Family of Projective Projections on Tensors and Connections

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Abstract

§1 finds the explicit expressions for all projective projections on the set of (1,2)-tensors. §2 analyses the action of extended projective projections on the set of connections and shows that in particular one gets the classical Thomas connection. §3 gives properties of the almost projective transformations of connections.

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Introduction

The theory of invariant decompositions of tensors and connections using global projections built with the Kronecker δ-tensor or the δ-tensor together with the Riemannian metric, almost complex structure, almost contact structure etc have been initiated by the first author in 1975. It was discussed by letters (1976-1977) with Prof.Dr. Lieven Vanhecke and was orally communicated as remarks at different Conferences in Geometry. A part of this theory is detailed in this paper on (1,2)-tensors and connections, but of course it can be generalized for (p,q)-tensors. The most interesting generalization is to apply the theory for the curvature (1,3)-tensor, and to relate the projection on connections with the projection on the corresponding curvature tensors, but this subject will be developed in another paper.

As was remarked by Krupka [4], [6] whose invariant trace decompositions are special cases of ours, the results can be applied in the representation theory of the orthogonal group, developed by Weyl [12]. Extensive literature on this subject can be found from different perspectives. For examples, N.Bokan [1] considers the case of a torsion free connection on a space endowed with a positive definite metric and finds a decomposition of the underlying tensor space, invariant with respect to the group SO(n).

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1 Family of projective projections on (1,2)-tensors

Let V be a real n-dimensional vector space, where $n \geq 2$, $T^1_2(V) = \{T^a_{bc}\}$ be the vector space of all tensors T of type (1,2), δ^i_j be the symbol of Kronecker, $I = \{\delta^r_a \delta^b_s \delta^c_t\}$ be the identity on $T^1_2(V)$.

A projection $P = \{P_a^{bc}{}^r_{st}\}$ on $T^1_2(V)$ of the form

$$P_a^{bc}{}^r_{st} = x_1 \delta^r_a \delta^b_s \delta^c_t + x_2 \delta^b_a \delta^r_s \delta^c_t + x_3 \delta^r_a \delta^c_s \delta^b_t + x_4 \delta^c_a \delta^b_s \delta^r_t + x_5 \delta^c_a \delta^r_s \delta^b_t + x_6 \delta^b_a \delta^c_s \delta^r_t$$

is called a *projective projection.* The adjective "projective" is justified by the fact that there exist induced projections P which transform a symmetric connection into the Thomas projective connection (see Section 2).

Of course, P is a projection iff $P^2 = P$ or $P_a^{bc}{}^r_{st} P_r^{st}{}^i_{jk} = P_a^{bc}{}^i_{jk}$, i.e.,

$$(1.1) \qquad \begin{cases} x_1^2 + x_3^2 = x_1 \\ 2x_1x_2 + nx_2^2 + x_2x_5 + x_3x_5 + x_2x_6 + x_3x_6 + nx_5x_6 = x_2 \\ 2x_1x_3 = x_3 \\ 2x_1x_4 + x_3x_6 + nx_4^2 + x_4x_6 + x_3x_5 + x_4x_5 + nx_5x_6 = x_4 \\ 2x_1x_5 + x_2x_3 + x_2x_4 + x_3x_4 + nx_4x_5 + nx_2x_5 + x_5^2 = x_5 \\ 2x_1x_6 + x_2x_3 + x_2x_4 + nx_2x_6 + x_3x_4 + nx_4x_6 + x_6^2 = x_6. \end{cases}$$

This algebraic system is easily obtained via the simplified expression $P = x_1 I_1 + ... + x_6 I_6$, the condition $P^2 = P$ and the table of compositions

*	I_1	I_2	I_3	I_4	I_5	I_6
I_1	I_1	I_2	I_3	I_4	I_5	I_6
I_2	I_2	nI_2	I_6	I_6	I_2	nI_6
I_3	I_3	I_5	I_1	I_6	I_2	I_4
I_4	I_4	I_5	I_5	nI_4	nI_5	I_4
I_5	I_5	nI_5	I_4	I_4	I_5	nI_4
I_6	I_6	I_2	I_2	nI_6	nI_2	I_6

If $P = \{P_a^{bc}{}^r_{st}\}$ is a projective projection, then its supplement $Q = I - P$ is also a projective projection. That in way the following theorem is true.

Theorem 1.1.. *If* $(x_1^0, ..., x_6^0)$ *is a solution for the algebraic system (1.1), then* $(1 - x_1^0, -x_2^0, ..., -x_6^0)$ *is also a solution.*

The projective projection P belongs to the class of invariant tensors studied by D.Krupka and J.Janyska [5] (a tensor $T \in T^r_r(V)$ being invariant iff $A \circ T = T$, for any $A \in GL(V)$).

Let us solve the system (1.1). For that reason we start with

$$(1.2) \qquad \begin{cases} x_1^2 + x_3^2 = x_1 \\ 2x_1x_3 = x_3. \end{cases}$$

This system is equivalent to

$$(1.3)\quad \begin{cases} x_1 + x_3 = 0 \\ 2x_1x_3 = x_3 \end{cases} \quad \text{or} \quad (1.4)\quad \begin{cases} x_1 + x_3 = 1 \\ 2x_1x_3 = x_3. \end{cases}$$

From (1.3) we obtain $\begin{cases} x_1 = 0 \\ x_3 = 0 \end{cases}$ or $\begin{cases} x_1 = \frac{1}{2} \\ x_3 = -\frac{1}{2}. \end{cases}$ The supplementary solutions $\begin{cases} x_1 = 1 \\ x_3 = 0 \end{cases}$ or $\begin{cases} x_1 = \frac{1}{2} \\ x_3 = \frac{1}{2}. \end{cases}$ are obtained from (1.4). We solve the initial system (1.1) for $x_1 = 0$, $x_3 = 0$ and for $x_1 = x_3 = \frac{1}{2}$. The other two cases are obtained taking supplementary solutions for the system (1.1).

I. In the case $x_1 = x_3 = 0$, from (1.1) we get the system

$$(1.5)\quad \begin{cases} x_2(1 - nx_2 - x_5) = x_6(x_2 + nx_5) \\ x_5(1 - nx_2 - x_5) = x_4(x_2 + nx_5) \\ x_4(1 - nx_4 - x_6) = x_5(x_4 + nx_6) \\ x_6(1 - nx_4 - x_6) = x_2(x_4 + nx_6). \end{cases}$$

Multiplying the first equation with the third equation and the second equation with the fourth equation, we get

$$(1.6)\quad \begin{cases} x_2x_4(1 - nx_2 - x_5)(1 - nx_4 - x_6) = x_5x_6(x_2 + nx_6)(x_4 + nx_6) \\ x_5x_6(1 - nx_2 - x_5)(1 - nx_4 - x_6) = x_2x_4(x_2 + nx_5)(x_4 + nx_6). \end{cases}$$

A). We study the case $x_2x_4 - x_5x_6 \neq 0$. From (1.5) we obtain

$$\begin{cases} nx_2 + x_5 = 1 \\ x_2 + nx_5 = 0 \end{cases} \quad \text{and} \quad \begin{cases} x_4 + nx_6 = 0 \\ x_6 + nx_4 = 1. \end{cases}$$

We get the solution

$$\begin{cases} x_2 = x_4 = -\dfrac{n}{1 - n^2} \\ x_5 = x_6 = \dfrac{1}{1 - n^2}. \end{cases}$$

B). We study the case $x_2x_4 = x_5x_6$.

B1). Let $x_2x_4 = x_5x_6 \neq 0$. The system (1.6) implies

$$(1 - nx_2 - x_5)(1 - nx_4 - x_6) = (x_2 + nx_5)(x_4 + nx_6).$$

We find the system

$$\begin{cases} n(x_2 + x_4) + x_5 + x_6 = 1 \\ x_2x_4 = x_5x_6. \end{cases}$$

B2). Let $x_2x_4 = x_5x_6 = 0$.

a) If $x_2 = x_5 = 0$, then (1.5) becomes $\begin{cases} x_4(1 - nx_4 - x_6) = 0 \\ x_6(1 - nx_4 - x_6) = 0. \end{cases}$ We get $x_2 = x_4 = x_5 = x_6 = 0$ or $\begin{cases} x_2 = x_5 = 0 \\ 1 = nx_4 + x_6. \end{cases}$

b) If $x_2 = x_6 = 0$, then (1.5) implies $\begin{cases} x_5(1 - x_5 - nx_4) = 0 \\ x_4(1 - x_5 - nx_4) = 0. \end{cases}$ We find $x_2 = x_4 = x_5 = x_6 = 0$ or $\begin{cases} x_2 = x_6 = 0 \\ 1 = nx_4 + x_5. \end{cases}$

c) If $x_4 = x_5 = 0$, then (1.5) becomes $\begin{cases} x_2(1 - nx_2 - x_6) = 0 \\ x_6(1 - nx_2 - x_6) = 0. \end{cases}$ Consequently $x_2 = x_4 = x_5 = x_6 = 0$ or $\begin{cases} x_4 = x_5 = 0 \\ 1 = nx_2 + x_6. \end{cases}$

d) If $x_4 = x_6 = 0$, then (1.5) gives $\begin{cases} x_2(1 - nx_2 - x_5) = 0 \\ x_5(1 - nx_2 - x_5) = 0. \end{cases}$ We get $x_2 = x_4 = x_5 = x_6 = 0$ or $\begin{cases} x_4 = x_6 = 0 \\ 1 = nx_2 + x_5. \end{cases}$

II. The case $x_1 = x_3 = \frac{1}{2}$. From (1.1) we get

$$\begin{cases} nx_2^2 + x_2x_5 + \frac{1}{2}x_5 + \frac{1}{2}x_6 + x_2x_6 + nx_5x_6 = 0 \\ nx_4^2 + x_4x_6 + \frac{1}{2}x_5 + \frac{1}{2}x_6 + x_4x_6 + nx_5x_6 = 0 \\ \frac{1}{2}x_2 + \frac{1}{2}x_4 + x_2x_4 + nx_4x_5 + nx_2x_5 + x_5^2 = 0 \\ \frac{1}{2}x_2 + \frac{1}{2}x_4 + x_2x_4 + nx_2x_6 + nx_4x_6 + x_6^2 = 0. \end{cases} \tag{1.7}$$

From the first two equations of the system (1.7) we obtain

$$(x_2 - x_4)[(x_2 + x_4)n + x_5 + x_6] = 0.$$

From the last equations of the system (1.7) we obtain $(x_5 - x_6)[(x_2 + x_4)n + x_5 + x_6] = 0$.

Let

$$\begin{cases} x_2 = x_4 \\ x_5 = x_6. \end{cases}$$

From the first equation and the third equation of (1.7)

$$\begin{cases} nx_2^2 + 2x_2x_5 + x_5 + nx_5^2 = 0 \\ x_2^2 + 2nx_2x_5 + x_2 + x_5^2 = 0 \end{cases}$$

we get $(x_2 - x_5)[(n - 1)(x_2 - x_5) - 1] = 0$. If $x_2 = x_5$, then $x_2[(2n + 2)x_2 + 1] = 0$. We obtain

$$(1.8_1) \begin{cases} x_2 = x_4 = 0 \\ x_5 = x_6 = 0, \end{cases}$$

or

$$x_2 = x_4 = x_5 = x_6 = -\frac{1}{2(n + 1)}. \tag{1.8$_2$}$$

If $x_2 = x_5 + \frac{1}{n - 1}$, then we obtain the solution

$$\begin{cases} x_2 = x_4 = \dfrac{1}{2(n-1)} \\ x_5 = x_6 = \dfrac{1}{2(1-n)} \end{cases} \quad \text{or} \quad \begin{cases} x_2 = x_4 = \dfrac{1}{n^2-1} \\ x_5 = x_6 = \dfrac{n}{1-n^2}. \end{cases}$$

The system

$$\begin{cases} n(x_2+x_4)+x_5+x_6=0 \\ nx_2^2+x_2x_5+\frac{1}{2}x_6+\frac{1}{2}x_5+x_2x_6+nx_5x_6=0 \\ \frac{1}{2}x_2+\frac{1}{2}x_4+x_2x_4+nx_4x_5+nx_2x_5+x_5^2=0 \end{cases}$$

is equivalent to

$$\begin{cases} n(x_2+x_4)+x_5+x_6=0 \\ x_5x_6=\frac{1}{2}(x_2+x_4)+x_2x_4. \end{cases}$$

Theorem 1.2. *The solutions of the quadratic system (1.1) and hence the set of all projective projections P on $T_2^1(V)$ are given by*

I.

a) $x_1=x_2=x_3=x_4=x_5=x_6=0$.

b) $x_1=0,\ x_3=0,\ x_2=x_4=\dfrac{n}{n^2-1},\ x_5=x_6=\dfrac{1}{1-n^2}$;

c) $x_1=x_3=0,\ x_2x_4=x_5x_6,\ 1=n(x_2+x_4)+x_5+x_6$.

Introducing the parameters $x_5=\lambda,\ x_6=\mu$ and imposing the condition $\left(\dfrac{\lambda+\mu-1}{n}\right)^2 \geq 4\lambda\mu,\ \lambda,\mu\in\mathbf{R}$, the values $x_2=\alpha,\ x_4=\beta$ are solutions of the equation $z^2+\dfrac{1}{n}(\lambda+\mu-1)z+\lambda\mu=0$.

The supplementary solutions for I are

I′.

a) $x_1=1,\ x_3=x_2=x_4=x_5=x_6=0$;

b) $x_1=1,\ x_3=0,\ x_2=x_4=\dfrac{n}{1-n^2},\ x_5=x_6=\dfrac{1}{n^2-1}$;

c) $x_1=1,\ x_3=0,\ x_2x_4=x_5x_6,\ -1=n(x_2+x_4)+x_5+x_6$.

Denoting $x_5=\lambda,\ x_6=\mu$, and imposing the condition $\left(\dfrac{\lambda+\mu+1}{n}\right)^2 \geq 4\lambda\mu,\ \lambda,\mu\in\mathbf{R}$, the values $x_2=\alpha,\ x_4=\beta$ are solutions of the equation $z^2+\dfrac{1}{n}(\lambda+\mu+1)z+\lambda\mu=0$.

II.

a) $x_1=x_3=\dfrac{1}{2},\ x_2=x_4=x_5=x_6=0$;

b) $x_1=x_3=\dfrac{1}{2},\ x_2=x_4=x_5=x_6=-\dfrac{1}{2(n+1)}$;

c) $x_1=x_3=\dfrac{1}{2},\ x_2=x_4=-x_5=-x_6=\dfrac{1}{2(n-1)}$;

d) $x_1 = x_3 = \frac{1}{2},\ x_2 = x_4 = \frac{1}{n^2-1},\ x_5 = x_6 = \frac{n}{1-n^2}$.

e) $x_1 = x_3 = \frac{1}{2},\ n(x_2+x_4)+x_5+x_6 = 0,\ x_5x_6 = \frac{1}{2}(x_2+x_4)+x_2x_4$.

With $x_5 = \lambda,\ x_6 = \mu,\ \left[\frac{1}{n}(\lambda+\mu)-1\right]^2 \geq 1+4\lambda\mu,\ \lambda,\mu \in \mathbf{R}$, *the values* $x_2 = \alpha,\ x_4 = \beta$ *are solutions of the equation* $z^2 + \frac{1}{n}(\lambda+\mu)z + \lambda\mu + \frac{1}{2n}(\lambda+\mu) = 0$.

The supplementary solutions for II are

II′.

a) $x_1 = -x_3 = \frac{1}{2},\ x_2 = x_4 = x_5 = x_6 = 0$;

b) $x_1 = -x_3 = \frac{1}{2},\ x_2 = x_4 = x_5 = x_6 = \frac{1}{2(n+1)}$;

c) $x_1 = -x_3 = \frac{1}{2},\ x_2 = x_4 = -x_5 = -x_6 = \frac{1}{2(1-n)}$;

d) $x_1 = -x_3 = \frac{1}{2},\ x_2 = x_4 = \frac{1}{1-n^2},\ x_5 = x_6 = \frac{n}{n^2-1}$.

e) $x_1 = -x_3 = \frac{1}{2},\ n(x_2+x_4)+x_5+x_6 = 0,\ x_5x_6 = -\frac{1}{2}(x_2+x_4)+x_2x_4$.

With $x_5 = \lambda,\ x_6 = \mu,\ \left[\frac{1}{n}(\lambda+\mu)+1\right]^2 \geq 1+4\lambda\mu,\ \lambda,\mu \in \mathbf{R}$, *the values* $x_2 = \alpha,\ x_4 = \beta$ *are solutions of the equation* $z^2 + \frac{1}{n}(\lambda+\mu)z + \lambda\mu - \frac{1}{2n}(\lambda+\mu) = 0$.

The images of a (1,2)-tensor by the precedent projections are obvious and contain the following generalizations of the results of Krupka [4].

Theorem 1.3. *Let* $T = (T^a_{bc}) \in T^1_2(V)$. *There exists an infinity of projective projections* $P = (P^{bc}_{a}{}^{r}_{st})$ *such that* $\Omega = PT = (P^{bc}_{a}{}^{r}_{st}T^a_{bc})$ *is a traceless tensor (i.e.* $\Omega^s_{st} = \Omega^s_{ts} = 0$*).*

I. $T^a_{at} = 0$.

1) $\Omega = 0$ *for* $x_1 = x_3 = x_4 = x_5 = 0,\ nx_2 + x_6 = 1$;

2) $\Omega^r_{st} = T^r_{st} - \frac{1}{1-n^2}(T^a_{ta}\delta^r_s - nT^a_{sa}\delta^r_t)$ *for*

$x_1 = 1,\ x_3 = 0,\ x_6 + nx_2 = 0,\ x_4 = \frac{n}{1-n^2},\ x_5 = \frac{1}{n^2-1}$;

3) $\Omega^r_{st} = \frac{1}{2}(T^r_{st}+T^r_{ts}) + \delta^r_s x_5 T^a_{ta} - \delta^r_t T^a_{sa}(\frac{1}{2}+nx_5)$, *for* $x_1 = x_3 = \frac{1}{2},\ \frac{1}{2}+x_4+nx_5 = 0,\ n(x_2+x_4)+x_5+x_6 = 0,\ x_5x_6 = \frac{1}{2}(x_2+x_4)+x_2x_4,\ \frac{1}{2}+x_6+nx_2 \neq 0$;

4) $\Omega^r_{st} = \frac{1}{2}(T^r_{st}-T^r_{ts}) + x_5\delta^r_s T^a_{ta} + (\frac{1}{2}-nx_5)\delta^r_t T^a_{sa}$, *for* $x_1 = -x_3 = \frac{1}{2};\ -\frac{1}{2}+x_4+nx_5 = 0,\ n(x_2+x_4)+x_5+x_6 = 0,\ x_5x_6 = -\frac{1}{2}(x_2+x_4)+x_2x_4,\ \frac{1}{2}+x_6+nx_2 \neq 0$.

II. $T^a_{at} = T^a_{ta} = 0$.

1) $\Omega = T$, *for* $x_1 = 1,\ x_2 = x_3 = x_4 = x_5 = x_6 = 0$;

2) $\Omega = 0$, *for* $x_1 = x_3 = 0,\ x_2 = x_4 = \frac{n}{n^2-1},\ x_5 = x_6 = \frac{1}{1-n^2}$;

3) $\Omega^r_{st} = \frac{1}{2}(T^r_{st} + T^r_{ts})$, *for* $x_1 = x_3 = \frac{1}{2},\ x_2 = x_4 = -x_5 = -x_6 = \frac{1}{2(n-1)}$;

4) $\Omega^r_{st} = \frac{1}{2}(T^r_{st} - T^r_{ts})$, *for* $x_1 = -x_3 = \frac{1}{2},\ x_2 = x_4 = x_5 = x_6 = \frac{1}{2(n+1)}$.

III. $T^a_{at} = -T^a_{ta}$.

1) $\Omega^r_{st} = \frac{1}{2}(T^r_{st} + T^r_{ts})$, *for* $x_1 = x_3 = \frac{1}{2},\ x_2 = x_4 = x_5 = x_6 = 0$;

2) $\Omega^r_{st} = \frac{1}{2}(T^r_{st} - T^r_{ts}) + \frac{1}{1-n}(\delta^r_s T^a_{at} - \delta^r_t T^a_{as})$, *for* $x_1 = -x_3 = \frac{1}{2},\ x_5 = x_6 = \frac{n}{n^2-1},\ x_2 = x_4 = \frac{1}{1-n^2}$.

IV. $T^a_{at} = T^a_{ta}$.

1) $\Omega^r_{st} = \frac{1}{2}(T^r_{st} + T^r_{ts}) - \frac{1}{n+1}(\delta^r_s T^a_{at} + \delta^r_t T^a_{as})$, *for* $x_1 = x_3 = \frac{1}{2}, x_2 = x_4 = \frac{1}{n^2-1},\ x_5 = x_6 = \frac{n}{1-n^2}$;

2) $\Omega^r_{st} = \frac{1}{2}(T^r_{st} - T^r_{ts})$, *for*

a) $x_1 = -x_3 = \frac{1}{2},\ x_2 = x_4 = x_5 = x_6 = 0$;

b) $x_1 = -x_3 = \frac{1}{2},\ x_2 = x_4 = -x_5 = -x_6 = \frac{1}{2(1-n)}$;

V. $T^a_{ta} = -\frac{x_6 + nx_2}{x_4 + nx_5} T^a_{at} \Rightarrow \Omega = 0$, *for*

$$x_4 \neq -nx_5 \quad x_2 x_4 = x_5 x_6,\ 1 = n(x_2 + x_4) + x_5 + x_6.$$

VI.

$$T^a_{ta} = -\frac{1 + x_6 + nx_2}{x_4 + nx_5} T^a_{at} \Rightarrow \Omega^r_{st} = T^r_{st} - \frac{x_4}{x_4 + nx_5} T^a_{as}\delta^r_t - \frac{x_5}{x_4 + nx_5} T^a_{at}\delta^r_s,$$

for

$$x_4 \neq -nx_5,\ x_2 x_4 = x_5 x_6,\ -1 = n(x_2 + x_4) + x_5 + x_6.$$

VII.

$$T^a_{ta} = -\frac{1 + 2x_6 + 2nx_2}{1 + 2x_4 + 2nx_5} T^a_{at}.$$

$$\Omega^r_{st} = \frac{1}{2}(T^r_{st} + T^r_{ts}) + \frac{x_6 + x_2}{1 + 2x_4 + 2nx_5} T^a_{as}\delta^r_t - \frac{x_4 + x_5}{1 + 2x_4 + 2nx_5} T^a_{at}\delta^r_s,$$

for

$$x_1 = x_3 = \frac{1}{2},\ n(x_2 + x_4) + x_5 + x_6 = 0,\ x_5 x_6 = \frac{1}{2}(x_2 + x_4) + x_2 x_4,\ x_4 + nx_5 \neq -\frac{1}{2}.$$

VIII.

$$T^a_{ta} = -\frac{1 + 2x_6 + 2nx_2}{1 + 2x_4 + 2nx_5} T^a_{at}.$$

$$\Omega^r_{st} = \frac{1}{2}(T^r_{st} - T^r_{ts}) + \frac{x_4 - x_5}{-1 + 2x_4 + 2nx_5}(T^a_{at}\delta^r_s - T^a_{as}\delta^r_t),$$

for

$x_1 = \frac{1}{2},\ x_3 = -\frac{1}{2},\ n(x_2+x_4)+x_5+x_6 = 0,\ x_5x_6 = -\frac{1}{2}(x_2+x_4)+x_2x_4,\ x_4+nx_5 \neq \frac{1}{2}.$

IX. *For any T having arbitrary traces one gets*

1) $\Omega^r_{st} = T^r_{st} + \frac{1}{n^2-1}[\delta^r_s(-nT^a_{at} + T^a_{ta}) + \delta^r_t(T^a_{as} - nT^a_{sa}]$, *for*

$$x_1 = 1,\ x_3 = 0,\ x_2 = x_4 = \frac{n}{1-n^2},\ x_5 = x_6 = \frac{1}{n^2-1}.$$

2) $\Omega^r_{st} = \delta^r_s(x_2T^a_{at} + \frac{1}{1-n^2}T^a_{ta}) + \delta^r_t(-nx_2T^a_{as} + \frac{n}{1-n^2}T^a_{sa})$, *for*

$$x_1 = x_3 = 0,\ x_5 = \frac{1}{1-n^2},\ x_4 = \frac{n}{n^2-1},\ x_6 = -nx_2;$$

3) $\Omega = 0$, for $x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 0$;

4) $\Omega^r_{st} = T^r_{st}+\delta^r_s x_2T^a_{at}-\delta^r_t(1+nx_2)T^a_{as}$, for $x_1 = 1, x_3 = x_4 = x_5 = 0, nx_2+x_6+1 = 0$;

5) $\Omega^r_{st} = \frac{1}{2}(T^r_{st} + T^r_{ts}) - \frac{1}{2(n+1)}(T^a_{at} + T^a_{ta})(\delta^r_s + \delta^r_t)$, *for*

$$x_1 = x_3 = \frac{1}{2},\ x_2 = x_4 = x_5 = x_6 = -\frac{1}{2(n+1)};$$

6) $\Omega^r_{st} = \frac{1}{2}(T^r_{st}+T^r_{ts})+\delta^r_s[x_2T^a_{at}+\frac{1}{2(1-n)}T^a_{ta}]+\delta^r_t[-(\frac{1}{2}+nx_2)T^a_{as}+\frac{1}{2(n-1)}T^a_{sa}]$, *for*

$$x_1 = x_3 = \frac{1}{2},\ nx_2 + x_6 + \frac{1}{2} = 0,\ x_4 = -x_5 = \frac{1}{2(n-1)};$$

7) $\Omega^r_{st} = \frac{1}{2}(T^r_{st}+T^r_{ts})+\delta^r_s[x_2T^a_{at}+\frac{1}{2(1+n)}T^a_{ta}]+\delta^r_t[-(\frac{1}{2}+nx_2)T^a_{as}+\frac{1}{2(n+1)}T^a_{sa}]$, *for*

$$x_1 = -x_3 = \frac{1}{2},\ x_6 = -nx_2 - \frac{1}{2},\ x_4 = x_5 = \frac{1}{2(n+1)};$$

8) $\Omega^r_{st} = T^r_{st} + \delta^r_s(\frac{n}{1-n^2}T^a_{at} - nx_4T^a_{ta}) + +\delta^r_t(\frac{1}{n^2-1}T^a_{as} + x_4T^a_{ta})$, *for*

$$x_1 = 1,\ x_3 = 0,\ x_2 = \frac{n}{1-n^2},\ x_6 = \frac{1}{n^2-1},\ x_5 = -nx_4.$$

9) $\Omega^r_{st} = T^r_{st} + \delta^r_s(1 - nx_4)T^a_{at} + +\delta^r_t x_4T^a_{ta})$, *for*

$$x_1 = x_3 = 0,\ x_2 = x_6 = 0,\ x_5 = 1 - nx_4.$$

Proof. Ω is a traceless tensor iff

$$\text{(1.9)}\qquad \begin{cases} (x_1 + x_6 + nx_2)T^a_{at} + (x_3 + x_4 + nx_5)T^a_{ta} = 0 \\ (x_1 + nx_4 + x_5)T^a_{ta} + (x_3 + nx_6 + x_2)T^a_{at} = 0. \end{cases}$$

This system with the unknowns $T^a_{at},\ T^a_{ta}$ is compatible. We study all the cases of the theorem 1.2.

I′ a) The system (1.9) is equivalent to $T^a_{at} = T^a_{ta} = 0$. We get II, 1);
b) We obtain the case IX, 1);
c) 1) $x_4 = -nx_5$.
1.1. $1 + x_6 + nx_2 = 0$. In this case we find IX, 4) and 8);
1.2. $T^a_{at} = 0$ and $1 + x_6 + nx_2 \neq 0$. We get I, 2).
2) If $x_4 \neq -nx_5$ we obtain VI.
I. a) We get IX, 3);
b) The system (1.9) is equivalent to $T^a_{ta} = T^a_{at} = 0$. We get II, 2);
c) 1) If $x_4 + nx_5 \neq 0$, then we obtain V.;
2) $x_4 + nx_5 = 0$. If $T^a_{at} = 0$ and $x_6 + nx_2 \neq 0$, then we get I, 1).
If $x_6 + nx_2 = 0$, then we arrive at IX, 2) and 9).
II. a) The system becomes $T^a_{ta} + T^a_{at} = 0$. We obtain III, 1);
b) (1.9) is identic satisfied and we have IX, 5);
c) (1.9) is equivalent with $T^a_{ta} = T^a_{at} = 0$. We get II, 3);
d) (1.9) becomes $T^a_{ta} = T^a_{at}$ and we obtain IV, 1)
e) 1) $\frac{1}{2} + x_4 + nx_5 = 0$. We get I, 3) for $T^a_{at} = 0$ and IX, 6) for $\frac{1}{2} + x_6 + nx_2 = 0$.
2) If $\frac{1}{2} + x_4 + nx_5 \neq 0$, then we find VII.
II′. a) The system (1.9) is equivalent to $T^a_{ta} = T^a_{at}$. We get the case IV, 2), a);
b) (1.9) becomes $T^a_{ta} = T^a_{at} = 0$. We have the case II, 4);
c) (1.9) is equivalent with $T^a_{ta} = T^a_{at}$. We get the case IV, 2), b);
d) (1.9) becomes $T^a_{ta} = -T^a_{at}$. We get the case III, 2);
e) 1) $x_4 + nx_5 = \frac{1}{2}$;
1.1. If $T^a_{at} = 0$, we get I, 4);
1.2. If $x_6 + nx_2 = -\frac{1}{2}$, then we get the case IX, 7);
2) If $x_4 + nx_5 \neq -\frac{1}{2}$, we get VIII.

Remark 1.1. a) If T is a traceless tensor, then $\Omega = PT$ is a traceless tensor, for any projective projection P.

b) The theorem " *Let V be a real n-dimensional vector space, where $n \geq 2$ and let $A = (A^i_{kl}) \in T^1_2(V)$. Then there exist a unique traceless tensor $B = (B^i_{kl}) \in T^1_2(V)$ and unique 1-forms $C = (C_k)$, $D = (D_k) \in \Lambda^1(V)$, such that $A^i_{kl} = B^i_{kl} + \delta^i_l D_k + \delta^i_k C_l$, where*

$$C_l = \frac{1}{n^2-1}(nA^t_{tl} - A^t_{lt},\ D_k = \frac{1}{n^2-1}(-A^t_{tk} + nA^t_{kt}),$$

$$B^i_{kl} = A^i_{kl} - \frac{1}{n^2-1}[\delta^i_k(nA^t_{tl} - A^t_{lt}) + \delta^i_l(-A^t_{tk} + nA^t_{kt})],\text{"}$$

proved by Krupka in [4], is a particular case of ours (IX, 1). Its trace decomposition problem corresponds to the case $x_1 = 1$, $x_3 = 0$ for our projective projections.

2 Family of projective projections on affine connections

Let M be a differentiable n-dimensional manifold and $\mathcal{T}^1_2 M$ be the bundle of (1,2)-tensor fields over M. The previous projection P extends to a global projection field on $\mathcal{T}_2^1 M$ denoted also by P whose extended coefficients $x_1, ..., x_6$ are scalar fields. Some of the scalar fields $x_1, .., x_6$ are arbitrary functions, others depend on the these arbitrary functions, and some of them are constant functions.

Denote with $\mathcal{A}_2^1(M)$ the set of all geometrical objects of type $(1,2)$ whose difference is a $(1,2)$-tensor field. The set $\mathcal{A}_2^1(M)$ is an affine vector space modelled on the vector space $\mathcal{T}_2^1(M)$. Obviously, the set $\mathcal{C}$ of all affine connections on M and $\mathcal{T}_2^1(M)$ are affine subspaces of $\mathcal{A}_2^1(M)$. Any projection on $\mathcal{T}_2^1(M)$ induces a projection on $\mathcal{A}_2^1(M)$.

Let $\Gamma = \{\Gamma^i_{jk}\}$ be an affine connection on M. The projective projections P of Theorem 1.2 work on $\mathcal{C}$ by the rule $\Pi = P\Gamma = (P_a^{bc}{}^r_{st}\Gamma^a_{bc})$. They produce almost projective connections Π iff $x_1 + x_3 = 1$. Otherwise $(x_1 + x_3 = 0)$, the image $P(\mathcal{C})$ consists of geometrical objects fields Π of type (1,2) which are not connections; particularly the torsion tensor $\frac{1}{2}(\Gamma^i_{jk} - \Gamma^i_{kj})$ is the image of Γ by the projective projection P having the coefficients

$$x_1 = \frac{1}{2},\ x_3 = -\frac{1}{2},\ x_2 = x_4 = x_5 = x_6 = 0.$$

Theorem 2.1. *Let $x_1 = 1,\ x_3 = 0$. The images of the projective projections P on $\mathcal{C}$ consist of the almost projective connections*

$$\Pi^r_{st} = \Gamma^r_{st} + \delta^r_s\psi_t + \delta^r_t\varphi_s,$$

where φ_s and ψ_t are defined by

a) *For $x_2 = x_4 = x_5 = x_6 = 0,\ \psi_t = \varphi_t = 0$;*

b) *For $x_2 = -x_4\dfrac{n}{1-n^2},\ x_5 = x_6 = \dfrac{1}{n^2-1}$,*

$$\psi_t = \frac{1}{n^2-1}(-n\Gamma^a_{at} + \Gamma^a_{ta}),\ \varphi_s = \frac{1}{n^2-1}(\Gamma^a_{as} - n\Gamma^a_{sa});$$

c) $x_5 = \lambda,\ x_6 = \mu,\ \left(\dfrac{\lambda+\mu+1}{n}\right)^2 \geq 4\lambda\mu,\ \lambda, \mu \in \mathbf{R},\ x_2 = \alpha,\ x_4 = \beta$ *solutions of the equation* $z^2 + \dfrac{1}{n}(\lambda+\mu+1)z + \lambda\mu = 0$,

$$\psi_t = \alpha\Gamma^a_{at} + \lambda\Gamma^a_{ta},\ \varphi_s = \beta\Gamma^a_{sa} + \mu\Gamma^a_{as}.$$

Remark. The geometrical objects φ_s and ψ_t are not 1-forms.

Corollary 2.1. *Let $x_1 = 1,\ x_3 = 0$ and P the corresponding projective projections acting on symmetric affine connections. The images of P consist of the almost projective connections*

$$\Pi^r_{st} = \Gamma^r_{st} + \delta^r_s\psi_t + \delta^r_t\varphi_s,$$

where ψ_t and φ_s are related to $\Gamma_t = \Gamma^a_{at}$ in the following ways:

Case b) $\Rightarrow \psi_t = \varphi_t = -\dfrac{1}{n+1}\Gamma_t$.

Case a) $\Rightarrow \psi_s = \varphi_s = 0$.

Case c) $\Rightarrow \psi_t = (\alpha+\lambda)\Gamma_t,\ \varphi_s = (\beta+\mu)\Gamma_s$.

In particular, for the case c) with $\alpha + \lambda = \beta + \mu = -\dfrac{1}{n+1}$ and for the case b) we find the Thomas projective connection [11]

$$\Pi^r_{st} = \Gamma^r_{st} - \frac{1}{n+1}(\delta^r_s \Gamma_t + \delta^r_t \Gamma_s),$$

corresponding to the projection

$$P_a^{bc}{}^r_{st} = \delta^r_a \delta^b_s \delta^c_t - \frac{1}{n+1}\delta^b_a \delta^r_s \delta^c_t - \frac{1}{n+1}\delta^b_a \delta^c_s \delta^r_t .$$

Theorem 2.1′. *Let $x_1 = x_3 = 0$. The images of the projective projections P on $\mathcal{C}$ consist of the objects of type (1,2)*

$$\Pi^r_{st} = \delta^r_s \psi_t + \delta^r_t \varphi_s,$$

where φ_s and ψ_t are defined by

a) *for $x_2 = x_5 = x_4 = x_6 = 0$,*

$$\psi_t = \varphi_t = 0$$

b) *for $x_2 = x_4 = \dfrac{n}{n^2-1}$, $x_5 = x_6 = \dfrac{1}{1-n^2}$,*

$$\psi_t = \frac{1}{n^2-1}(n\Gamma^a_{at} - \Gamma^a_{ta}),\ \varphi_s = \frac{1}{n^2-1}(n\Gamma^a_{sa} - \Gamma^a_{as});$$

c) *for $x_5 = -\lambda$, $x_6 = -\mu$, $\left(\dfrac{1-\lambda-\mu}{n}\right)^2 \geq 4\lambda\mu$, $\lambda, \mu \in \mathbf{R}$, $x_2 = -\alpha$, $x_4 = -\beta$ solutions of the equation $z^2 - \dfrac{1}{n}(1-\lambda-\mu)z + \lambda\mu = 0$,*

$$\psi_t = -\alpha\Gamma^a_{at} - \lambda\Gamma^a_{ta},\ \psi_s = -\beta\Gamma^a_{sa} - \mu\Gamma^a_{as}.$$

Theorem 2.2. *Let $x_1 = x_3 = \dfrac{1}{2}$. The images of the projective projections P on $\mathcal{C}$ consist of the almost projective connections*

$$\Pi^r_{st} = \frac{1}{2}(\Gamma^r_{st} + \Gamma^r_{ts}) + \delta^r_s \psi_t + \delta^r_t \varphi_s,$$

where φ_s and ψ_t are defined by

a) *for $x_2 = x_4 = x_5 = x_6 = 0$, $\psi_t = \varphi_t = 0$;*

b) *for $x_2 = x_4 = x_5 = x_6 = -\dfrac{1}{2(n+1)}$, $\psi_t = \varphi_t = -\dfrac{1}{2(n+1)}(\Gamma^a_{at} + \Gamma^a_{ta})$;*

c) *for $x_2 = x_4 = -x_5 = -x_6 = \dfrac{1}{2(n-1)}$,*

$$\psi_t = -\varphi_t = \frac{1}{2(n-1)}(\Gamma^a_{at} - \Gamma^a_{ta});$$

d) *for $x_2 = x_4 = \dfrac{1}{n^2-1}$, $x_5 = x_6 = \dfrac{n}{1-n^2}$,*

$$\psi_t = \frac{1}{n^2-1}(\Gamma^a_{at} - n\Gamma^a_{ta}),\ \varphi_s = \frac{1}{n^2-1}(\Gamma^a_{sa} - n\Gamma^a_{as});$$

e) *for* $x_5 = \lambda,\ x_6 = \mu,\ \left[\frac{1}{n}(\lambda+\mu)-1\right]^2 \geq 1+4\lambda\mu,\quad \lambda,\mu\in\mathbf{R},\ x_2=\alpha,\ x_4=\beta$ *solutions of the equation* $z^2+\frac{1}{n}(\lambda+\mu)z+\lambda\mu+\frac{1}{2n}(\lambda+\mu)=0,$

$$\psi_t = \alpha\Gamma^a_{at}+\lambda\Gamma^a_{ta},\quad \varphi_s=\beta\Gamma^a_{sa}+\mu\Gamma^a_{as}.$$

Corollary 2.2. *Let* $x_1 = x_3 = \frac{1}{2}$ *and* P *the corresponding projective projections working on symmetric connections. The images of* P *consist of the next almost projective connections*

a) $x_2 = x_4 = x_5 = x_6 = 0 \Rightarrow \Pi^r_{st} = \Gamma^r_{st}$

b) $x_2 = x_4 = x_5 = x_6 = -\frac{1}{2(n+1)} \Rightarrow \Pi^r_{st} = \Gamma^r_{st} - \frac{1}{n+1}(\delta^r_s\Gamma_t + \delta^r_t\Gamma_s)$

c) $x_2 = x_4 = -x_5 = -x_6 = \frac{1}{2(n-1)} \Rightarrow \Pi^r_{st} = \Gamma^r_{st};$

d) $x_2 = x_4 = \frac{1}{n^2-1}, x_5 = x_6 = \frac{n}{1-n^2} \Rightarrow \Pi^r_{st} = \Gamma^r_{st} - \frac{1}{n+1}(\delta^r_s\Gamma_t + \delta^r_t\Gamma_s).$

e) $x_5 = \lambda,\ x_6 = \mu,\ x_2 = \alpha,\ x_4 = \beta,\quad \alpha,\beta,\lambda,\mu\in\mathbf{R}$, *satisfying*

$$n(\alpha+\beta)+\lambda+\mu = 0,\quad \lambda\mu = -\frac{1}{2}(\alpha+\beta)+\alpha\beta \Rightarrow$$

$$\Rightarrow \Pi^r_{st} = \Gamma^r_{st} + \delta^r_s(\lambda+\alpha)\Gamma_t + \delta^r_t(\beta+\mu)\Gamma_s).$$

Remark 2.1. In particular, the case e) $\lambda+\alpha = \beta+\mu = -\frac{1}{n+1}$ and the cases b) and d) produce the Thomas projective connection.

Theorem 2.2′. *Let* $x_1 = \frac{1}{2},\ x_3 = -\frac{1}{2}$. *The images of the projective projections* P *on* $\mathcal{C}$ *consist of the objects of type (1,2)*

$$\Pi^r_{st} = \frac{1}{2}(\Gamma^r_{st} - \Gamma^r_{ts}) + \delta^r_s\psi_t + \delta^r_t\varphi_s,$$

where φ_s *and* ψ_t *are defined by:*

a) *for* $x_2 = x_4 = 0,\ x_5 = x_6 = 0,\ \psi_t = \varphi_t = 0;$

b) *for* $x_2 = x_4 = x_5 = x_6 = \frac{1}{2(n+1)},\ \varphi_t = \psi_t = \frac{1}{2(n+1)}(\Gamma^a_{at}+\Gamma^a_{ta});$

c) *for* $x_2 = x_4 = -x_5 = -x_6 = \frac{1}{2(1-n)},$

$$\psi_t = -\varphi_t = \frac{1}{2(1-n)}(\Gamma^a_{at}-\Gamma^a_{ta});$$

d) *for* $x_2 = x_4 = \frac{1}{1-n^2},\ x_5 = x_6 = \frac{n}{n^2-1},$

$$\psi_t = \frac{1}{1-n^2}(\Gamma^a_{at}-n\Gamma^a_{ta}),\ \varphi_s = \frac{1}{1-n^2}(\Gamma^a_{sa}-n\Gamma^a_{as});$$

e) *for* $x_5 = -\lambda,\ x_6 = -\mu,\ \left[\frac{1}{n}(\lambda+\mu)+1\right]^2 \geq 1+4\lambda\mu,\ \lambda,\mu\in\mathbf{R},\ x_2 = -\lambda,\ x_4 = -\beta$ *solutions of the equations* $z^2+\frac{1}{n}(\lambda+\mu)z-\frac{1}{2n}(\lambda+\mu)+\lambda\mu = 0,$

$$\psi_t = -\alpha\Gamma^a_{at}-\lambda\Gamma^a_{ta},\ \varphi_s = -\beta\Gamma^a_{sa}-\mu\Gamma^a_{as}.$$

3 Almost projective transformations of connections

Let M be a finite dimensional differentiable manifold endowed with the affine connection Γ. The class $\bar{\Gamma}$ of the almost projective transformations (apt) of the connection Γ is defined by [3], [9]

$$\bar{\Gamma} = \Gamma + \eta \otimes I + I \otimes \xi,$$

where $\eta, \xi \in \wedge^1(M)$.

Theorem 3.1. *For each connection Γ and each projective projection P in Theorems 2.1, 2.2, 2.1′, 2.2′ there exists a class of connections $\bar{\Gamma}$ satisfying the commutative diagram*

$$\begin{array}{ccc} \Gamma & \stackrel{apt}{\longrightarrow} & \bar{\Gamma} \\ P\downarrow & & \downarrow P \\ \Pi & = & \bar{\Pi}, \end{array}$$

where $\Pi = P\Gamma$, $\bar{\Pi} = P\bar{\Gamma}$. This diagram reflects also the invariance of Π with respect to $\bar{\Gamma}$ (the gauge invariance of Π with respect to the projective group).

Proof. We fix a projective projection $P = (P_{a\ \ st}^{bc\ r})$ by $(x_1, ..., x_6)$. Since

$$\bar{\Gamma}_{bc}^a = \Gamma_{bc}^a + \eta_b \delta_c^a + \xi_c \delta_b^a$$

it is enough to prove that there exist the 1-forms $\eta = (\eta_b)$, $\xi = (\xi_c)$ such that

$$P_{a\ \ st}^{bc\ r}(\eta_b \delta_c^a + \xi_c \delta_b^a) = [(x_1 + nx_4 + x_6)\eta_s + (x_3 + x_4 + nx_6)\xi_s]\delta_t^r +$$

$$+[(x_2 + x_3 + nx_5)\eta_t + (x_1 + nx_2 + x_5)\xi_t]\delta_s^r = 0.$$

This condition is equivalent to the linear system

$$\begin{cases} (x_1 + nx_4 + x_6)\eta_s + (x_3 + x_4 + nx_6)\xi_s = 0 \\ (x_2 + x_3 + nx_5)\eta_s + (x_1 + nx_2 + x_5)\xi_s = 0, \end{cases}$$

with $2n$ unknows $(\eta_1, ..., \eta_n,\ \xi_1, ..., \xi_n)$ and with $2n$ equations. The determinant of the matrix of this linear system is

$$\Delta = -[(x_1 + nx_4 + x_6)(x_1 + nx_2 + x_5) - (x_2 + x_3 + nx_5)(x_3 + x_4 + nx_6)]^n.$$

For each $(x_1, ..., x_6)$ in Theorems 2.1, 2.2, 2.1′, 2.2′ one proves that Δ is as a rule zero, excepting few cases in which $\Delta \neq 0$. In other words the preceding linear system is as a rule compatible undetermined, excepting few cases in which $\eta = 0$, $\xi = 0$.

In the sequel we suppose that Γ is a symmetric connection, and we identify the connection Γ with the induced covariant derivative ∇. The class of the almost projective transformations of ∇ is characterized by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(X)Y + \xi(Y)X, \quad \forall X, Y \in \mathcal{X}(M), \quad \eta, \xi \in \wedge^1(M).$$

The curvature tensor fields $\bar{R}$ of $\bar{\nabla}$ and R of ∇ are related by

$$\bar{R}(X,Y)Z = R(X,Y)Z - C(Y,Z)X + C(X,Z)Y + d\eta(X,Y)Z,$$

where

$$C(X,Y) = (\nabla_X \xi)(Y) - \xi(X)\xi(Y).$$

Let (M,g) be a Riemannian space. Denote by g_{ij} the components of the metric g, and by R_{ijkl} the components of the curvature tensor field. Introduce the symbols $R \cdot R$ and $Q(g,R)$ by

$$\begin{aligned}(R\cdot R)_{hijklm} &= -R^s_{hlm}R_{sijk} - R^s_{ilm}R_{hsjk} - R^s_{jlm}R_{hisk} - R^s_{klm}R_{hijs},\\ Q(g,R)_{hijklm} &= -g_{mh}R_{lijk} + g_{hl}R_{mijk} - g_{mi}R_{hljk} + g_{il}R_{hmjk} - \\ &-g_{jm}R_{hilk} + g_{jl}R_{himk} - g_{km}R_{hijl} + g_{kl}R_{hijm}.\end{aligned} \tag{3.1}$$

Pseudo-symmetric manifolds [2], i.e., Riemannian spaces (M,g) for which the fields $(*)$ $R \cdot R,\ Q(g,R)$ are linearly dependent at every point of the manifold, constitute a generalization of spaces of constant sectional curvature, along the line of locally symmetric and semi-symmetric spaces $R \cdot R = 0$, studied by Szabo in [8]), consecutively.

The linear dependence of the fields $(*)$ is equivalent to

$$R \cdot R = LQ(g,R) \quad \text{on} \quad U = \{x \in M | R \neq R(1) \quad \text{at} \quad x\}, \tag{$**$}$$

where

$$R(1)_{hijk} = \frac{k}{n(n-1)}(-g_{ik}g_{jh} + g_{ij}g_{kh}),$$

k being the scalar curvature. Similarly to (3.1) we can define $Q(g,A)$, $R \cdot A,\ Q(D,A)$, where D, A are tensors of type (0,2).

Let us consider the square matrix whose entries are R_{ijkl}, where ij indicate the rows and kl indicate the colums. The rank of this symmetric matrix will be denoted by $q(x)$. Obviously $q(x) \leq \dfrac{n(n-1)}{2}$, $\forall x \in M$ ([10]).

Theorem 3.2. *Let ∇ be the Levi-Civita connection of the Riemannian space (M,g) and $\bar{\nabla}$ its almost projective transformation*

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(X)Y + \xi(Y)X,$$

such that η is a closed 1-form and $C = fg,\ f \in \mathcal{F}(M)$. If $\bar{\nabla}$ is a metrique connection (i.e. there exists $\bar{g} \in \mathcal{T}^0_2(M)$, symmetric and positive definite such that $\bar{\nabla}\bar{g} = 0$) and (M,g) is a pseudo-symmetric manifold with L a constant function, then

$$(f+L)\left[\bar{g} - \frac{1}{n}\,Trace(\bar{g})g\right] = 0 \tag{3.2}$$

holds on the open set U.

Proof. Because $\bar{\nabla}\bar{g} = 0$, we get

$$\bar{g}_{ij,k} = 2\eta_k\bar{g}_{ij} + \xi_i\bar{g}_{kj} + \xi_j\bar{g}_{ik},$$

where the comma denotes covariant differentiation with respect to the Levi-Civita connection. The second covariant derivative is

$$\begin{aligned}\bar{g}_{ij,kl} &= 2\eta_{k,l} + \xi_{i,l}\bar{g}_{kj} + \xi_{j,l}\bar{g}_{ik} + 2\eta_k(2\xi_l\bar{g}_{ij} + \xi_i\bar{g}_{lj} + \xi_j\bar{g}_{il}) + \\ &+\xi_i(2\eta_l\bar{g}_{kj} + +\xi_k\bar{g}_{lj} + \xi_j\bar{g}_{kl}) + \xi_j(2\eta_l\bar{g}_{ik} + \xi_i\bar{g}_{lk} + \xi_k\bar{g}_{il}).\end{aligned} \tag{3.3}$$

From (3.3) we get

$$\bar{g}_{ij,kl} - \bar{g}_{ij,lk} = \bar{g}_{kj}(\xi_{i,l} - \xi_i\xi_l) + \bar{g}_{ik}(\xi_{j,l} - \xi_j\xi_l) - \bar{g}_{lj}(\xi_{i,k} - \xi_k\xi_i) - \bar{g}_{il}(\xi_{j,k} - \xi_j\xi_k),$$

which is equivalent to $(R \cdot \bar{g})_{ijkl} = -Q(C, \bar{g})_{ijkl} = -Q(g, f\bar{g})_{ijkl}$.

Using the Theorem 1 of [2] we find

$$(f + L)\left[\bar{g} - \frac{1}{n}\mathrm{Trace}(\bar{g})g\right] = 0 \quad \text{on} \quad U.$$

Proposition 3.1. *In the same hypothesis of the Theorem 3.2, if moreover (U, g) is not conformally related to $(U, \bar{g})$, then $(\bar{R} \cdot \bar{R})^h_{ijklm} = 0$ holds on U.*
Proof.

$$(\bar{R} \cdot \bar{R})^h_{ijklm} = \bar{R}^r_{ijk}\bar{R}^h_{rlm} - \bar{R}^h_{rkl}\bar{R}^r_{ilm} - \bar{R}^h_{irk}\bar{R}^r_{jlm} - \bar{R}^h_{ijr}\bar{R}^r_{klm}.$$

This relation is equivalent to

$$(\bar{R} \cdot \bar{R})^h_{ijklm} = (L + f)Q(g, R)^h_{ijklm}.$$

Using (3.2) we find $L = -f$ and hence $(\bar{R} \cdot \bar{R})^h_{ijklm} = 0$ on the set U.

Proposition 3.2. *In the same hypothesis of the previous Proposition, if moreover $\bar{\nabla}$ is a symmetric connection and the rank of the matrix $(\bar{R}_{ijkl})$ is $q(x) = \dfrac{n(n-1)}{2}$, then $(\mathcal{U}, \bar{g})$ has constant curvature.*

Proof. The proposition is a direct consequence of the Proposition 3.1 and Theorem 2 of [10].

Remark 3.1. If $\eta = \xi$, then $\bar{\nabla}$ is the Levi-Civita connection associated to $\bar{g}$ and hence (M, g) and $(M, \bar{g})$ are special geodesically related spaces. The Theorem 3.2 generalizes the Theorem 2 of [2]. In this special case $(\mathcal{U}, g)$ has also constant curvature.

Theorem 3.3. *Let $\bar{\nabla}$ be the almost projective transformation of the Levi-Civita connection of the Riemannian space (M, g),*

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(X)Y + \xi(Y)X$$

such that η is a closed 1-form and ξ a closed, non vanishing 1-form. If (M, g) is an Einstein space and $\bar{\nabla}$-recurrent (i.e. $\bar{\nabla}_X \bar{R} = \omega(X)\bar{R}$, ω being a 1-form, and $\bar{R}$ the curvature tensor field), then the two connections are flat projective (i.e. the projective curvature tensor $W^i_{jkl} = R^i_{jkl} + \dfrac{1}{n-1}(\delta^i_k S_{jl} - \delta^i_l S_{jk})$ is 0 and also $\bar{W}^i_{jkl} = 0$, where S is the Ricci tensor field).

Proof. The projective curvature tensor field is invariant with respect to this special almost projective transformation of connections. The relations $\bar{R}^i_{jkl;r} = \omega_r \bar{R}^i_{jkl}$ and $\bar{S}_{jl;r} = \omega_r \bar{S}_{jl}$ imply

$$W^i_{jkl;r} = \omega_r W^i_{jkl}. \tag{3.4}$$

The relation (3.4) is equivalent to

$$W_{ijkl,r} + g_{ir}\xi_s W^s_{jkl} - \xi_k W_{ijrl} - \xi_l W_{ijkr} - \xi_j W_{irkl} = (\omega_r + 2\eta_r)W_{ijkl}. \tag{3.5}$$

Because (M, g) is an Einstein space, we have $W_{ijkl} + W_{jikl} = 0$. From (3.5) we get

$$g_{ir}\xi_s W^s_{jkl} + g_{jr}\xi_s W^s_{ikl} - \xi_j W_{irkl} - \xi_i W_{jrkl} = 0 \tag{3.6}$$

Contracting with g^{ir} in (3.6) we obtain

$$\xi_s W^s_{jkl} = 0. \tag{3.7}$$

From (3.6) and (3.7) we get $\xi_j W_{irkl} + \xi_i W_{jrkl} = 0$ and hence $W = \bar{W} = 0$.

Theorem 3.3. *Let $\bar{\nabla}$ be the almost projective transformation of the affine connection ∇,*

$$\bar{\nabla}_X Y = \nabla_X Y + k\xi(X)Y + \xi(Y)X, \quad k \in \mathbf{Z} \setminus \left\{-1, \frac{n}{2}\right\},$$

ξ being a closed 1-form. If ∇ and $\bar{\nabla}$ are projective recurrent so that $\nabla_X W = \mu(X)W, \bar{\nabla}_X \bar{W} = \mu(X)\bar{W}$, μ being a 1-form, then the two connections $\nabla, \bar{\nabla}$ are flat projective.

Proof. From the relation

$$\begin{aligned}(\bar{\nabla}_U \bar{W})(X,Y)Z = (\nabla_U W)(X,Y)Z + \xi(W(X,Y)Z)U - \xi(X)W(U,Y)Z - \\ -\xi(Y)W(X,U)Z - \xi(Z)W(X,Y)U - 2k\xi(U)W(X,Y)Z\end{aligned} \tag{3.8}$$

we obtain

$$\begin{aligned}2k\xi(U)W(X,Y)Z + \xi(X)W(U,X)Z + \xi(Y)W(X,U)Z + \\ +\xi(Z)W(X,Y)U = \xi(W(X,Y),Z)U.\end{aligned} \tag{3.9}$$

If $\{\lambda^i\} \subset \wedge^1(M)$ and $\{X_i\} \subseteq \mathcal{X}(M)$ are dual local bases, let us take $U = X_i$ in (3.9). Contracting the resulting formula with λ^i we get

$$(n - 2k)\xi(W(X,Y)Z) = 0. \tag{3.10}$$

From (3.9) and (3.10) it follows

$$2k\xi(U)W(X,Y)Z + \xi(X)W(U,X)Z + \xi(Y)W(X,U)Z + \xi(Z)W(X,Y)U = 0. \tag{3.11}$$

Taking $U = Z = X$ in (3.11) we have

$$2(k+1)\xi(X)W(X,Y)X = 0. \tag{3.12}$$

There is $T \in \mathcal{X}(M)$ so that $\xi(T) \neq 0$ and hence $W(T,Y)T = 0$. Using (3.11) we get $W(X,Y)Z = 0$.

Remark 3.2. If we suppose that M is endowed with two affine connections $\nabla, \bar{\nabla}$ and $A = \bar{\nabla} - \nabla$, we can construct the deformation algebra $\mathcal{U}(M, \bar{\nabla}, \nabla)$ considering $X \star Y = A(X,Y)$. An element $X \in \mathcal{U}(M, \bar{\nabla}, \nabla)$ is called a *characteristic vector field* if there exists $\lambda \in \mathcal{F}(M)$ such that $A(X,X) = \lambda X$ and is called an *almost principal vector field* if there are $f \in \mathcal{F}(M)$ and $\omega \in \wedge^1(M)$ such that $A(Z,X) = fZ + \omega(Z)X$, $\forall X, Z \in \mathcal{X}(M)$ [7].

Theorem 3.4. *Let $\bar{\nabla}$ be the almost projective transformation of the affine connection ∇, $\bar{\nabla} = \nabla + I \otimes \xi + \eta \otimes I$, η, ξ being arbitrary 1-forms. All the elements of the deformation algebra $\mathcal{U}(M, \nabla, \bar{\nabla},)$ are characteristic vector fields and almost principal vector fields.*

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