Landsberg Spaces Satisfying the T-Condition

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Abstract

The T-tensor of Finsler spaces was studied by several authors ([1], [3], [4], [8], [10], [12] and [13]). Especially, Szabó [12], Watanabe and the present author [13] proved that a positive definite Finsler space with T-condition (i.e. T-tensor=0) is a Riemannian space. And non- Riemannian Finsler spaces with T-condition are treated by Asanov and Kirnasov [1].

The purpose of the present paper is to study non-Riemannian Landsberg spaces satisfying the T-condition and to consider conformal flatness of non-Riemannian Finsler spaces with T-condition. In §1, we shall define the T-tensor of Finsler spaces and show a Ricci identity and two Bianchi identities which play important roles. And, in §2, we shall prove that certain Landsberg spaces satisfying the T-condition become locally Minkowski spaces. Moreover, we shall show that Finsler spaces with T-condition are conformally flat if and only if they should be locally Minkowski spaces in §3. Finally, we shall apply above considerations to semi C-reducible Finsler spaces in §4.

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1 Preliminaries

Let M^n be an $n(\geq 3)$-dimensional Finsler space endowed with a fundamental function $L = L(x,y)$ and C_{ijk} be the $(h)hv$-torsion tensor of M^n, where $x = (x^i)$ is a point and $y = (y^i)$ is a supporting element of M^n, respectively. Then, the T-tensor T_{ijkl} of M^n is defined by the following relation

$$T_{ijkl} = C_{ijk}\,|_l + L^{-1}(C_{ijk}l_l + C_{ljk}l_i + C_{ilk}l_j + C_{ijl}l_l), \tag{1.1}$$

where $l_i = L^{-1}g_{ir}y^r$, $g_{ir} = (\partial L^2/\partial y^i \partial y^r)/2$ is called the *fundamental tensor* and the symbol $|$ means the v-covariant derivative with respect to the Cartan connection $C\Gamma$ of M^n.

Definition 1.1. If the T-tensor T_{ijkl} of an n-dimensional Finsler space M^n vanishes identically, then M^n satisfies the T-condition.

Transvecting (1.1) by the reciprocal tensor g^{ij} of g_{ij}, we obtain

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$$T_{kl} = C_k\,|_l + L^{-1}(C_k l_l + C_l l_k), \tag{1.2}$$

where T_{kl} and C_k are called the T'*-tensor* and the *torsion vector* of M^n, respectively.

For the later use, we show a Ricci identity with respect to the torsion vector C_i and two Bianchi identities

$$C_{i|j|k} - C_{i|k|j} = -C_r R_i{}^r{}_{jk} - C_i\,|_r\,R^r{}_{jk}, \tag{1.3}$$

$$\begin{aligned} R^h{}_{jr}C_k{}^r{}_i &- R^h{}_{kr}C_j{}^r{}_i + P^h{}_{jr}P^r{}_{ki} - P^h{}_{kr}P^r{}_{ji} + P^h{}_{ki|j} - \\ &- P^h{}_{ji|k} + R^h{}_{jk}\,|_i - R_i{}^h{}_{jk} = 0, \end{aligned} \tag{1.4}$$

$$\begin{aligned} R_l{}^h{}_{jr}C_k{}^r{}_i &- R_l{}^h{}_{kr}C_j{}^r{}_i + P_l{}^h{}_{jr}P^r{}_{ki} - P_l{}^h{}_{kr}P^r{}_{ji} + P_l{}^h{}_{ki|j} - \\ &- P_l{}^h{}_{ji|k} + R_l{}^h{}_{jk}\,|_i + S_l{}^h{}_{ir}R^r{}_{jk} = 0, \end{aligned} \tag{1.5}$$

where $R_i{}^r{}_{jk}, P_i{}^r{}_{jk}, S_i{}^r{}_{jk}, R^r{}_{jk}$ and $P^r{}_{jk}$ are the h-curvature tensor, the hv-curvaure tensor, the v-curvature tensor, the $(v)h$-torsion tensor and the $(v)hv$-torsion tensor of M^n, respectively. And the symbol $_|$ represents the h-covariant derivative with respect to $C\Gamma$.

2 Landsberg spaces

In this section, we shall deal with $n(\geq 3)$-dimensional Landsberg spaces defined by

Definition 2.1. An n-dimensional Finsler space M^n is called a *Landsberg space*, if the $(v)hv$-torsion tensor $P^r{}_{jk}$ of M^n vanishes identically.

Let M^n be an $n(\geq 3)$-dimensional Landsberg space with T-condition, then (1.2), (1.4) and (1.5) reduce to

$$C_k\,|_l = -L^{-1}(C_k l_l + C_l l_k), \tag{2.1}$$

$$R^h{}_{jr}C_k{}^r{}_i - R^h{}_{kr}C_j{}^r{}_i + R^h{}_{jk}\,|_i - R_i{}^h{}_{jk} = 0, \tag{2.2}$$

$$R_l{}^h{}_{jr}C_k{}^r{}_i - R_l{}^h{}_{kr}C_j{}^r{}_i + R_l{}^h{}_{jk}\,|_i + S_l{}^h{}_{ir}R^r{}_{jk} = 0, \tag{2.3}$$

because the hv-curvature tensor $P_i{}^r{}_{jk}$, the $(v)hv$-torsion tensor $P^r{}_{jk}$ and the T'-tensor T_{kl} vanish identically.

It is well-known that Landsberg spaces satisfying the T-condition become Berwald spaces ([4] and [8]), then the h-covariant derivative of the $(h)hv$-torsion tensor $C_i{}^r{}_k$ vanishes identically, and so $C_{i|j} = 0$ holds. Thus, (1.3) and (2.1) yield

$$-C_r R_i{}^r{}_{jk} + L^{-1}C_r R^r{}_{jk} l_i = 0, \tag{2.4}$$

because of $l_r R^r{}_{jk} = 0$.

V-covariantly differentiating (2.4) and using (2.1), we find $L^{-1}C_r R_i{}^r{}_{jk} l_l - L^{-2} R_{ijk} C_l - C_r R_i{}^r{}_{jk} \mid_l - 2L^{-2} C_r R^r{}_{jk} l_i l_l + L^{-1} C_r R^r{}_{jk} \mid_l l_i + L^{-2} C_r R^r{}_{jk} h_{il} = 0$, which leads, by virtue of (2.2) and (2.3), to

$$L^{-2} C_r R^r{}_{jk} h_{il} - L^{-2} R_{ijk} C_l + C_r S_i{}^r{}_{lt} R^t{}_{jk} = 0. \tag{2.5}$$

where $h_{il} (= g_{il} - l_i l_l)$ is the angular metric tensor of M^n.

Now we consider Landsberg spaces whose v-curvature tensor S_{ijkl} has a certain form. First, suppose that the v-curvature tensor S_{ijkl} of our Landsberg spaces vanishes identically. Then (2.5) reduces to

$$L^{-2} C_r R^r{}_{jk} h_{il} - L^{-2} R_{ijk} C_l = 0. \tag{2.6}$$

Transvection of (2.6) by g^{il} yields $(n-2)L^{-2} C_r R^r{}_{jk} = 0$, from which, via (2.2) and (2.6), $R_i{}^r{}_{jk} = 0$ is derived. Therefore, we have

Theorem 2.1. *If v-curvature tensor $S_i{}^r{}_{jk}$ of an $n(\geq 3)$-dimensional Landsberg space M^n with T-condition vanishes identically, then M^n is a locally Minkowski space.*

Since the v-curvature tensor S_{ijkl} of a $C2$-like Landsberg space M^n vanishes identically, it is evident the following corollary:

Corollary 2.2. *If an $n(\geq 3)$-dimensional $C2$-like Landsberg space M^n satisfies the T-condition, then M^n is a locally Minkowski space.*

Next, we assume that our Landsberg spaces are $S3$-like, and then the v-curvature tensor S_{ijkl} is represented by

$$S_{ijkl} = S(h_{ik} h_{jl} - h_{il} h_{jk}). \tag{2.7}$$

Substituting (2.7) into (2.5) and then transvecting it by g^{il}, we have

$$(L^{-2} + S)(C_r R^r{}_{jk} h_{il} - R_{ijk} C_l) = 0. \tag{2.8}$$

Thus, if $L^{-2} \neq -S$ holds, then (2.2) and (2.8) lead to $R_i{}^j{}_{kl} = 0$. Therefore, we get

Theorem 2.3. *Let M^n be an $n(\geq 3)$-dimensional $S3$-like Landsberg space M^n with T-condition. If the function S of (2.7) is not equal to $-L^{-2}$, then M^n is a locally Minkowski space.*

Since the v-curvature tensor S_{ijkl} of three-dimensional Finsler spaces is written as (2.7), the following corollary is evident:

Corollary 2.4. *If a three-dimensional Landsberg space M^3 satisfies the T-condition and the function S of (2.7) is not equal to $-L^{-2}$, then M^3 is a locally Minkowski space.*

3 Conformal flatness of Finsler spaces with T-condition

In this section, we shall deal with conformal flatness of Finsler spaces. For this concept, Kikuchi proved a very important theorem under the Kikuchi condition that the tensor $W^i{}_j$ is regular, where $W^i{}_j = (\partial L^2 C^2/\partial y^r) B^{ri}_j$, $B^{ri}_j = \partial B^{ri}/\partial y^j$ and $B^{ri} = L^2(g^{rj} - l^r l^i)/2$ [**8**]. If Finsler spaces satisfy the T-condition, then the above tensor $W^i{}_j$ is equal to zero. So, Finsler spaces with T-condition don't satisfy the Kikuchi condition.

We have already known that if an $n(\geq 3)$-dimensional Finsler space M^n satisfies the T-condition, then the $(v)hv$-torsion tensor P_{ijk} of M^n is conformally invariant [**2**]. Thus, it is clear that an $n(\geq 3)$-dimensional Finsler space M^n satisfying the T-condition is conformal to a Landsberg space if and only if M^n is a Landsberg space. Therefore, from above results, Theorem 2.1 and Theorem 2.3, we have the following theorems:

Theorem 3.1. *If the v-curvature tensor $S_i{}^r{}_{jk}$ of an $n(\geq 3)$-dimensional Finsler space M^n with T-condition vanishes identically, then a necessary and sufficient condition for M^n to be conformally flat is that M^n itself is a locally Minkowski space.*

Theorem 3.2. *If an $n(\geq 3)$-dimensional Finsler space M^n with T-condition is $S3$-like and the function S of (2.7) is not equal to $-L^{-2}$, then M^n is conformally flat if and only if M^n itself is a locally Minkowski space.*

Theorem 3.3. *If a three-dimensional Landsberg space M^3 is the T-condition and the function S of (2.7) is not equal to $-L^{-2}$, then M^3 is conformally flat if and only if M^n itself is a locally Minkowski space.*

4 Semi C-reducible Finsler spaces

In this section, we shall consider semi C-reducible Finsler spaces whose $(h)hv$-torsion tensor C_{ijk} is written as

$$C_{ijk} = p(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j) + qC_iC_jC_k, \tag{4.1}$$

where p, q are functions satisfying $(n+1)p + qC^2 = 1$ and $C^2 = g_{ij}C^iC^j$.

From the definition of the v-curvature tensor $S_{ijkl}(= C_{ilt}C_j{}^t{}_k - C_{ikt}C_j{}^t{}_l)$ and the equation (4.1), it is derived

$$\begin{aligned} S_{ijkl} &= p^2C^2(h_{il}h_{jk} - h_{ik}h_{jl}) + \\ &+ (p^2 + pqC^2)(h_{il}C_jC_k + h_{jk}C_iC_l - h_{ik}C_jC_l - h_{jl}C_iC_k)). \end{aligned} \tag{4.2}$$

Substituting (4.2) into (2.5) and then transvecting it by g^{il}, we have

$$(n-2)(L^{-2} - pC^2 + (n-1)p^2C^2)(C_rR^r{}_{jk}h_{il} - R_{ijk}C_l) = 0. \tag{4.3}$$

Therefore, we obtain

Theorem 4.1. *Let M^n be an $n(\geq 3)$-dimensional semi C-reducible Landsberg space. If M^n satisfies the T-condition and $L^{-2} - pC^2 + (n-1)p^2C^2 \neq 0$, then M^n is a locally Minkowski space.*

Theorem 4.2. *Let M^n be an $n(\geq 3)$-dimensional semi C-reducible Finsler space with T-condition. If M^n satisfies $L^{-2} - pC^2 + (n-1)p^2C^2 \neq 0$, then M^n is conformally flat if and only if M^n itself is a locally Minkowski space.*

Since the function p of C-reducible Finsler spaces is $(n+1)^{-1}$, the following theorems are evident.

Theorem 4.3. *Let M^n be an $n(\geq 3)$-dimensional C- reducible Landsberg space. If M^n satisfies the T-condition and $L^2C^2 \neq (n+1)^2/2$, then M^n is a locally Minkowski space.*

Theorem 4.4. *Let M^n be an $n(\geq 3)$-dimensional C-reducible Finsler space with T-condition. If M^n satisfies $L^2C^2 \neq (n+1)^2/2$, then M^n is conformally flat if and only if M^n itself is a locally Minkowski space.*

Next, we shall state special semi C-reducible Finsler spaces whose $(h)hv$-torsion tensor represented by

$$C_{ijk} = (h_{ij}C_k + h_{jk}C_i + h_{ki}C_j)/(n-1) - 2C^{-2}C_iC_jC_k/(n-1). \tag{4.4}$$

Such semi C-reducible Finsler spaces were treated by Izumi, Sakaguchi and Yoshida in their paper [**7**]. The function $L^{-2} - pC^2 + (n-1)p^2C^2$ is equal to L^{-2}, because of $p = (n-1)^{-1}$. Therefore, from Theorem 4.1 and 4.2, we have

Theorem 4.5. *Let M^n be an $n(\geq 3)$-dimensional Landsberg space whose $(h)hv$-torsion tensor C_{ijk} is represented by (4.4). If M^n satisfies the T-condition, then M^n is a locally Minkowski space.*

Theorem 4.6. *Let M^n be an $n(\geq 3)$-dimensional Finsler space with T-condition. If the $(h)hv$-torsion tensor C_{ijk} is presented by (4.4) then a necessary and sufficient condition for M^n to be conformally flat is that M^n itself is a locally Minkowski space.*

Finally, we shall consider three-dimensional Finsler spaces whose $(h)hv$- torsion tensor C_{ijk} is represented by (4.4). Let (l_i, m_i, n_i) be a Moór frame and H, I and J be main scalas, then (4.4) rewritten as

$$LC_{ijk} = I(m_im_jm_k + m_in_jn_k + n_im_jn_k + n_in_jm_k), \tag{4.5}$$

which shows that $H = I$ and $J = 0$ hold. From the previous paper [5], $I\,|_t\, n^t = 0$ is satisfied because of $J = 0$, and then the T-tensor T_{ijkl} is rewritten as

$$\begin{aligned} T_{ijkl} &= I\,|_t\, m^t(m_im_jm_km_l - 3n_in_jn_kn_l + m_im_jn_kn_l + m_in_jm_kn_l + \\ &+ m_in_jn_km_l + n_in_jm_km_l + n_im_jn_km_l + n_im_jm_kn_l), \end{aligned} \tag{4.6}$$

and the V-curvature tensor S_{ijkl} vanishes identically.

On the other hand, the Kikuchi condition for conformal flatness of Finsler spaces becomes $II\,|_i \neq 0$, which is equivarent to $T_{ijkl} \neq 0$. Summarizing these results, we have

Theorem 4.7 *Let M^3 be a three-dimensional Finsler spaces whose $(h)hv$-torsion tensor C_{ijk} is written as (4.4). If M^3 does not satisfy the Kikuchi condition for conformal flatness, then a necessary and sufficient condition for M^3 to be conformally flat is that M^3 itself is a locally Minkowski space.*

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