Complex Lagrange Spaces

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Abstract

The geometry of Lagrange spaces is a field of large interest that knows a permanent development [6].

In the present paper our desire is to extend the results from the geomety of complex Finsler manifolds, recently obtained by T. Aikou [1], to the case of manifolds endowed with a complex Lagrangian. We shall make a study of holomorphic bundle of (1,0) vector fields using a nonlinear connection determined by the complex Lagrange metric. Finally, the discovered results will be applied to the case of complex Lagrangian of electromagnetic field.

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1 The geometry of $T'M$ holomorphic bundle

We consider M be a complex manifold, $\dim_C M = n$, $(U, (z^i))$ complex coordinates in a local map. The complexification $T_C M$ of tangent bundle TM is decomposed at each point $z \in U$ after the (1, 0) vector fields and their conjugate of (0,1) type ([3], [5]), $T_C M = T'M \oplus T''M$.

The bundle $T'M$ is a holomorphic bundle and the projection $\pi_T : T'M \to M$ is a holomorphic map, $\dim_C T'M = 2n$, and $T''M$ is the conjugate of $T'M$.

We denote by $(\pi_T^{-1}(U), z^i, \eta^i)$ the induced complex coordinates on $T'M$. The transition functions of the holomorphic tangent bundle $T'(T'M)$, of the complex manifold $T'M$, are given by

$$\begin{pmatrix} \partial z'^j/\partial z^i & 0 \\ (\partial^2 z'^j/\partial z^i \partial z^k)\eta^k & \partial z'^j/\partial z^i \end{pmatrix}. \tag{1.1}$$

According to [1], a vector bundle $\tilde{E}$ over $T'M$ is said to be a *Finsler complex bundle* if $\tilde{E} = \pi_T^{-1}(E)$, i. e. $\tilde{E}$ is the pull-back of a complex bundle E over M. If E is holomorphic, then $\tilde{E}$ is called a *holomorphic Finsler vector bundle.*

The vertical subbundle

$$V(T'M) = \{\xi \in T'(T'M) \ / \ \pi_{T*}(\xi) = 0\}$$

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is a vector bundle of rank n over $T'M$; a base of local fields is $\{\partial/\partial\eta^i\}$ and from (1.1) it follows that $V(T'M)$ is a holomorphic bundle of Finsler type.

Let $H(T'M)$ be a supplementary subundle of $V(T'M)$ in $T'(T'M)$ i. e. $T'(T'M) = H(T'M) \oplus V(T'M)$. This spliting determines a distribution $N : u = (z^i, \eta^i) \to H_u(T'M)$ which will be called a *complex nonlinear connection*. Let us consider

$$\frac{\delta}{\delta z^i} = \frac{\partial}{\partial z^i} - N_i^j \frac{\partial}{\partial \eta^j}, \tag{1.2}$$

a local base on $H_u(T'M)$. The complex functions $N_i^j(z, \eta)$ are called the coefficients of the complex nonlinear connection. After (1.1) the functions N_i^j satisfy the following transformation law

$$N_k'^i \frac{\partial z'^k}{\partial z^j} = \frac{\partial z'^i}{\partial z^k} N_j^k - \frac{\partial^2 z'^i}{\partial z^j \partial z^k} \eta^k. \tag{1.2'}$$

From the spliting of $T'(T'M)$ it results that $H(T'M)$ is a Finsler complex vector bundle of rank n therefore the adapted base $\{\delta/\delta z^i\}$ changes by the rule

$$\frac{\delta}{\delta z^i} = \frac{\partial z'^j}{\partial z^i} \frac{\delta}{\delta z'^j}. \tag{1.3}$$

The dual base of $(\delta/\delta z^i, \partial/\partial\eta^i)$ will be denoted by $(dz^i, \delta\eta^i)$.

Consider the conjugates $\overline{H(T'M)}$ and $\overline{V(T'M)}$. These are subbundles of $T''(T'M)$. It results the following decomposition of the complexified tangent bundle

$$T_C(T'M) = T'(T'M) \oplus T''(T'M) = H(T'M) \oplus V(T'M) \oplus \overline{H(T'M)} \oplus \overline{V(T'M)} \tag{1.4}$$

The corresponding projectors on the distributions are denoted by $h, v, \overline{h}$ and $\overline{v}$.

Taking into account that $\frac{\partial}{\partial \eta^i} = \frac{\partial z'^j}{\partial z^i} \frac{\partial}{\partial \eta'^j}$, by conjugation it results

Proposition 1.1. *A local base on the conjugate* $T''(T'M)$ *is*

$$\left(\frac{\delta}{\delta \overline{z}^i} = \frac{\partial}{\partial \overline{z}^i} - \overline{N}_i^j \frac{\partial}{\partial \overline{\eta}^j}; \quad \frac{\partial}{\partial \overline{\eta}^i} \right)$$

and it is changed by the rules

$$\frac{\delta}{\delta \overline{z}^i} = \frac{\partial \overline{z}'^j}{\partial \overline{z}^i} \frac{\delta}{\delta \overline{z}'^j}; \quad \frac{\partial}{\partial \overline{\eta}^i} = \frac{\partial \overline{z}'}{\partial \overline{z}^i} \frac{\partial}{\partial \overline{\eta}'^j}, \tag{1.5}$$

where $\overline{N}_i^j(z, \eta)$ *are the conjugates of coefficients of the nonlinear connection* N_i^j *and are transformed by the rule obtained from (1.2′) by conjugation.*

If $E = T'M$ then $\widetilde{T'M}$ is a holomorphic bundle of Finsler type that can be identified with $H(T'M) \oplus V(T'M)$.

Since $T'M$ is a holomorphic manifold, on the complexification $T_C(T'M)$ there exists a natural complex strucure J defined by

$$J(\partial/\partial z^k) = i\partial/\partial z^k \ ; \quad J(\partial/\partial \eta^k) = i\partial/\partial \eta^k \ ;$$

$$J(\partial/\overline{z}^k) = -i\partial/\partial \overline{z}^k \ ; \quad J(\partial/\partial \overline{\eta}^k) = -i\partial/\partial \overline{\eta}^k \ , (i^2 = -1),$$

which is globaly defined.

For simplicity, in the following we shall use for the adapted bases the abreviations

$$\delta_k = \delta/\delta z^k \quad ; \quad \partial_k = \partial/\partial\eta^k \quad \text{and} \quad \delta_{\overline{k}} = -i\delta/\delta\overline{z}^k \quad ; \quad \partial_{\overline{k}} = \partial/\partial\overline{\eta}^k.$$

Hence, in adapted bases the complex structure are written

$$J(\delta_k) = i\delta_k \;\; ; \;\; J(\partial_k) = i\partial_k \;\; ; \;\; J(\delta_{\overline{k}}) = -i\delta_{\overline{k}} \;\; ; \;\; J(\partial_{\overline{k}}) = -i\partial_{\overline{k}}. \tag{1.6}$$

Moreover, let us consider the adjoint almost tangent structures F and F^*

$$\begin{aligned} &F(\delta_k) = \partial_k; \quad F(\partial_k) = 0; \quad F(\delta_{\overline{k}}) = \partial_{\overline{k}}; \quad F(\partial_{\overline{k}}) = 0 \\ &F^*(\delta_k) = 0; \quad F^*(\partial_k) = \delta_k; \quad F^*(\delta_{\overline{k}}) = 0; \quad F^*(\partial_{\overline{k}}) = \delta_{\overline{k}} \end{aligned} \tag{1.7}$$

globaly defined as it results from (1.3) and (1.5).

The following relations hold

$$F \cdot F^* = v + \overline{v} \quad ; \quad F^* \cdot F = h + \overline{h} \;\; ; \;\; J \cdot F = F \cdot J \;\; ; \;\; J \cdot F^* = F^* \cdot J$$

and

$$F^2 = 0 \quad F^* = 0 \quad J^2 = -I,$$

i.e., the structures $(J, F, J \cdot F)$ and $(J, F^*, J \cdot F^*)$ are commutative almost semiqaternionic structures ([8]) , adjoint one to the other.

Let us consider D a derivative law on the complexification $T_C(T'M)$.

Through analogy to the real case ([6]), we say that D is a *N-linear complex connection* (shortly N-l.c.c.) if it preserves the four distributions determined by the complex nonlinear connection N.

In adapted base $(\delta_k, \partial_k, \delta_{\overline{k}}, \partial_{\overline{k}})$ the N-l.c.c. D has the local expression

$$\begin{aligned} &D_{\delta_k}\delta_j = \overset{1}{L}{}^i_{jk}\,\delta_i; \quad D_{\partial_k}\delta_j = \overset{1}{C}{}^i_{jk}\,\delta_i; \quad D_{\delta_{\overline{k}}}\delta_j = \overset{3}{L}{}^i_{j\overline{k}}\,\delta_i; \quad D_{\partial_{\overline{k}}}\delta_j = \overset{3}{C}{}^i_{j\overline{k}}\,\delta_i \\ &D_{\delta_k}\partial_j = \overset{2}{L}{}^i_{jk}\,\partial_i; \quad D_{\partial_k}\partial_j = \overset{2}{C}{}^i_{jk}\,\partial_i; \quad D_{\delta_{\overline{k}}}\partial_j = \overset{4}{L}{}^i_{j\overline{k}}\,\partial_i; \quad D_{\partial_{\overline{k}}}\partial_j = \overset{4}{C}{}^i_{j\overline{k}}\,\partial_i \\ &D_{\delta_k}\delta_{\overline{j}} = \overset{3}{L}{}^{\overline{i}}_{\overline{j}k}\,\delta_{\overline{i}}; \quad D_{\partial_k}\delta_{\overline{j}} = \overset{3}{C}{}^{\overline{i}}_{\overline{j}k}\,\delta_{\overline{i}}; \quad D_{\delta_{\overline{k}}}\delta_{\overline{j}} = \overset{1}{L}{}^{\overline{i}}_{\overline{j}\overline{k}}\,\delta_{\overline{i}}; \quad D_{\partial_{\overline{k}}}\delta_{\overline{j}} = \overset{1}{C}{}^{\overline{i}}_{\overline{j}\overline{k}}\,\delta_{\overline{i}} \\ &D_{\delta_k}\partial_{\overline{j}} = \overset{4}{L}{}^{\overline{i}}_{\overline{j}k}\,\partial_{\overline{i}}; \quad D_{\partial_k}\partial_{\overline{j}} = \overset{4}{C}{}^{\overline{i}}_{\overline{j}k}\,\partial_{\overline{i}}; \quad D_{\delta_{\overline{k}}}\partial_{\overline{j}} = \overset{2}{L}{}^{\overline{i}}_{\overline{j}\overline{k}}\,\partial_{\overline{i}}; \quad D_{\partial_{\overline{k}}}\partial_{\overline{j}} = \overset{2}{C}{}^{\overline{i}}_{\overline{j}\overline{k}}\,\partial_{\overline{i}}. \end{aligned} \tag{1.8}$$

Because $\overline{D_X Y} = D_{\overline{X}}\overline{Y}$ it results that some of the coefficients in (1.8) are the conjugate of the others, for instance $\overset{1}{L}{}^{\overline{i}}_{\overline{j}\overline{k}} = \overline{\overset{1}{L}{}^i_{jk}}$, etc.

Let us note that a N-l.c.c. is decomposed in $D = D' + D''$, with $D'' = \overline{D'}$ and in order $D' = D'^h + D'^v$; $D'' = D''^{\overline{h}} + D''^{\overline{v}}$, where $D'^h = D_{\delta_k}$, $D'^v = D_{\partial_k}$, $D''^{\overline{h}} = D_{\delta_{\overline{k}}}$, $D''^{\overline{v}} = D_{\partial_{\overline{k}}}$.

Proposition 1.2. *D is a N-l.c.c. if and only if* $DJ = 0$ *and* $Dv = 0$.

Proof. $DJ = 0$ implies that D' is a derivative law of (1,0)-type, and D'' of (0,1)-type. $Dv = 0$ is the condition to preserve the $H(T'M)$ and $V(T'M)$ on $T'(T'M)$. By conjugation it results $D\overline{v} = 0$.

The N-l.c.c. D is said to be *normal* if in addition the connection coefficients satisfy the following conditions

$$\overset{1}{L}{}^{i}_{jk}=\overset{2}{L}{}^{i}_{jk}\ ;\ \overset{3}{L}{}^{i}_{\bar{j}k}=\overset{4}{L}{}^{\bar{i}}_{\bar{j}k}\ ;\ \overset{1}{C}{}^{i}_{jk}=\overset{2}{C}{}^{i}_{jk}\ ;\ \overset{3}{C}{}^{\bar{i}}_{\bar{j}k}=\overset{4}{C}{}^{\bar{i}}_{\bar{j}k}$$

(and their conjugates). In the case of a normal N-l.c.c. we shall use a simple written form for the coefficients

$$\overset{1}{L}{}^{i}_{jk}=L^{i}_{jk}\ ;\ \overset{3}{L}{}^{i}_{\bar{j}k}=L^{\bar{i}}_{\bar{j}k}\ ;\ \overset{1}{C}{}^{i}_{jk}=C^{i}_{jk}\ ;\ \text{and}\ \overset{3}{C}{}^{\bar{i}}_{\bar{j}k}=C^{\bar{i}}_{\bar{j}k}.$$

Theorem 1.3. *D is a normal N-l.c.c. if and only if it is a commutative semiquaternionic connection with respect to both structures, i.e.,*

$$DJ=0,\ DF=0,\ DF^{*}=0.$$

The proof results from the fact $F\cdot F^{*}=v$ and $D_X(FY)=FD_XY$ for any X and Y vector fields of the adapted base. The general family of such connections is given in [8] for the anticommutative case.

For a normal N-l.c.c. we can write without any difficuly the components of torsions and curvatures. The nonvanishing torsions are

$$\begin{array}{lll}
T^{k}_{ij}\delta_k=hT(\delta_j,\delta_i) & \Theta^{k}_{i\bar{j}}\partial_k=vT(\delta_{\bar{j}},\delta_i) & \tau^{k}_{i\bar{j}}\delta_k=hT(\delta_{\bar{j}},\delta_i)\\
R^{k}_{ij}\partial_k=vT(\delta_j,\delta_i) & \Theta^{\bar{k}}_{i\bar{j}}\partial_{\bar{k}}=\overline{v}T(\delta_{\bar{j}},\delta_i) & \tau^{\bar{k}}_{i\bar{j}}\delta_{\bar{k}}=\overline{h}T(\delta_{\bar{j}},\delta_i)\\
Q^{k}_{ij}\delta_k=hT(\partial_j,\delta_i) & \rho^{k}_{i\bar{j}}\partial_k=vT(\partial_{\bar{j}},\delta_i) & \Upsilon^{k}_{i\bar{j}}\delta_k=hT(\partial_{\bar{j}},\delta_i)\\
P^{k}_{ij}\partial_k=vT(\partial_j,\delta_i) & \Sigma^{\bar{k}}_{i\bar{j}}\partial_{\bar{k}}=\overline{v}T(\delta_{\bar{j}},\partial_i) & \chi^{k}_{i\bar{j}}\partial_k=vT(\partial_{\bar{j}},\partial_i)\\
S^{k}_{ij}\partial_k=vT(\partial_j,\delta_i) & \Sigma^{k}_{i\bar{j}}\partial_k=vT(\delta_{\bar{j}},\partial_i) & \chi^{\bar{k}}_{i\bar{j}}\partial_{\bar{k}}=\overline{v}T(\partial_{\bar{j}},\partial_i)
\end{array}\tag{1.9}$$

and their conjugates.

The local expresions of torsions are obtained by direct calculation

$$\begin{array}{lll}
T^{k}_{ij}=L^{k}_{ij}-L^{k}_{ji}, & \Theta^{k}_{i\bar{j}}=\frac{\delta N^{k}_{i}}{\delta\bar{z}^{j}}, & \tau^{k}_{i\bar{j}}=L^{k}_{ij}\\
R^{k}_{ij}=\frac{\delta N^{k}_{j}}{\delta z^{i}}-\frac{\delta N^{k}_{i}}{\delta z^{j}}, & \Theta^{\bar{k}}_{i\bar{j}}=-\frac{\delta\overline{N}^{k}_{j}}{\delta z^{i}}, & \tau^{\bar{k}}_{i\bar{j}}=-L^{\bar{k}}_{i\bar{j}}\\
Q^{k}_{ij}=C^{k}_{ij}, & \rho^{k}_{i\bar{j}}=-\frac{\partial N^{k}_{i}}{\partial\overline{\eta}^{j}} & \Upsilon^{k}_{i\bar{j}}=C^{k}_{i\bar{j}}\\
P^{k}_{ij}=\frac{\partial N^{k}_{i}}{\partial\eta^{j}}-L^{k}_{ji}, & \Sigma^{\bar{k}}_{i\bar{j}}=-\frac{\partial\overline{N}^{k}_{j}}{\partial\eta^{i}}, & \chi^{k}_{i\bar{j}}=C^{k}_{i\bar{j}}\\
S^{k}_{ij}=C^{k}_{ij}-C^{k}_{ji}, & \Sigma^{k}_{i\bar{j}}=-L^{k}_{i\bar{j}}, & \chi^{\bar{k}}_{i\bar{j}}=-C^{\bar{k}}_{\bar{j}i}
\end{array}\tag{1.10}$$

and their conjugates.

The nonvanishing curvatures are

$$\begin{aligned}
&R^i_{jkh}\delta_i = R(\delta_h,\delta_k)\delta_j \quad R^{\bar i}_{\bar j kh}\delta_{\bar i} = R(\delta_h,\delta_k)\delta_{\bar j} \quad R^{\bar i}_{\bar j\bar k h}\delta_{\bar i} = R(\delta_h,\delta_{\bar k})\delta_{\bar j}\\
&S^i_{jkh}\partial_i = R(\partial_h,\partial_k)\partial_j \quad S^{\bar i}_{\bar j kh}\partial_{\bar i} = R(\partial_h,\partial_k)\partial_{\bar j} \quad S^{\bar i}_{\bar j\bar k h}\partial_{\bar i} = R(\partial_h,\partial_{\bar k})\partial_{\bar j}\\
(1.11)\quad &P^i_{jkh}\partial_i = R(\partial_h,\delta_k)\partial_j \quad P^{\bar i}_{\bar j kh}\partial_{\bar i} = R(\partial_h,\delta_k)\partial_{\bar j} \quad P^{\bar i}_{\bar j\bar k h}\partial_{\bar i} = R(\partial_h,\delta_{\bar k})\partial_{\bar j}\\
&\Omega^i_{j\bar k h}\partial_i = R(\delta_h,\delta_{\bar k})\partial_j \quad \Omega^{\bar i}_{\bar j\bar k h}\partial_{\bar i} = R(\delta_h,\delta_{\bar k})\partial_{\bar j} \quad \Omega^{\bar i}_{\bar j kh}\partial_{\bar i} = R(\delta_h,\delta_k)\partial_{\bar j}\\
&\Pi^i_{j\bar k h}\partial_i = R(\delta_h,\partial_{\bar k})\partial_j \quad \chi^{\bar i}_{\bar j k\bar h}\partial_{\bar i} = R(\delta_{\bar h},\partial_k)\partial_{\bar j}
\end{aligned}$$

and conjugates.

The local expressions of curvatures are easily obtained from (1.11).

A system of functions $W^{k_1\ldots k_p\bar k_1\ldots\bar k_k}_{h_1\ldots h_q\bar h_1\ldots\bar h_s}(z,\eta)$ is a *complex d-tensor field* if at (1,1) transformations are changed by the rule

$$(1.12)\qquad W'^{i_1\ldots\ldots i_p\bar i_1\ldots\ldots\bar i_r}_{j_1\ldots\ldots j_q\bar j_1\ldots\ldots\bar j_s}(z',\eta') = \frac{\partial z'^{i_1}}{\partial z^{k_1}}\cdot\ldots\cdot\frac{\partial z'^{i_p}}{\partial z^{k_p}}\cdot\frac{\partial z^{h_1}}{\partial z'^{j_1}}\cdot\ldots\cdot\frac{\partial z^{h_r}}{\partial z'^{j_r}}\cdot$$

$$\cdot\frac{\partial\bar z'^{i_1}}{\partial\bar z^{k_1}}\cdot\ldots\cdot\frac{\partial\bar z'^{i_r}}{\partial\bar z^{k_r}}\cdot\frac{\partial\bar z^{h_1}}{\partial\bar z'^{j_1}}\cdot\ldots\cdot\frac{\partial\bar z^{h_s}}{\partial\bar z'^{j_s}}W^{k_1\ldots k_p\bar k_1\ldots\overline{kr}}_{h_1\ldots h_q\bar h_1\ldots\bar h_s}(z,\eta).$$

Examples of complex d-tensor fields are: δ_k , $\delta_{\bar k}$, ∂_k , $\partial_{\bar k}$, C^i_{jk} , $C^{\bar i}_{\bar j\bar k}$, $C^{\bar i}_{\bar j k}$ the components of torsions and curvatures, etc.

A Hermitean metric structure on $T_C(T'M)$ is a nondegenerate 2-field $G(X,Y) = \overline{G(Y,X)}$ and with respect to the commutative almost semiquaternion structure $(J\ ,F)$ it satisfies the conditions

$$(1.13)\qquad \begin{aligned} &G(JX,JY) = G(X,Y)\\ &G(FX,FY) = G((h+\bar h)X(h+\bar h)Y).\end{aligned}$$

Expressing G in adapted base and taking into account the required conditions (1.13), we obtain

Proposition 1.3. *G defines a Hermitian metric structure on $T_C(T'M)$ if and only if*

$$(1.14)\qquad G = g_{i\bar j}dz^i\otimes d\bar z^j + g_{i\bar j}\delta\eta^i\otimes\delta\bar\eta^j,$$

where $g_{i\bar j}(z,\eta)$ is a complex d-tensor field of $\begin{pmatrix}0 & \bar 0\\ 1 & \bar 1\end{pmatrix}$ type and nondegenerate.

From $F\cdot F^* = v+\bar v$ it results that $G(F^*X,F^*Y) = G((v+\bar v)X,(v+\bar v)Y)$.

A complex N-linear conection D is said to be *metric* with respect to the Hermitian metric structure G if $DG = 0$.

Let's denote by " † " and by " | "the h- and respectively v-covariant derivatives of one complex d- tensor field with respect to D', and by " † †" and " || ", the $\bar h$- and respectively $\bar v$-derivatives with respect to D''. It follows that D is a complex metric N-linear conection if and only if:

$$(1.15)\qquad g_{i\bar j\dagger k} = 0\quad ,\quad g_{i\bar j|\,k} = 0\quad ,\quad g_{i\bar j\dagger\dagger\bar k} = 0\quad ,\quad g_{i\bar j||\,\bar k} = 0$$

Analogous reasons as in [7] show that if D is a complex metric d-linear connection then its distributions are ortogonal, and if D is compatible with respect to one metric semiquaternionic structure (G, J, F), then D is also, with respect to the other (G, J, F^*) structure. Therefore, D is also a normal d-l.c.c.

In [7] is given the general set of metric almost semiquaternionic connections for the anticommutative case. In the present commutative case similar calculus gives:

Theorem 1.4. *If* $\overset{\circ}{\nabla}$ *is the Levi-Civita connection of the metric* G*, then a metric connection with respect to the structure* (G, J, F) *is:*

$$2\overset{c}{D}_X Y = (v+\overline{v})\overset{\circ}{\nabla}_X (v+\overline{v})Y - (v+\overline{v})J\overset{\circ}{\nabla}_X (v+\overline{v})JY + F^*\overset{\circ}{\nabla}_X FY - JF^*\overset{\circ}{\nabla}_X JF^*Y. \tag{1.16}$$

It is easly to find that $\overset{c}{D}$ is a normal d-l.c.c. and is metric because $\overset{\circ}{\nabla}$ is metric. It's local expression will be given in the next section.

2 Complex Lagrange spaces

The notion of complex Finsler space is already well-known from the papers of S. Kobayashi [4], H. Rund [10], H. Royden [9], M. Fukui [2], and T. Aikou [1]. Giving up to the homogeneity condition with respect to η, we shall define the notion of complex Lagrange metric.

Definition 2.1. A *complex Lagrangian* on $T'M$ is a smooth real function $L : T'M \to R$, with the property that

$$g_{i\overline{j}} = \frac{1}{2}\frac{\partial^2 L}{\partial\eta^i\partial\overline{\eta}^j} \tag{2.1}$$

is a nondegenerate complex d-tensor field.

The pair $(T'M, L)$ is called a *complex Lagrange space.* In the particular case when $L : (z,\eta) \to L(z,\eta)$ is absolutely homogeneous of degree 1 in η we obtain the notion of *complex Finsler space.*

As it is known, the existence of a Lagrange function involves the study of the variational problem on curves. Let $L(z,\eta)$ be a complex Lagrangian on $T'M$ and $c : [0,1] \to M$ be a holomorphic curve. The Euler-Lagrange equation for a geodesic is ([1], [9]):

$$\frac{\partial L}{\partial z^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \eta^i}\right) = 0 \quad \text{with} \quad \eta^i = \frac{dz^i}{dt}. \tag{2.2}$$

Developing the calculus in (2.2), we obtain

$$(2g_{i\overline{j}}\frac{d^2\overline{z}^j}{dt^2} + \frac{\partial^2 L}{\partial\overline{z}^j\partial\eta^i}\overline{\eta}^j) + (2a_{ij}\frac{d^2 z^j}{dt^2} + \frac{\partial^2 L}{\partial z^j\partial\eta^i}\eta^j - \frac{\partial l}{\partial z^i}) = 0, \tag{2.3}$$

where $a_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial\eta^i\partial\eta^j}$ is a symmetric d-tensor field.

In [1], [9] is defined the notion of complex geodesic in a complex Finsler space as being a curve $\gamma(t) = \varphi(e^{i\theta}\cdot t)$ independent of θ, and φ is a holmorphic map on the small disk $\Delta(r)$ to M satisfying $\varphi(0) = z$ and $d\varphi(0) = \eta$, where (z,η) is a fixed point.

Taking into account the independence on θ of γ, we shall define a *geodesic complex Lagrange space* as being a curve that satisfies the system of equations

$$2g_{i\bar{j}}\frac{d^2\overline{z}^j}{dt}+\frac{\partial^2 L}{\partial\overline{z}^j\partial\eta^i}\overline{\eta}^j=0 \tag{2.4}$$

$$2a_{ij}\frac{d^2 z^j}{dt^2}+\frac{\partial^2 L}{\partial z^j\partial\eta^i}\eta^j-\frac{\partial L}{\partial z^i}=0. \tag{2.5}$$

Theorem 2.1. *Let's consider the functions* H^j *given by*

$$H^j=\frac{1}{4}g^{\bar{i}j}\frac{\partial^2 L}{\partial z^k\partial\overline{\eta}^i}\eta^k. \tag{2.6}$$

Then $N_i^j=\frac{\partial H^j}{\partial\eta^i}$ *are the coefficients of one complex nonlinear connection.*

The proof consists in verifying by direct calculus that N_i^j satisfies the law (1.2), and hence these are the coefficients of a complex nonlinear connection on $T'(T'M)$. The conjugate $\overline{N}_i^j=\frac{\partial\overline{H}^j}{\partial\overline{\eta}^i}$, where $\overline{H}^j=\frac{1}{4}g^{\bar{i}j}\frac{\partial^2 L}{\partial\overline{z}^k\partial\eta^i}\overline{\eta}^k$ are the coefficients of a complex nonlinear connection on $T''(T'M)$.

Let's note that if (a_{ij}) is a nondegenerate complex matrix, then $N_i^j=\frac{\partial G^j}{\partial\eta^i}$, with $G^j=\frac{1}{4}a^{ij}(\frac{\partial^2 L}{\partial z^k\partial\eta^i}\eta^k-\frac{\partial L}{\partial z^i})$ determine another complex nonlinear connection on $T'(T'M)$.

In the real case it is known that a geodesic determines a spray and conversely ([6]). We shall extend this topic to the complex case.

Let be $\Gamma=\eta^i\frac{\partial}{\partial\eta^i}+\overline{\eta}^i\frac{\partial}{\partial\overline{\eta}^i}$ the complex Liouville d-field and T the natural almost tangent structure on $T'M$, $T(\frac{\partial}{\partial z^i})=\frac{\partial}{\partial\eta^i}$, $T(\frac{\partial}{\partial\eta^i})=0$, $T(\frac{\partial}{\partial\overline{z}^i})=\frac{\partial}{\partial\overline{\eta}^i}$, $T(\frac{\partial}{\partial\overline{\eta}^i})=0$ (globally defined because $T'M$ is a holomorphic manifold). A *complex spray* is avector field $S\in\chi(T'M)$ with the property $T(S)=\Gamma$. It follows that a complex spray is locally given in the form

$$S=\eta^i\frac{\partial}{\partial z^i}-2G^i\frac{\partial}{\partial\eta^i}+\overline{\eta}^i\frac{\partial}{\partial\overline{z}^i}-2\overline{H}^i\frac{\partial}{\partial\overline{\eta}^i}, \tag{2.7}$$

where G^i and $\overline{H}^i$ are the coefficients of the spray and have the following rules of transformation:

$$2G'^i=2\frac{\partial z'^i}{\partial z^j}G^j-\frac{\partial^2 z'^i}{\partial z^k\partial z^j}\eta^k\eta^j \quad \textit{and} \quad 2\overline{H}'^i=2\frac{\partial\overline{z}'^i}{\partial\overline{z}^j}\overline{H}^j-\frac{\partial^2\overline{z}'^i}{\partial\overline{z}^k\partial\overline{z}^j}\overline{\eta}^k\overline{\eta}^j. \tag{2.8}$$

Proposition 2.2. *The functions* $\overset{(1)}{N}{}_j^i=\frac{\partial G^i}{\partial\eta^j}$ *and* $\overset{(2)}{N}{}_j^i=\frac{\partial\overline{H}^i}{\partial\overline{\eta}^j}$ *define complex nonlinear connections.*

Remark. From (2.7) and (2.8) it results in particular that $\overline{H}^i$ can be taken the conjugate of G^i.

Proposition 2.3. *A complex spray* S *determines a complex differential system*

$$\frac{d^2 z^i}{dt^2}+2G^i(z,\eta,\overline{z},\overline{\eta})+\frac{d^2\overline{z}^i}{dt^2}+2\overline{H}^i(z,\eta,\overline{z},\overline{\eta})=0 \tag{2.9}$$

and coversely.

Proof. Let's consider the complex spray S and the real curves $c : t \to (z + \overline{z})$ with the tangent vectors $\frac{d(z^i + \overline{z}^i)}{dt} = \eta^i + \overline{\eta}^i$. By derivation is obtained: $\frac{d^2 z^i}{dt^2} + \frac{d^2 \overline{z}^i}{dt^2} = \frac{d}{dt}(\eta^i + \overline{\eta}^i) = -2x^i$, where x^i is a real number that can be of the form $x^i = G^i + \overline{H}^i$ (with $ImG^i = -Im\overline{H}^i$). Conversely, G^i and $\overline{H}^i$ from (2.9) determines the complex spray given by (2.7).

Proposition 2.4. *A complex spray S determines a complex geodesic of the complex Lagrange space (M, L) if and only if there exist such local maps on $T'M$, in wich the coordinates η^i are real.*

Proof. The equation (2.3) is a complex spray iff $g_{i\overline{j}} = a_{ij}$, and therefore $\frac{\partial}{\partial \eta^i}(\frac{\partial L}{\partial \eta^j} - \frac{\partial L}{\partial \overline{\eta}^j}) = 0 \Rightarrow \frac{\partial}{\partial \eta^j} = \frac{\partial}{\partial \overline{\eta}^j} \Rightarrow \eta^j \in R$.

In the next we shall make some references to the particular case of complex Finsler spaces.

From the homogeneity condition $L(z, \lambda\eta) =\mid \lambda \mid^2 L(z, \eta)$, it results the complex Euler Theorem, $\frac{\partial L}{\partial \eta^i}\eta^i = 2L$ and the conjugate $\frac{\partial L}{\partial \overline{\eta}^i}\overline{\eta}^i = 2L$. Deriving, it results that

$$\frac{\partial g_{k\overline{j}}}{\partial \eta^i}\eta^i = \frac{\partial g_{i\overline{j}}}{\partial \eta^k}\eta^i = g_{k\overline{j}} \quad \text{and} \quad \frac{\partial g_{j\overline{k}}}{\partial \overline{\eta}^i}\overline{\eta}^i = \frac{\partial g_{j\overline{i}}}{\partial \overline{\eta}^k}\overline{\eta}^i = g_{j\overline{k}}. \tag{2.10}$$

Let us consider $\Gamma^i_{jk} = \frac{1}{2}g^{\overline{l}j}(\frac{\partial g_{j\overline{l}}}{\partial z^k} + \frac{\partial g_{k\overline{l}}}{\partial z^j})$ the first Christoffel symbols of the Hermitean metric $g_{i\overline{j}}$ ([3], [5]) and let's denote by $\mathcal{E} = g_{i\overline{j}}\eta^i\overline{\eta}^j$ the *complex energy* of complex Finsler space (M, L).

Theorem 2.5. *The functions*

$$\overset{c}{N}{}^j_i = \frac{1}{2}\frac{\partial \Gamma^j_{00}}{\partial \eta^i}, \quad \text{where} \quad \Gamma^j_{00} = \Gamma^j_{kl}\eta^k\eta^l, \tag{2.11}$$

are the coefficients of a complex nonlinear connection on $T'M$, called the Cartan nonlinear connection of the Finsler space (M, L).

Proof. Using (2.10), by direct calculus it is found out that H^j from (2.6) is: $H^j = \frac{1}{2}\Gamma^j_{00} = \frac{1}{2}g^{\overline{i}j}\frac{\partial g_{l\overline{i}}}{\partial z^k}\eta^l\eta^k$.

From (2.4) and (2.5) it results that:

Proposition 2.6. *The autoparallel curves of the Cartan nonlinear connection of the complex Finsler space (M, L) are just the extremal curves of the energy action $\mathcal{E}$.*

Now, let's return to the general case of the complex Lagrangian space (M, L), endowed with a fixed nonlinear connection, for example given by (2.6). From Prop. 1.3 it results that G defined by (1.14) with the Lagrange metric $g_{i\overline{j}}$ given by (2.1), is a Hermitean metric structure on $T_C(T'M)$. Then the set (M, G, J, F) defined by (1.6), (1.7) and (1.13) is called the *metric commutative almost semiquater nionic* model of the complex Lagrange space (M, L).

We are interested to determine a metric d-l.c.c. of the complex Lagrange space (M, L).

Theorem 2.7. *The following normal N-linear complex connection $\overset{c}{D}$ is metric*

$$\begin{array}{ll} L^i_{jk} = \frac{1}{2}g^{\overline{l}i}(\frac{\delta g_{j\overline{l}}}{\delta z^k} + \frac{\delta g_{k\overline{l}}}{\delta z^j}) & C^i_{jk} = \frac{1}{2}g^{\overline{l}i}(\frac{\partial g_{j\overline{l}}}{\partial \eta^k} + \frac{\partial g_{k\overline{l}}}{\partial \eta^j}) \\ L^{\overline{i}}_{\overline{j}k} = \frac{1}{2}g^{\overline{i}l}(\frac{\delta g_{l\overline{j}}}{\delta z^k} - \frac{\delta g_{k\overline{j}}}{\delta z^l}) & C^{\overline{i}}_{\overline{j}k} = \frac{1}{2}g^{\overline{i}l}(\frac{\partial g_{l\overline{j}}}{\partial \eta^k} - \frac{\partial g_{k\overline{j}}}{\partial \eta^l}) \end{array} \tag{2.12}$$

Moreover, $\overset{c}{D}$ *has the following vanishing torsions:* $T^i_{jk} = 0$ *and* $S^i_{jk} = 0$.

Proof. Let $\overset{\circ}{\nabla}$ be the Levi-Civita connection of the metric G given by (1.14) with the complex Lagrange metric $g_{i\bar{j}}$. Since G is Hermitean with respect to J, the coefficients Γ^i_{jk} , $\Gamma^i_{j\bar{k}}$ and $\Gamma^i_{\bar{j}k}$ and their conjugates are known from complex manifolds ([3], [5]). Replacing this in (1.16) it results the coefficients (2.12) of $\overset{c}{D}$ connection in the adapted base. The connection (2.12) is metric and normal. From the symmetry of Γ^i_{jk} it results that $T^i_{jk} = S^i_{jk} = 0$.

The connection (2.12) will be called the *Miron canonic N-linear complex connection* of the complex Lagrange space (M, L).

3 The complex Lagrangian of electrodynamics

In [6] it is given a Lagrangian model for real electrodinamycs. In the present section we shall consider a similar model for the complex case. The complex Lagrangians frequently appear in quantum mechanics and gauge theories ([11]). Inspired from these models, let's consider the complex Lagrangian

$$L_0(z,\eta) = mc\gamma_{i\bar{j}}(z)\eta^i\bar{\eta}^j + \frac{e}{m}A_i(z)(\eta^i + \bar{\eta}^i), \tag{3.1}$$

where $\gamma_{i\bar{j}}$ is a Hermitian metric on the complex manifold M ; m, c, e are real constants and A_i is a real covector.

The metric $\gamma_{i\bar{j}}$ can be taken as the extension through linearity of Lorentz metric to the complexification $T_C M$.

From (2.1), it results that L_0 determine the following Hermitian metric on $T'M$:

$$g_{i\bar{j}} = mc\gamma_{i\bar{j}}(z). \tag{3.1'}$$

The metric $g_{i\bar{j}}$ determines a nonlinear connection $N^j_i = \frac{\partial H^j}{\partial \eta^i}$, where H^j is given by (2.6). So, we have:

$$N^j_i = \frac{1}{2}\frac{\partial \Gamma^j_{00}}{\partial \eta^i} + \frac{e}{4m^2 c}\gamma^{\bar{k}j}\frac{\partial A_k}{\partial z^i}, \tag{3.2}$$

where Γ^i_{jk} are the first Christoffel coefficients of $\gamma_{i\bar{j}}$ metric and $\Gamma^i_{00} = \Gamma^i_{jk}\eta^j\eta^k$. Accordingly, the Miron canonic connection (2.12) has the components:

$$\begin{aligned} L^i_{jk} &= \Gamma^i_{jk} = \tfrac{1}{2}\gamma^{\bar{l}i}\left(\tfrac{\partial \gamma_{j\bar{l}}}{\partial z^k} + \tfrac{\partial \gamma_{k\bar{l}}}{\partial z^j}\right) \quad C^i_{jk} = 0 \\ L^{\bar{i}}_{\bar{j}k} &= \Gamma^{\bar{i}}_{\bar{j}k} = \tfrac{1}{2}\gamma^{\bar{i}l}\left(\tfrac{\partial \gamma_{l\bar{j}}}{\partial z^k} + \tfrac{\partial \gamma_{k\bar{j}}}{\partial z^l}\right) \quad C^{\bar{i}}_{\bar{j}k} = 0. \end{aligned} \tag{3.3}$$

From (1.10) we obtain the following vanishing torsions: $T^k_{ij} = S^k_{ij} = Q^k_{ij} = \Upsilon^k_{ij} = \chi^k_{i\bar{j}} = \chi^{\bar{k}}_{i\bar{j}} = 0$ and their conjugates. Because $C^i_{jk} = C^{\bar{I}}_{\bar{j}k} = 0$, it results that the Miron canonic connection has the next nonvanishing components of curvatures:

$$R^i_{jkh} = r^i_{jkh}; \quad R^{\bar{i}}_{\bar{j}kh} = r^{\bar{i}}_{\bar{j}kh}; \quad R^{\bar{i}}_{\bar{j}\bar{k}h} = r^{\bar{i}}_{\bar{j}\bar{k}h}$$

(and the conjugates) where r^i_{jkh} ,..., are the curvatures of the linear connection Γ^i_{jk}.

For the real Lagrangian space from [6] we know the Einstein equations with respect to the canonic connection. They result from the Einstein equations

$\mathrm{R}_{\alpha\beta} - \frac{1}{2}\mathrm{R}\mathrm{G}_{\alpha\beta} = \chi \mathrm{T}_{\alpha\beta}$ ($\mathrm{R}_{\alpha\beta}$=Ricci tensor, R=Ricci curvature, $\mathrm{G}_{\alpha\beta}$=the metric structure, χ=universal constant, $\mathrm{T}_{\alpha\beta}$ = the energy-momentum tensor), written in the adapted base of the canonic nonlinear connection. Acting in a similar manner for the complex Lagrangian, it results the following set of Einstein equations

$$\begin{aligned} &R_{i\bar{j}} - \tfrac{1}{2}\rho g_{i\bar{j}} = \chi \overset{h\bar{h}}{T}_{i\bar{j}}, \quad R_{ij} = \chi \overset{hh}{T}_{ij}, \quad P_{ij} = \chi \overset{hv}{T}_{ij} \\ &S_{i\bar{j}} - \tfrac{1}{2}\rho g_{i\bar{j}} = \chi \overset{v\bar{v}}{T}_{i\bar{j}}, \quad S_{ij} = \chi \overset{vv}{T}_{ij}, \quad \pi_{i\bar{j}} = \chi \overset{h\bar{v}}{T}_{i\bar{j}}, \end{aligned} \tag{3.4}$$

where $R_{i\bar{j}} = \overline{R}_{\bar{i}j} = -R^k_{i\bar{j}k}$, $e.t.c., \rho = r + s \in R$, $r = g^{\bar{i}j} R_{\bar{i}j}$, $s = g^{\bar{i}j} S_{\bar{i}j}$, $\overset{h\bar{h}}{T}_{i\bar{j}}, e.t.c.$= the components of energy-momentum tensor, $\chi \in R$.

The conservation law with respect to the one X_α field of adapted base is

$$\overset{c}{D}_{X_\alpha} (R^\alpha_\beta - \frac{1}{2} R\delta^\alpha_\beta) = 0, \quad \text{where} \quad R^\alpha_\beta = G^{\alpha\gamma} R_{\gamma\beta}.$$

In the particular case of the complex Lagrangian L_0 for electrodynamics given by (3.1), taking into account the nonvanishing curvatures, the first (3.4) Einstein equation is just that for the Levi-Civita connection of $\gamma_{i\bar{j}}$ metric, in the secound equation $S_{i\bar{j}} = 0$, and the last three equations reduce to $\overset{vv}{T} = \overset{hv}{T} = \overset{h\bar{v}}{T} = 0$. The conservation law is identically satisfied.

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