac{\partial f^a(k,t)}{\partial q^{1i}(k,t)}\right) = p, \ \forall (k,t) \in \mathcal{R}. \tag{54}$$

An element $\eta(k,t) \in T_{q(k,t)}\Omega$ is called a *continuous virtual variation* of the constrained lagrangian system (Ω, S) if

$$\frac{\partial f^a(k,t)}{\partial \dot{q}^i(k,t)}\eta^i_c(k,t) = 0, \quad a = \overline{1,p}, \qquad \text{rang}\left(\frac{\partial f^a(k,t)}{\partial \dot{q}^i(k,t)}\right) = p, \quad \forall (k,t) \in \mathcal{R}. \tag{55}$$

The Lagrange-d'Alembert principle is:

a) The admissible element $q(k,t) \in \Omega$ is called a *discrete motion* of the constrained lagrangian system (Ω, L, S) if for all discrete virtual variation the equations are

$$E_i(L)\eta_d^i(k,t) = 0, \qquad \forall (k,t) \in \mathcal{R}.$$

b) The admissible element $q(k,t) \in \Omega$ is called a *continuous motion* of the constrained lagrangian system (Ω, L, S) if for all continuous variations we can write the equations

$$E_i(L)\eta_c^i(k,t) = 0, \qquad \forall (k,t) \in \mathcal{R}.$$

Let S_d be the constraint distribution. The constraints in general are nonintegrable. Choose a local coordinate chart for the constraint such that

$$f^a(k,t) = q^{1a}(k,t) + A_\alpha^a(k,t)q^{1\alpha}(k,t) - \gamma^a(k,t) = 0, \;\; a = \overline{1,p}, \tag{56}$$

where

$$A_\alpha^a(k,t) = A_\alpha^a(q(k,t)), \qquad \gamma^a(k,t) = \gamma^a(q(k,t)).$$

The discrete virtual variation for (Ω, S_d) is

$$\eta^a(k,t) + A_\alpha^a(k,t)\eta^\alpha(k,t) = 0.$$

The Lagrange-d'Alembert equation of motion for the discrete motion are

$$E_\alpha(L)(k,t) = A_\alpha^a(k,t)E_a(L)(k,t) \tag{58}$$

$$q^{1a}(k,t) + A_\alpha^a(k,t)q^{1\alpha}(k,t) - \gamma^a(k,t) = 0.$$

Now we define the constrained lagrangian as

$$L_c(k,t) = L(k,t,q(k,t), -A_\alpha^a(k,t)q^{1\alpha}(k,t) + \gamma^a(k,t), q^{1\alpha}(k,t), \dot{q}(k,t)) =$$

$$L_c(k,t,q(k,t), q^{1\alpha}(k,t), \dot{q}(k,t)), \quad a = \overline{1,p}, \quad \alpha = \overline{p+1,m}. \tag{59}$$

Proposition 1. *The discrete equations of motion are*

$$\begin{aligned} E_\alpha(L_c)(k,t) \quad &- \quad A_\alpha^a(k,t)\left[\frac{\partial L_c(k,t)}{\partial q^a(k,t)} - \frac{d}{dt}\left(\frac{\partial L_c(k,t)}{\partial \dot{q}^a(k,t)}\right)\right] = \\ &= \quad [A_\alpha^a(k,t) - A_\alpha^a(k-1,t)]\frac{\partial L(k-1,t)}{\partial q^{1a}(k-1,t)} + \\ &+ \quad B_{\alpha\beta}^b(k,t)\frac{\partial L(k,t)}{\partial q^{1b}(k,t)}q^{1\beta}(k,t) + \gamma_\alpha^b(k,t)\frac{\partial L(k,t)}{\partial q^{1b}(k,t)}, \end{aligned} \tag{60}$$

where

$$B_{\alpha\beta}^b(k,t) = A_\alpha^a(k,t)\frac{\partial A_\beta^b(k,t)}{\partial q^a(k,t)} - \frac{\partial A_\beta^b(k,t)}{\partial q^\alpha(k,t)}$$

$$\gamma_\alpha^b(k,t) = \frac{\partial \gamma^b(k,t)}{\partial q^\alpha(k,t)} - A_\alpha^a(k,t)\frac{\partial \gamma^b(k,t)}{\partial q^a(k,t)}.$$

Remark. If $L(k,t) = L(k, q(k), q^1(k))$ and $q^{1a}(k) + A_\alpha^a(k)q^{1\alpha}(k) - \gamma^a(k) = 0$, $a = \overline{1,p}$, we can write the constrained lagrangian

$$L_c(k) = L(k, q(k), -A_\alpha^a(k)q^{1a}(k) + \gamma^a(k,t), \; q^{1\alpha}(k)) \tag{61}$$

and the discrete equations of motion are

$$E_\alpha(L_c)(k) - A^a_\alpha(k)\frac{\partial L_c(k)}{\partial q^a(k)} = [A^b_\alpha(k) - A^b_\alpha(k-1)]\frac{\partial L(k-1)}{\partial q^{1b}(k-1)} + \tag{62}$$

$$+\left[A^a_\alpha(k)\frac{\partial A^b_\beta(k)}{\partial q^a(k)} - \frac{\partial A^b_\beta(k)}{\partial q^\alpha(k)}\right]\frac{\partial L(k)}{\partial q^{1b}(k)}q^{1\beta}(k) + \left[\frac{\partial \gamma^b(k)}{\partial q^\alpha(k)} - A^a_\alpha(k)\frac{\partial \gamma^b(k)}{\partial q^a(k)}\right]\frac{\partial L(k)}{\partial q^{1b}(k)}.$$

If the constraint distribution is given by

$$f^a(k,t) = \dot{q}^a(k,t) + A^a_\alpha(k,t)\dot{q}^\alpha(k,t) - \gamma^a(k,t) = 0, \quad a = \overline{1,p}, \tag{63}$$

the continuous distribution is given by

$$\eta^a(k,t) + A^a_\alpha(k,t)\eta^\alpha(k,t) = 0, \tag{64}$$

the Lagrange-d′Alembert equation of continuous motion for the system are

$$E_\alpha(L)(k,t) = A^a_\alpha(k,t)E_a(L)(k,t). \tag{65}$$

Let L_c be the constraint time independent lagrangian

$$L_c(k,t) = L(k, q(k,t), -A^a_\alpha(k,t)\dot{q}^\alpha(k,t) + \gamma^a(k,t), q^{1\alpha}(k,t), \dot{q}^\alpha(k,t)), \tag{66}$$

$$a = \overline{1,p}, \qquad \alpha = \overline{p+1,m}$$

Proposition 2. *The continuous equations of motion are*

$$E_\alpha(L_c)(k,t) - A^a_\alpha(k,t)\left[\frac{\partial L_c(k,t)}{\partial q^a(k,t)} + \frac{\partial L_c(k-1,t)}{\partial q^{1a}(k-1,t)} - \frac{\partial L_c(k,t)}{\partial q^{1a}(k,t)}\right] =$$

$$= B^a_{\alpha\beta}(k,t)\frac{\partial L(k,t)}{\partial \dot{q}^a(k,t)}\dot{q}^\beta(k,t) + \frac{\partial L(k,t)}{\partial \dot{q}^a(k,t)} + \gamma^a_\alpha(k,t), \tag{67}$$

where

$$B^a_{\alpha\beta}(k,t) = \frac{\partial A^a_\alpha(k,t)}{\partial q^\beta(k,t)} - \frac{\partial A^a_\beta(k,t)}{\partial q^\alpha(k,t)} + A^b_\alpha(k,t)\frac{\partial A^a_\beta(k,t)}{\partial q^b(k,t)} - A^b_\beta(k,t)\frac{\partial A^a_\alpha(k,t)}{\partial q^b(k,t)}$$

$$\gamma^b_\alpha(k,t) = \frac{\partial \gamma^b(k,t)}{\partial q^\alpha(k,t)} - A^a_\alpha(k,t)\frac{\partial \gamma^b(k,t)}{\partial q^a(k,t)} + \gamma^b(k,t)\frac{\partial A^a_\alpha(k,t)}{\partial q^b(k,t)}. \tag{68}$$

Remark. If $L(k,t) = L(q(t), \dot{q}(t))$ and $\dot{q}^a + A^a_\alpha\dot{q}^\alpha - \gamma^a = 0$, $a = \overline{1,p}$, we obtain

$$L_c(t) = L(q(t), -A^a_\alpha(t)\dot{q}^\alpha(t) + \gamma^a, \dot{q}^\alpha(t)). \tag{69}$$

The motion of the system is given by

$$E_\alpha(L_c)(t) - A^a_\alpha(t)\frac{\partial L_t(k)}{\partial q^a(t)} = B^a_{\alpha\beta}(t)\frac{\partial L(t)}{\partial \dot{q}^a(t)}\dot{q}^\beta(t) + \gamma^a_\alpha(t)\frac{\partial L(t)}{\partial \dot{q}^a(t)}. \tag{70}$$

These equations are derived in [4].

Examples.

We now consider two detailed examples to illustrate the theory developed above. Although these examples are of strictly academic interest, they ilustrate the basic concepts and indicate how more complicated examples should be attacked.

1 A nonholonomically constrained particle

Consider a particle with the lagrangian

$$L(k) = \frac{1}{2}\left\{[q^{11}(k)]^2 + [q^{12}(k)]^2 + [q^{13}(k)]^2\right\}, \qquad k \in [0, N],$$

where $q^{1i}(k) = q^i(k+1) - q^i(k)$, $i = 1, 2, 3$, $k \in [0, N]$ and the nonholonomic discrete constraint

$$\text{(1)} \qquad \mathcal{S}: \quad q^{13}(k) = q^2(k) q^{11}(k).$$

We obtain $A_1^3(k) = -q^2(k)$, $A_2^3(k) = 0$ and the constraint lagrangian is

$$\text{(2)} \qquad L_c(k) = \frac{1}{2}[1 + (q^2(k))][q^{11}(k)]^2 + \frac{1}{2}[q^{12}(k)]^2.$$

The corresponding equation of motion are

$$\frac{\partial L_c(k)}{\partial q^\alpha(k)} - \frac{\partial L_c(k)}{\partial q^{1\alpha}(k)} + \frac{\partial L_c(k-1)}{\partial q^{1\alpha}(k-1)} - A_\alpha^a(k)\frac{\partial L_c(k)}{\partial q^a(k)} =$$

$$= \left[A_\alpha^a(k)\frac{\partial A_\beta^b(k)}{\partial q^a(k)} - \frac{\partial A_\beta^b(k)}{\partial q^\alpha(k)}\right] q^{1\beta}(k)\frac{\partial L(k)}{\partial q^{1b}(k)} - (A_\alpha^a(k) - A_\alpha^a(k-1))\frac{\partial L(k-1)}{\partial q^{1a}(k-1)}.$$

Here we take $a = 3$, $\alpha = 1, 2$, and so

$$\text{(3)} \qquad \frac{\partial L_c(k)}{\partial q^1(k)} - \frac{\partial L_c(k)}{\partial q^{11}(k)} + \frac{\partial L_c(k-1)}{\partial q^{11}(k-1)} = -(A_1^3(k) - A_1^3(k-1))\frac{\partial L(k-1)}{\partial q^{13}(k-1)}$$

$$\frac{\partial L_c(k)}{\partial q^2(k)} - \frac{\partial L_c(k)}{\partial q^{12}(k)} + \frac{\partial L_c(k-1)}{\partial q^{12}(k-1)} = -\frac{\partial A_1^3(k)}{\partial q^2(k)} q^{11}(k)\frac{\partial L(k)}{\partial q^{13}(k)}.$$

The equations of motion now become

$$[1 + q^2(k)^2]q^{11}(k) - [1 + q^2(k-1)^2]q^{11}(k-1) = (q^2(k) - q^2(k-1))q^{13}(k-1)$$

$$\text{(4)} \qquad q^2(k)[q^{11}(k)]^2 - q^{12}(k) + q^{12}(k-1) = q^{11}(k)q^{13}(k).$$

Together with the constrained equation (1) this completely specifies the motion. Using (1) and (4) we obtain

$$q^{12}(k) - q^{12}(k-1) = 0$$

$$\text{(5)} \qquad [1 + q^2(k-1)q^2(k)]q^{11}(k) + [1 + q^2(k-1)^2]q^{11}(k-1) = 0.$$

Then $q^{12}(k) = q^{12}(k-1)$, gives $q^2(k) = q^2(0) + kc$. The constraints and the lagrangian are invariant under the $\mathbf{R}^2$ action on Ω given by

$$(q^1(k), q^2(k), q^3(k)) \to (q^1(k) + \lambda, q^2(k), q^3(k) + \mu) \quad \forall(\lambda, \mu) \in \mathbf{R}^2, \forall k \in [0, N].$$

We obtain

$$T_{q(k)} orb(q(k)) = span\,\{(1,0,0);(0,0,1)\}.$$

The tangent space to the orbits of this action is given by

$$\mathcal{S}_{q(k)} = span\left\{(1,0,q^2(k)),(0,1,0)\right\},$$

and the constraint distribution is

$$\mathcal{S}_{q(k)} \cap T_{q(k)} orb(q(k)) = span\left\{(1,0,q^2(k))\right\},$$

and thus

$$\xi_\Omega^{q(k)} = (1,0,q^2(k)), \qquad \xi^{q(k)} = (1,q^2(k)).$$

We obtain

$$K_1^1(k)=1 \;\; K_2^1(k)=0 \;\; K_1^2(k)=0 \;\; K_2^2(k)=0 \;\; K_1^3(k)=0 \;\; K_2^3(k)=1 \;\; . \tag{6}$$

The nonholonomic momentum map is

$$\begin{aligned} J^{nhd}(\xi^q) &= \tfrac{\partial L(k-1)}{\partial q^{1i}(k-1)}\xi_\Omega^{iq(k)} = \\ &= <(q^{11}(k-1),q^{12}(k-1),q^{13}(k-1)),(1,0,q^2(k))>= \\ &= <(q^{11}(k),q^{12}(k),q^{13}(k)),(0,0,q^{12}(k))>= q^{13}(k)q^{12}(k). \end{aligned} \tag{7}$$

Hence the momentum equation becomes

$$q^{11}(k)+q^2(k+1)q^{13}(k)-q^{11}(k-1)-q^2(k)q^{13}(k-1)=\frac{\partial L(k)}{\partial q^{1i}(k)}K_\alpha^i(k)\xi^{1\alpha}(k)=$$

$$=<(q^{11}(k),q^{12}(k),q^{13}(k)),(0,0,q^{12}(k))>= q^{13}(k)q^{12}(k). \tag{8}$$

$$q^{11}(k)-q^{11}(k-1)+q^2(k)[q^{13}(k)-q^{13}(k-1)]=0. \tag{9}$$

Using (1) in (9) the momentum equation map be rewritten as

$$q^{11}(k)-q^{11}(k-1)+\frac{q^2(k)}{1+q^2(k)^2}q^{11}(k-1)q^{12}(k-1)=0. \tag{10}$$

We can illustrate the momentum equation in a moving basis. Choose a fix basis for the Lie algebra of $\mathcal{G}=\mathbf{R}^2$ namely $e_1=(1,0)$, $e_2=(0,1)$; we have $\xi^q = e_1 + q^2(k)e_2$ and the moving basis

$$e_1(q(k)) = (1,q^2(k)), \qquad e_2(q(k)) = (0,1).$$

2 A homogeneuous ball on a rotating plate

The configuration space of the sphere is $Q = \mathbf{R}^2 \times SO(3)$ parameterized by (x, y, R), $R \in SO(3)$. Let $\omega = (\omega_x, \omega_y, \omega_z)$ be the angular velocity vector of the sphere measured with respect to the inertial frame (Ω constant about z axis), let m the mass of the sphere, mk^2 its inertia about the Oz axis and a its radius. The lagrangian of the system is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}mk^2(\omega_x^2 + \omega_y^2 + \omega_z^2), \tag{1}$$

with the affine nonholonomic constraints

$$\dot{x} - a\omega_y = -\Omega y, \quad \dot{y} + a\omega_x = \Omega x. \tag{2}$$

The angular momentum of the ball about the z-axis is conserved since the lagrangian is invariant under rotations about the z-axis.

The conservation law and the constraints are

$$\omega_x + \frac{1}{a}\dot{y} = \frac{\Omega x}{a}, \quad \omega_y - \frac{1}{a}\dot{x} = \frac{\Omega y}{a}, \quad \omega_z = c \tag{3}$$

(up to an irrelevant constant)

$$L_c = \frac{1}{2a^2}\left\{m(a^2 + k^2)(\dot{x}^2 + \dot{y}^2) + mk^2\Omega^2(x^2 + y^2) + 2mk^2(\Omega y\dot{x} - \Omega x\dot{y}).\right\}$$

The equation of motion are

$$\ddot{x} + \frac{k^2\Omega}{a^2 + k^2}\dot{y} = 0, \quad \ddot{y} - \frac{k^2\Omega}{a^2 + k^2}\dot{x} = 0. \tag{5}$$

These equations may also be derived by considering the momentum equations associated with the system. The constraint distribution given by the two kinematic constraints is

$$\mathcal{S}_q = span\left\{(a, 0, 0) + \xi_y, (0, -a, 0) + \xi_x, \xi_z\right\}, \tag{6}$$

where ξ_x, ξ_y, ξ_z denote the infinitesimal generators of the rotations about the Ox, Oy 'si Oz axes of the ball. Now we consider the action of the group $\mathbf{R}^2 \times SO(3)$ on the configuration space (clearly the lagrangian is invariant under this action). We have

$$\mathcal{S}_q \cap T_q(orb\ q) = \mathcal{S}_q. \tag{7}$$

The nonholonomic momentum map J^{nhc} has three components:

$$\begin{aligned} J_1 &= \frac{\partial L}{\partial \dot{q}^j}(\xi_x)^j_\Omega = am\dot{x} + mk^2\omega_y \\ J_2 &= \frac{\partial L}{\partial \dot{q}^j}(\xi_y)^j_\Omega = -am\dot{y} + mk^2\omega_x \\ J_3 &= \frac{\partial L}{\partial \dot{q}^j}(\xi_z)^j_\Omega = mk^2\omega_z. \end{aligned} \tag{8}$$

In all cases ξ^q are independent of q so the momentum equations are simply

$$\frac{dJ_1}{dt} = 0, \qquad \frac{dJ_2}{dt} = 0, \qquad \frac{dJ_3}{dt} = 0. \tag{9}$$

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