Filed of Cones on (k, r)-Covelocities Vector Bundle on a Manifold

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Abstract

The study of the some important fields of geometrical objects (Riemannian or Lorenzian metrics) leads to some fields of cones defined on the tangent bundles.

In this Note, field of cones on (k,r)-covelocities vector bundle on a manifold and special case of r-th order tangent vector bundle are studied.

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Dennis Sullivan ([8],1976) associated to a field of convex and compact cones (cone of structure) in the bundles of tangent p-vectors on a manifold M a field of cones (cone structure) of currents (linear functional on C^{∞}-forms) generated by the Dirac currents.

The currents are "directed" by an a-priori given field of cones in the space of tangent p-vectors and a positivity condition leads to a compact convex cone of currents with a compact convex subcone of cycles (closed currents).

Dan I. Papuc ([5],1992) introduced the notion of field of cones in a regular vector bundle and studied the geometry of this kind of general structure with direct application to the particular case of the tangent bundle to a maniflod (see also [2]-1988, [3]-1988, [4]-1990).

Definition 1. (Dennis Sullivan [8], 1976) A *compact convex* cone C in a (locally convex topological) vector space V over $\mathbf{R}$ is a convex cone which for some (continuous) linear functional $L : V \to \mathbf{R}$ satisfies $L(x) > 0,\ \forall x \in C,\ x \neq 0$ and $L^{-1} \cap C$ is compact.

The set $L^{-1} \cap C$ is called a *base* for the cone often identified with the set of rays in the cone, and the kernel of L is called a *strictly supporting hyperplane* of the cone C.

Definition 2. A *cone structure* (field of cones) on a closed subset N of a manifold M is a map

$$C : x \in N \to C_x,$$

where C_x is a compact convex cone in the vector space $A_p(x)$ of tangent p-vectors on $M,\ x \in M$.

Continuity of cones is defined by the Hausdorf metric:

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$$d(K,K') = \max(\sup_{x\in K} \rho\,(x,K'),\ \sup_{x'\in K'} \rho\,(x',K'))$$

$$K = L^{-1}(1) \cap C,\ K' = L'^{-1} \cap C',$$

where ρ is a convenient metric on rays in some local trivialization of Λ_p.

Definition 3. A differential p-form ω (of class C^∞) on M is transversal to the cone structure C if

$$\forall\ \nu \neq 0, \nu \in C_x,\ (x \in N) \Rightarrow \omega\,(\nu) > 0.$$

Definition 4. A current (linear functional on C^∞-forms) is *Dirac current* if is one determined by the evaluation of p-forms on a single p-vector at one point.

Definition 5. The *cone of structure currents* C^* associated to the cone structure C is the closed convex cone of currents generated by the Dirac currents associted to elements of $C_x,\ x \in N$.

If N is compact the cone of structure currents C^* associated to the cone structure C on N is a compact convex cone.

Definition 6. (Dan I. Papuc [5], 1992) A field of cones on a regular vector bundle $\xi = (E,p,M)$ (E,M paracompact connected manifolds without boundary) is a map

$$K : x \in M \to K_x \subset E_x \subset E\ (E_x = p^{-1}(x))$$

such that the following two axioms to be satisfied:

1) $\forall x \in M,\ K_x$ is a convex pointed closed cone having interior points in the space E_x).

2) The set $\bigcup_{x\in M} Int K_x$ and $\bigcup_{x\in M} Int(E_x - K_x)$ are open subsets of E.

The pair (E_x, K_x) is a Krein space, the norm of E_x is arbitrarily chosen, and all considerations concerning Krein spaces are available for (E_x, K_x).

An example of structure (ξ, K) is the tangent vector bundle of a Lorentzian time-oriented manifold (K_x is the quadratic cone of non space -like time- oriented tangent vectors).

The geometry of structure (ξ, K) was studied in [4],[5],[6].

In [8] to a regular vector bundle endowed with a field of cones there are associated the r-th jet prolongation and the initial field of cones are lifted in a natural way.

Remark 1. ([6], 1992, p.44) A regular vector bundle $\xi = (E,\pi,M)$ has a field of cones if and only if there is a continuous global section of non-zero vectors

$$s : M \to E,\ \pi \circ s = 1_M,\ \forall x \in M,\ s(x) \neq 0.$$

In the following we shall use for fields of cones the Definition 6 (with the supplementary condition as the section from previous remark to be of class C^∞ if it is necessary) because such fields of cones also are fields of cones introduced by the Definition 1.

Let M be a manifold. The space $T_k^{r^*}M = J^r(M,R^k)_0$ (the set of all r-jets of M into R^k with target 0) is called the space of (k,r)-*covelocities* on M.

Since R^k is a vector space, the fibber bundle $(T_k^{r^*}M, \pi_k^{r^*}, M)$ is a vector bundle which verifies

$$j^r{}_x f + j^r{}_x g \ \stackrel{def}{=}\ j^r{}_x(f+g)$$

$$j^r{}_x(\lambda f)\ \stackrel{def}{=}\ \lambda(j^r{}_x f),\ \ \forall j^r{}_x f, j^r{}_x g \in E_x = (\pi_k^{r^*})^{-1}(x),\ \lambda \in R,\ x \in M.$$

It is called the vector bundle of (k,r)- *covelocities* on M. Every local diffeomorfism $\phi : M \to N$ is extended to a vector bundle morphism $T_k^{r^*}\varphi : j^r_x \in T_k^{r^*}M \to j^r_{\varphi(x)}(f \circ \varphi^{-1}) \in T_k^{r^*}N$, where φ^{-1} is constructed locally.

In this sense $T_k^{r^*} : Man(m) \to FV$ is a functor.
($Man(m)$ is the category of m- dimensional manifolds whose morphisms- locall diffeomorfisms and FV is the category of vector bundle).

For $k = r = 1$ we obtain the construction of the cotangent bundle as a functor $T_1^{1^*} = T^*$ on $Man(m)$.

Proposition 1. For every field of cones K on tangent bundle (TM, π, M) of the manifold M we can define a "canonical" field of cones K_k^{1*} on the vector bundle $(T_k^{1*}M, \pi_k^{1*}, M)$.

Proof. Every field of cones K on the tangent bundle (TM, π, M) defines a field of cones K^* on cotangentfibre bundle (T^*M, π^*, M),

$$K^* : x \in M \to K^*(x) \subset (T^*M)_x,$$

$$K^*(x) \stackrel{def}{=} \{\omega_x / \ \omega_x \in (T^*M)_x,\ \omega_x(X_x) \geq 0,\ \forall X_x \in K(x)\}.$$

By virtue of Remark 1 there is a global section of nonzero covectors

$$\omega : x \in M \to \omega_x \in T^*M,$$

$$\omega_x = j^1_x f \in T^*M = J^1(M, R)_0.$$

Let by $f_k : x \in M \to (f(x), f(x), ..., f(x)) \in R^k$. Obviously, $j^1_x f_k$ depends on the germ of f at x only.

Then $\omega^k : x \in M \to \omega^k_x = j^1_x f_k \in T_k^{1*}M$ is a global section nonzero $(1,k)$ - covelocities and according to Remark 1 there is a field of cones K_k^{1*} on vector bundle $(T_k^{1*}M, \pi_k^{1*}, M)$.

Proposition 2. to every field of cones K_k^{r*} on the vector bundle $(T_k^{r*}M, \pi_k^{r*}, M)$ we can associate a field of cones K_k^{p*} $(p \leq r)$ on the vector bundle $(T_k^{p*}M, \pi_k^{p*}, M)$.

Proof. Indeed $K_k^{p*} : x \in M \to K_k^{p*} \stackrel{def}{=} \{j^p_x f / j^r_x f \in K_k^{r*}(x)\}$ is a field of cones on the vector bundle $(T_k^{p*}M, \pi_k^{p*}, M)$.

Definition 7. The field of cones K_k^{r*} and C_k^{r*} on the vector bundle $(T_k^{r*}M, \pi_k^{r*}, M)$ are *p-equivalent* $0 < p \leq r;\ p \in N)$ if the fields of cones induced by them on the vector bundle $(T_k^{p*}M, \pi_k^{p*}, M)$ coincide.

Evidently if two fields of cones are p- equivalent they are s-equivqlent ,$s < p \leq r$.

The greatest natural number p from the Definition 7 is called *order of equvalence* for the field of cones K_k^{r*} and C_k^{r*}.

The relation introduced by the Definition 7 determines a partition for fields of cones on the vector bundles $(T_k^{p*}M, \pi_k^{p*}, M)$ in class of orders of equivalence.

We remark that for $k = 1$ the fibre of vector bundle $(T_1^{r*}M, \pi_1^{r*}, M) = (T^{r*}M, \pi^{r*}, M)$ is an algebra, and the multiplication in every space $T_x^{r*}M$ is given by

$$j^r_x f \cdot j2\,_x g \stackrel{def}{=} j^r_x (f \circ g).$$

The projection $\pi_{r,r-1} : T^{r*}M \to T^{r-1*}M$ is a linear morphism of vector bundles.

Let $f_1, f_2, ... f_r$ be differentiable functions on M with values 0 at $x \in M$.

The r-jet at x of their product $j^r_x(f_1 f_2 ... f_r)$ depends only on $J^1_x f_1,\ J^1_x f_2, ... J^1_x f_r$ and lies in $\ker(\pi_{r,r-1})$.

We can extended uniquely the map

$$(f_1, f_2, ...f_r) \in T^*M \times T^*M \times ... \times T^*M \to j^r_x(f_1 f_2 ... f_r) \in T^{r*}M$$

into a linear isomorphism of $S^r T^*M$ (the r-th symmetric tensor power of T^*M) in $\ker(\pi_{r,r-1})$.

So, we have the following exact sequence of vector bundles over M :

$$0 \to S^r T^*M \to T^{r*}M \, - \, \pi_{r,r-1} \to T^{r-1*}M \to 0 \tag{1}$$

Remark 2. To every field of cones K on tangent bundle (TM, π, M) we can define a field of cones on $S^r T^*M$:

$$S^{r*}K : x \in M \to S^r K^*(x) \in S^r T^*M$$

$$S^{r*}K(x) \overset{def}{=} \{\rho_x / \rho_x \in S^r T^*M, \rho_x(X_{1x}, X_{2x}, ..., X_{rx}) \geq 0,$$
$$\forall X_{1x}, X_{2x}, ..., X_{rx} \in K(x)\}.$$

Because the sequence (1) of vector bundle over M is exact we can associated to field of cones $S^{r*}K$ a field of cones on $\ker(\imath_{r,r-1} \subset T^{r*}M$ whose projection on $(T^{r-1*}M, \pi^{r-1*}, M)$ is zero vector.

Let by $T^{r\triangleright}_k M = (T^{r*}_k M)^*$,the dual vector bundle of the $(k,r)-$covelocities bundle on M.

For every map $\varphi : M \to N$ we can define the linear map

$$j^r_x f \in (T^{r*}_k)_{f(x)} \to j^r_x(f \circ \varphi) \in (T^{r*}_k M)_x.$$

Its dual map

$$(T^{r\triangleright}_k \varphi)_x : (T^{r\triangleright}_k M)_x \to (T^{r\triangleright}_k N)_{f(x)}$$

determines a functor $T^{r\triangleright}_k$ on *Man* (the category of manifold) with values in the category of vector bundles FV.

In the most important case $k = l$ we shall write $T^{r\triangleright}_k = T^{r\triangleright}$. The element of $T^{r\triangleright}M$ are called *r-th order* tangent vectors on M.

We remark that for $r = l$ we have $\mathrm{T}M = (T^*M)^*$.

Dualizing the exact sequence (1), we obtain

$$0 \to T^{r-1\triangleright}M \to T^{r\triangleright}M \to S^r TM \to 0. \tag{2}$$

This shows that there is a natural injection of the $(r-1)$-st order tangent vectors into the r-th order one.

Proposition 3. To every field of cones K on the tangent bundle $(TM\pi, M)$ we can define a *canonical* field of cones K^r on the vector bundle

$$(T^{r\triangleright}M, \pi^r, M).$$

Because of natural injection of the $(r-1)$-st order tangent vectors into the r-th order one, for a section in tangent bundle TM we can associate to a section of non-zero vectors in vector bundle $(T^{p-1\triangleright}M, \pi^{p-1}, M)$ a section of non-zero vectors in the vector bundle $(T^{p\triangleright}M, \pi^p, M)$ $(p = 2...r)$.

So, there is a section of non-zero vectors in $(T^{r\triangleright}M, \pi^r, M)$ and according with Remark 1 the above proposition is true.

For every map $\varphi : M \to N$ the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T^{r-1\triangleright}M & \longrightarrow & T^{r\triangleright}M & \longrightarrow & S^rTM & \longrightarrow & 0 \\ & & \downarrow T^{r-1\triangleright}\varphi & & \downarrow T^{r\triangleright}\varphi & & \downarrow S^rT\varphi & & \\ 0 & \longrightarrow & T^{r-1\triangleright}N & \longrightarrow & T^{r\triangleright}N & \longrightarrow & S^rTN & \longrightarrow & 0 \end{array} \tag{3}$$

Remark 3. The functors $T^{r-1\triangleright}$ and $T^{r\triangleright}$ from *Man (M)* to FV are natural functors.

The comutativity of diagram (3) and the fact that the sequences are exact make possible the construct a natural *canonical* operators $\Theta : T^{r-1\triangleright} \gg\Rightarrow T^{r\triangleright}$ from the natural functor $T^{r-1\triangleright}$ to the natural functor $T^{r\triangleright}$.

If on the tangent bundle there is a field of cones and we consider on the vector bundle of r-th order tangent vectors the field of associated cones, the *canonical* operators are positive.

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