Complete Minimal Hypersurfaces in a Locally Symmetric Space

Hyang Sook Kim and Yong-Soo Pyo

Abstract

In this paper we shall consider the Riemannian manifold under some supposed curvature conditions and investigate some properties about a complete minimal hypersurface in a locally symmetric space. In particular, we obtain characterizations of a totally geodesic minimal hypersurface in a locally symmetric space.

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1 Introduction

Let $M^m(c)$ be an m-dimensional Riemannian manifold of constant curvature c which is called a *space form* and M be a hypersurface of $M^m(c)$. Many differential geometers have studied M from various points of view. For example, the study of hypersurfaces with constant mean curvature of $M^m(c)$ was initiated by Nomizu and Smyth [4], who proved some results. Later, Okumura [5] and Hasanis [2] characterized totally umbilical hypersurfaces of $M^m(c)$, $c > 0$, under a certain condition given by an inequality between the length of the second fundamental form and the mean curvature. Moreover, Cheng and Nakagawa [1] also generalized their results.

In consideration of these subjects, it seems interesting to the authors to investigate some properties about complete hypersurfaces with constant mean curvature in a locally symmetric space. This is closely relevant to the generalization of the so-called *Bernstein problem.*

The purpose of this paper is to obtain the characterization of totally geodesic hypersurfaces in a locally symmetric space under some supposed curvature conditions and also to characterize complete minimal hypersurfaces in the same one.

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2 Preliminaries

We begin with recalling basic properties of hypersurfaces of a Riemannian manifold. Let (M', g') be an $(n+1)$-dimensional Riemannian manifold with the Riemannian metric g'. Throughout this paper, manifolds are always assumed to be connected and the differentiability of any geometric object is assumed to be of class C^∞. For any point x in M' we choose a local field of orthonormal frames $\{e_A\} = \{e_0, e_1, \cdots, e_n\}$ around x adapted the Riemannian metric g'. We denote $\{\omega_A\} = \{\omega_0, \omega_1, \cdots, \omega_n\}$ by the dual frames on M' which consists of 1-forms on M' such that $\omega_A(e_B) = \delta_{AB}$. Here and in the sequel, the following convention on the range of indices is used, unless otherwise stated :

$$A, B, \cdots = 0, 1, \cdots, n,$$

$$i, j, \cdots = 1, \cdots, n.$$

Associated with the frame field $\{e_A\}$, there exist 1-forms $\{\omega_{AB}\}$. They are usually called *connection forms* on M' so that they satisfy the structure equations of M' :

$$d\omega_A + \sum_A \omega_{AB} \wedge \omega_B = 0, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} + \sum_c \omega_{BC} \wedge d\omega_{CB} = \Omega'_{AB}, \tag{2.2}$$

$$\Omega'_{AB} = -\frac{1}{2} \sum_{C,D} R'_{ABCD} \omega_C \wedge \omega_D, \tag{2.3}$$

where $\Omega' = (\Omega'_{AB})$ (resp. R'_{ABCD}) denotes the curvature form (resp. the components of the Riemannian curvature tensor R') of M'.

Now, relative to the frame field chosen above, the Ricci tensor S' of M' can be expressed as follows :

$$S' = \sum_{A,B} S'_{AB} \omega_A \otimes \omega_B, \tag{2.4}$$

where $S'_{AB} = \sum_C R'_{CABC} = S'_{BA}$. The scalar curvature r' is also given by

$$r' = \sum_A S'_{AA}. \tag{2.5}$$

The components R'_{ABCDE} of the covariant derivative of the Riemannian curvature tensor R' are obtained by

$$\begin{aligned} \sum_E R'_{ABCDE} \omega_E &= dR'_{ABCD} \\ -\sum_E (R'_{EBCD} \omega_{EA} &+ R'_{AECD} \omega_{EB} + R'_{ABED} \omega_{EC} + R'_{ABCE} \omega_{ED}). \end{aligned} \tag{2.6}$$

By $M^m(c)$ we denote an m-dimensional Riemannian manifold of constant curvature c, which is called a *space form*. Then the Riemannian curvature tensor R'_{ABCD} of $M^m(c)$ is given by

$$R'_{ABCD} = c(\delta_{AD}\delta_{BC} - \delta_{AC}\delta_{BD}). \tag{2.7}$$

Next, let (M', g') be an $(n+1)$-dimensional Riemannian manifold and let M be an n-dimensional hypersurface of M'. We choose a local field of orthonormal frames $\{e_A\} = \{e_0, e_j\} = \{e_0, e_1, \cdots, e_n\}$ on a neighborhood of M' in such a way that restricted to M, $e_1, \cdots, e_n$ are tangent to M and e_0 is normal to M. With respect to this frame field, let $\{\omega_A\} = \{\omega_0, \omega_j\}$ be its dual frame field. Then the Riemannian metric tensor g' of M' is given by $g' = \sum_A \omega_A \otimes \omega_A$. The connection forms on M' are denoted by ω_{AB}. Restricting these forms to the submanifold M, we have

$$\omega_0 = 0, \tag{2.8}$$

and the induced Riemannian metric tensor g of M is given by $g = \sum_j \omega_j \otimes \omega_j$. Then $\{e_j\}$ is a local field of orthonormal frames with respect to this metric and $\{\omega_j\}$ is a local dual frame field due to $\{e_j\}$, which consists if 1-forms on M. Moreover, $\omega_1, \cdots, \omega_n$ are linearly independent, and they are said to be *canonical 1-forms* on M. It follows from (2.8) and the Cartan lemma that the exterior derivative of (2.8) gives rise to

$$\omega_{0i} = \sum_j h_{ij}\omega_j, \qquad h_{ij} = h_{ji}. \tag{2.9}$$

The quadratic form $\alpha = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j \otimes e_0$ with values in the normal bundle is called the *second fundamental form* of the submanifold M. From the structure equations of M' it follows that the structure equations for M are similarly given by

$$d\omega_i + \sum_j \omega_{ij} \wedge \omega_j = 0, \qquad \omega_{ij} + \omega_{ji} = 0, \tag{2.10}$$

$$\begin{aligned} d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} &= \Omega_{ij}, \\ \Omega_{ij} &= \sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l. \end{aligned} \tag{2.11}$$

For the Riemannian curvature tensor R and R' of M and M', respectively, it follows from (2.3) and (2.11) that we have the Gauss equation

$$R_{ijkl} = R'_{ijkl} + (h_{il}h_{jk} - h_{ik}h_{jl}). \tag{2.12}$$

The components of the Ricci tensor S and the scalar curvature r of M are given by

$$S_{ij} = \sum_k R'_{kijk} + hh_{ij} - h^2_{ij}, \tag{2.13}$$

(2.14) $$r=\sum_{j,k}R'_{kjjk}+h^2-h_2,$$

where $h=\sum_i h_{ii}$, $h^2_{ij}=\sum_r h_{ir}h_{rj}$ and $h_2=\sum_j h^2_{jj}$.

Now, the components h_{ijk} of the covariant derivative of the second fundamental form α on M are given by

(2.15) $$\sum_k h_{ijk}\omega_k=dh_{ij}-\sum_k(h_{kj}\omega_{ki}+h_{ik}\omega_{kj}).$$

Using (2.1) $\sim$ (2.3), and (2.9) $\sim$ (2.11), it follows from the coefficients of $\omega_j\wedge\omega_k$ that

(2.16) $$h_{ijk}-h_{ikj}=R'_{0ijk}.$$

Similarly the components h_{ijkl} of the covariant derivative of h_{ijk} can be defined by

(2.17) $$\sum_l h_{ijkl}\omega_l=dh_{ijk}-\sum_l(h_{ljk}\omega_{li}+h_{ilk}\omega_{lj}+h_{ijl}\omega_{lk}).$$

On the other hand, differentiating (2.15) exteriorly and using the properties $d^2=0$, (2.10) and (2.11) we see

$$\begin{aligned}&\sum_k(dh_{ijk}\wedge\omega_k-h_{ijk}\sum_l\omega_{kl}\wedge\omega_l)\\&=-\sum_k\Big\{dh_{kj}\wedge\omega_{ki}-h_{jk}(\sum_l\omega_{kl}\wedge\omega_{li}+\frac{1}{2}\sum_{l,m}R_{kilm}\omega_l\wedge\omega_m)\\&+dh_{ik}\wedge\omega_{kj}-h_{ik}(\sum_l\omega_{kl}\wedge\omega_{lj}+\frac{1}{2}\sum_{l,m}R_{kjlm}\omega_l\wedge\omega_m)\Big\}.\end{aligned}$$

Substituting (2.15) and (2.17) into the above equation, we have the following Ricci formula for the second fundamental form on M

(2.18) $$h_{ijkl}-h_{ijlk}=-\sum_r(h_{ir}R_{rjkl}+h_{jr}R_{rikl})$$

from the coefficient of $\omega_k\wedge\omega_l$.

Let us denote the covariant derivative of R'_{ABCD}, as a curvature tensor of M', by $R'_{ABCD;E}$. Then, restricting on M, $R'_{oijk;l}$ is given by

(2.19) $$R'_{0ijk;l}=R'_{0ijkl}-R'_{0i0k}h_{jl}-R'_{0ij0}h_{kl}+\sum_m R'_{mijk}h_{ml},$$

where R'_{0ijkl} denotes the covariant derivative of R'_{0ijk} as a tensor on M so that

(2.20) $$\begin{aligned}\sum_l R'_{0ijkl}\omega_l=dR'_{0ijk}&-\sum_l R'_{0ljk}\omega_{li}\\&-\sum_l R'_{0ilk}\omega_{lj}-\sum_l R'_{0ijl}\omega_{lk}.\end{aligned}$$

For the sake of brevity, a tensor h_{ij}^m and a function h_m on M for any integer $m(\geq 2)$ are introduced as follows :

$$h_{ij}^m = \sum_{i_1,\cdots,i_{m-1}} h_{ii_1}h_{i_1i_2}\cdots h_{i_{m-1}j}, \tag{2.21}$$

$$h_m = \sum_i h_{ii}^m. \tag{2.22}$$

Here we introduce a fundamental property for the generalized maximal principle due to Omori [6] and Yau [7] and quote the following in order to prove our results.

Theorem 2.1. *Let M be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below. Let f be a C^2-function bounded from above on M, then for any $\varepsilon > 0$, there exists a point p in M at which it satisfies*

$$\sup f - \varepsilon < f(p), \ \ |grad\ f(p)| < \varepsilon \ \ and \ \ \Delta f(p) < \varepsilon, leqno(2.23)$$

where Δ is the Laplacian operator on M.

3 The Laplacian operator

In this section we shall consider the Laplacian of the second fundamental form. Let M be a hypersurface of an $(n+1)$-dimensional Riemannian manifold M' and let us denote by α the second fundamental form on M. Then the Laplacian $\triangle h_{ij}$ of the components h_{ij} of α is defined by

$$\triangle h_{ij} = \sum_k h_{ijkk}. \tag{3.1}$$

From (2.16) and (2.18) it follows that

$$\begin{aligned} \triangle h_{ij} &= \sum_k h_{ikjk} + \sum_k R'_{0ijkk} \\ &= \sum_k h_{kijk} + \sum_k R'_{0ijkk} \\ &= \sum_k \{h_{kikj} - \sum_l (h_{il}R_{lkjk} + h_{kl}R_{lijk}) + R'_{0ijkk}\}. \end{aligned}$$

Replacing h_{kikj} with $h_{kkij} + R'_{0kikj}$ in the above equation, we get

$$\begin{aligned} \triangle h_{ij} = \sum_k h_{kkij} &+ \sum_k (R'_{0ijkk} + R'_{0kikj}) \\ &- \sum_{k,l} (h_{il}R_{lkjk} + h_{kl}R_{lijk}). \end{aligned}$$

Combining (2.12) and (2.19) with the above equation, we obtain

$$\begin{aligned}\triangle h_{ij} &= \sum_k h_{kkij} + \sum_k (R'_{0ijk;k} + R'_{0kik;j}) \\ &+ \sum_k (h_{jk}R'_{0i0k} + h_{kk}R'_{0ij0} - \sum_l h_{kl}R'_{lijk}) \\ &+ \sum_k (h_{ij}R'_{0k0k} + h_{kj}R'_{0ki0} - \sum_l h_{jl}R'_{lkik}) \\ &- \sum_{k,l} (R'_{kjkl} + h_{kl}h_{jk} - h_{kk}h_{jl})h_{il} \\ &- \sum_{k,l} (R'_{kjil} + h_{kl}h_{ij} - h_{lj}h_{ik})h_{kl}\end{aligned}$$

and hence we have

$$\begin{aligned}\Delta h_{ij} = \sum_k h_{kkij} &+ \sum_k (R'_{0ijk;k} + R'_{0kik;j}) \\ &+ (hR'_{0ij0} + \sum_k h_{ij}R'_{0k0k}) \\ &- \sum_{k,l} (2h_{kl}R'_{kijl} + h_{jl}R'_{lkik} + h_{il}R'_{kjkl}) \\ &+ hh^2_{ij} - h_2 h_{ij}.\end{aligned} \tag{3.2}$$

4 The curvature conditions

In this section we shall consider the Riemannian manifold under some supposed curvature conditions and estimate Δh_2 from below. Let M' be an $(n+1)$-dimensional Riemannian manifold and let M be a hypersurface of M'. For a point x in M' let $\{e_0, e_1, \cdots, e_n\}$ be a local field of orthonormal frames of M' on a neighborhood of x in such a way that, restricted to M, the vectors $e_1, \cdots, e_n$ are tangent to M and the other is normal to M. For linearly independent vectors u and v in the tangent space T_xM' we denote by $K'(u,v)$ the sectional curvature of the section spanned by u and v in M' and by R' or $Ric'(u,u)$ the Riemannian curvature tensor in M' or the Ricci curvature in the direction of u in M', respectively. Let us denote by ∇' the Riemannian connection on M'. We assume that the ambient space M' satisfies the following conditions :

$$Ric'(e_0, e_0) \leq c_1, \tag{4.1}$$

$$K'(e_i, e_j) \geq c_2 \text{ for any } i \text{ and } j \text{ such that } i \neq j \tag{4.2}$$

and

$$|\nabla' R'| \leq \frac{c_3}{n} \tag{4.3}$$

for some constant c_1, c_2 and c_3.

When M' satisfies the above conditions (4.1), (4.2) and (4.3), it is said simply for M' to satisfy *the condition* $(*)$.

Remark 4.1. If $c_3 = 0$, then the ambient space M' is locally symmetric. Conversely if the ambient space M' is locally symmetric, then without loss of generality, we may regard as $c_3 = 0$.

Remark 4.2. If M' is a space form $M^{n+1}(c)$ of constant curvature c, then it satisfies the condition $(*)$, where $\dfrac{c_1}{n} = c_2 = c$.

In the sequel, we assume that the ambient space M' satisfies the condition $(*)$ and hypersurface M is minimal. Namely we assume that the function h vanishes. Then the Laplacian of the squared norm h_2 of the second fundamental form α of M is given by

$$\begin{aligned}\Delta h_2 &= \Delta(\sum_{i,j} h_{ij}h_{ij}) = 2\sum_{i,j,k}(h_{ijk}h_{ij})_k \\ &= 2\sum_{i,j,k}(h_{ijkk}h_{ij} + h_{ijk}h_{ijk}) \\ &= 2|\nabla\alpha|^2 + 2\sum_{i,j,k} h_{ijkk}h_{ij},\end{aligned}$$

where ∇_α is the covariant derivative of the second fundamental form α and $|\nabla\alpha|^2$ is the squared norm of $\nabla\alpha$. i.e., $|\nabla\alpha|^2 = \sum_{i,j,k} h_{ijk}h_{ijk}$. Hence from (3.1) and (3.2) it follows that

$$\begin{aligned}\Delta h_2 = 2|\nabla\alpha|^2 \quad &+ \quad 2\sum_{i,j}\left\{\sum_k (R'_{0kik;j} + R'_{0ijk;k}) + \sum_k h_{ij}R'_{0k0k}\right. \\ &- \quad \left. 2\sum_{k,l}(h_{kl}R'_{lijk} + h_{lj}R'_{lkik} + h_{li}R'_{lkjk}) - h_2 h_{ij}\right\} h_{ij}.\end{aligned}$$

Thus we have

$$\begin{aligned}\Delta h_2 = 2|\nabla\alpha|^2 \quad &+ \quad 2\left\{\sum_{i,j,k}(R'_{0kik;j} + R'_{0ijk;k})h_{ij} + h_2\sum_k R'_{0k0k}\right. \\ &- \quad \left.\sum_{k,l} 2(h_{ij}h_{kl}R'_{lijk} + h_{lj}^2 R'_{lkjk}) - h_2^2\right\}.\end{aligned} \tag{4.4}$$

On the other hand, since the matrix $H = (h_{ij})$ can be diagonalized, we can expressed by

$$h_{ij} = \lambda_i\delta_{ij}, \tag{4.5}$$

where λ_i is the principal curvature on M. By definition, we see

$$h_2 = \sum_i \lambda_i^2 \geq \lambda_i^2$$

and hence we have

$$-\sqrt{h_2} \leq \lambda_i \leq \sqrt{h_2}, \tag{4.6}$$

$$-h_2 \leq \lambda_i \lambda_j \leq h_2. \tag{4.7}$$

Now in order to prove our results, we need to estimate Δh_2 of (4.4) from below. First of all, we treat with the second term of (4.4). It is seen that

$$\begin{aligned} 2\sum_{i,j,k}(R'_{0kik;j} + R'_{0ijk;k})h_{ij} &= 2\sum_{j,k}\lambda_j(R'_{0kjk;j} + R'_{0jjk;k}) \\ &\geq -2\sum_{j,k}|\lambda_j|(|R'_{0kjk;j}| + |R'_{0jjk;k}|). \end{aligned}$$

So by (4.3) and (4.6) we have

$$\text{the second term of (4.4)} \geq -4c_3\sqrt{h_2}. \tag{4.8}$$

Next, we consider the third term of (4.4). It is estimated as follows :

$$\begin{aligned} 2h_2\sum_k R'_{0k0k} &= -2h_2\sum_k R'_{0kk0} \\ &= -2h_2\sum_k R'_{0kk0} \geq -2h_2c_1, \end{aligned}$$

where we have used (4.1). Hence we have

$$\text{the third term of (4.4)} \geq -2c_1h_2. \tag{4.9}$$

It is evident that if the ambient space M' is a space form $M^{n+1}(c)$ of constant curvature c, then (4.9) also holds and hence

$$\text{the third term of (4.4)} \geq -2c\, n\, h_2.$$

Last, we estimate the fourth term of (4.4). We have by (4.2)

$$\begin{aligned} &-4\sum_{k,l}(h_{ij}h_{kl}R'_{lijk} + h^2_{lj}R'_{lkjk}) = -4\sum_{j,k}(\lambda_j\lambda_kR'_{kjjk} - \lambda_j^2R'_{kjjk}) \\ &= -4\sum_{j,k}(\lambda_j\lambda_k - \lambda_j^2)R'_{kjjk} = 2\sum_{j,k}(\lambda_j - \lambda_k)^2R'_{kjjk} \\ &\geq 2c_2\sum_{j,k}(\lambda_j - \lambda_k)^2. \end{aligned}$$

Accordingly we obtain

$$\text{the fourth term of (4.4)} \geq 4c_2nh_2. \tag{4.10}$$

Thus, substituting (4.8), (4.9) and (4.10) into (4.4), we can prove the following

Lemma 4.1. *Let M' be an $(n+1)$-dimensional Riemannian manifold satisfying the condition $(*)$ and let M be a minimal hypersurface of M' Then we have*

$$\Delta h_2 \geq -4c_3\sqrt{h_2} + 2(2nc_2 - c_1)h_2 - 2h_2^2. \tag{4.11}$$

In particular, if $M' = M^{n+1}(c)$, then substituting $\frac{c_1}{n} = c_2 = c$ and $c_3 = 0$ into (4.11) we obtain

$$\Delta h_2 \geq 2(n\ c\ h_2 - h_2^2). \tag{4.12}$$

5 Locally symmetric spaces

Let M' be an $(n+1)$-dimensional locally symmetric space which satisfies the condition $(*)$ and let M be a complete minimal hypersurface of M'. For a point x in M let $\{e_0, e_1, \cdots, e_n\}$ be a local field of orthonormal frames of M' around of x in such a way that, restricted to M, the vectors $e_1, \cdots, e_n$ are tangent to M and the other is normal to M. Then for two constants c_1 and c_2

$$Ric'(e_0, e_0) \leq c_1,$$

$$K'(e_i, e_j) \geq c_2 \text{ for any } i \text{ and } j \text{ such that } i \neq j.$$

Moreover, since M' is locally symmetric, we have $c_3 = 0$ and the inequalities (4.11) implies

$$\Delta f^2 \geq 2\{(2nc_2 - c_1) - nh_2\}f^2, \tag{5.1}$$

where f is the non-negative function defined by $f^2 = h_2$.

Theorem 5.1. *Let M' be a locally symmetric space which satisfies the condition $(*)$ and let M be a complete minimal hypersurface. If it satisfies*

$$n\ \sup h_2 \leq 2nc_2 - c_1, \tag{5.2}$$

then M is totally geodesic.

Proof. We define a non-negative function f by $f^2 = h_2$. Then it is bounded by the assumption. Let $\lambda_1, \cdots, \lambda_n$ be principal curvatures on M. It is easily seen that $f^2 = \sum_i \lambda_i^2$ and hence the function f vanishes identically on M if and only if $\lambda_i = 0$ for any indices i, i.e., M is totally geodesic.

First of all, we will show that the Ricci tensor is bounded from below. The Ricci tensor S_{ij} is expressed by

$$\begin{aligned} S_{ij} &= \sum_k (R'_{kijk} + h_{ij}h_{kk} - h_{ik}h_{jk}) \\ &\geq (n-1)c_2 - \lambda_i^2 \delta_{ij}. \end{aligned}$$

Since the principal curvatures satisfy $\lambda_i \lambda_j \geq -h_2$, we have

$$S_{ij} \geq (n-1)c_2 - h_2,$$

which yields the Ricci curvature of M is bounded from below under the assumption. This means that Theorem 2.1 due to Omori [5] and Yau [6] can be applied to the function f^2. Therefore given any positive number ε there exists a point p in M at which f^2 satisfies (2.23). Accordingly, for a converge sequence $\{\varepsilon_m\}$ so that it converges to zero as m tends to infinity, there exists a point sequence $\{p_m\}$ so that the sequence $\{f(p_m)\}$ converges to f_0 by taking a subsequence, if necessary. By (2.23) we see

$$f(p_m) \to f_0 = \sup f.$$

By (5.1) we can estimate Δf^2 from the below, and hence by (2.23) we obtain

$$\varepsilon_m \geq \Delta f^2(p_m) \geq 2\{(2nc_2 - c_1) - nf^2(p_m)\}f^2(p_m),$$

from which it follows that we have

$$2\{(2nc_2 - c_1) - nf_0^2\}f_0^2 \leq 0$$

as m tends to infinity. Under this inequality we have

$$f_0 = 0 \quad \text{or} \quad nf_0^2 \geq (2nc_2 - c_1).$$

Under the assumption of the present theorem, the above restriction of the supremum of the function f yields that $f_0 = 0$, which implies f vanishes identically on M and hence M is totally geodesic. □

Remark 5.1. An $(n+1)$-Euclidean space $M' = \mathbf{R}^{n+1}$ satisfies the condition $(*)$ with $c_1 = c_2 = c_3 = 0$. The complete hypersurface $M = S^{n-1}(c) \times \mathbf{R}^1$ in $M' = \mathbf{R}^{n+1}$ is not umbilic and it satisfies $h_2 = \dfrac{h^2}{n-1} > 0$. In this example, the condition (5.2) is equivalent to $\sup h_2 < \dfrac{h^2}{n}$.

6 Minimal hypersurfaces

Let M' be an $(n+1)$-dimensional Riemannian manifold which satisfies the condition $(*)$. That is, for any point x in M let $\{e_0, e_1, \cdots, e_n\}$ be a local field of orthonormal frames of M' around of x in such a way that, restricted to M, the vectors $e_1, \cdots, e_n$ are tangent to M and the other is normal to M. Then for three constants c_1, c_2 and c_3 we have

$$Ric'(e_0, e_0) \leq c_1,$$

$$k'(e_i, e_j) \geq c_2 \quad \text{for any} \quad i \quad \text{and} \quad j \quad \text{such that} \quad i \neq j,$$

$$|\nabla R'| \leq \frac{c_3}{n}.$$

Let M be a complete minimal hypersurface of M'. From the inequality (4.11) it follows that

$$\Delta f^2 \geq -4c_3 f + 2(2nc_2 - c_1)f^2 - 2f^4, \tag{6.1}$$

where f is the non-negative function defined by $f^2 = h_2$.

Theorem 6.1. *Let M' be an $(n+1)$-dimensional Riemannian manifold which satisfies the condition $(*)$ and let M be a complete minimal hypersurface. If the Ricci curvatures of M is bounded from below and if it satisfies*

$$(2nc_2 - c_1)^3 > 27c_3^2 > 0, \tag{6.2}$$

then there exists constants a_2 and a_3 depending only on c_1, c_2, c_3 and n such that

$$0 \leq \sup h_2 \leq a_2 \quad \text{or} \quad \sup h_2 \geq a_3.$$

Proof. Let $\lambda_1, \cdots, \lambda_n$ be principal curvatures on M. Since the hypersurface M is minimal, the Ricci tensor S_{ij} is given by

$$\begin{aligned} S_{ij} &= \sum_k (R'_{kijk} - h_{ik}h_{jk}) \\ &= \sum_k R'_{kijk} - \lambda_i^2 \delta_{ij} \geq (n-1)c_2 - \lambda_i^2 \delta_{ij}. \end{aligned}$$

Here, we suppose that the squared norm h_2 is not bounded from above, then there exists at least one index j so that the principal curvature λ_j is not bounded, from this fact together with the above inequality it follows that the Ricci curvature of M is not bounded from below. Thus we have a contradiction. Therefore the squared norm h_2 is bounded from above, which implies Theorem 2.1 due to Omori [5] and Yau [6] can be applied to the function f^2. Given any positive number ε there exists a point p in M, at which f^2 satisfies (2.23). Accordingly, for a converge sequence $\{\varepsilon_m\}$ so that it converges to zero as m tends to infinite, there exists a point sequence $\{p_m\}$ so that the sequence $\{f(p_m)\}$ converges to f_0, by taking a subsequence, if necessary. By (2.23) we see

$$f(p_m) \to f_0 = \sup f.$$

By means of (6.1) we can estimate Δf^2 from the below, and hence by (2.23) we obtain

$$\varepsilon_m \geq \Delta f^2(p_m) \geq 2\{-2c_3 + (2nc_2 - c_1)f(p_m) - f^3(p_m)\}f(p_m),$$

which implies

$$\{-2c_3 + (2nc_2 - c_1)f_0 - f_0^3\}f_0 \leq 0 \tag{6.3}$$

as m tends to infinity. On the other hand, the quadrant expression y is defined by

$$y = y(x) = -2\{x^3 - (2nc_2 - c_1)x + 2c_3\}x, \tag{6.4}$$

then we have

$$\begin{aligned} y' &= -4\{2x^3 - (2nc_2 - c_1)x + c_3\}, \\ y'' &= -4\{6x^2 - (2nc_2 - c_1)\}, \\ y''' &= -48x, \end{aligned} \tag{6.5}$$

which implies that y' has the relative maximum on the interval $(0, \infty)$, whose value is $4\left\{\frac{2}{3\sqrt{6}}(2nc_2 - c_1)^{3/2} - c_3\right\}$, which implies that if $2(2nc_2 - c_1)^3 > 27c_3^2$, then

$y'\left(\sqrt{\dfrac{2nc_2-c_1}{6}}\right) > 0$. Thus under the assumption (6.2) we get $y'\left(\sqrt{\dfrac{2nc_2-c_1}{6}}\right) > 0$. This yields that there exist two positive roots b_2 and b_3 $(b_2 < b_3)$ of the algebraic equation $y'(x) = 0$. Thus from (6.4) it follows that

$$y(b_3) = \{(2nc_2 - c_1)b_3 - 3c_3\}b_3. \tag{6.6}$$

On the other hand, by the direct calculation, we have

$$y\left(\frac{3c_3}{2nc_2 - c_1}\right) = -6c_3^2\frac{27c_3^2 - (2nc_2 - c_1)^3}{(2nc_2 - c_1)^4}.$$

So, under the assumption (6.2), the value is positive. Because $y'(b_3) = 0$ and $\lim_{x\to\infty} y'(x) = -\infty$ and $y'|_{[b_3,\infty)} \leq 0$, we obtain

$$\frac{3c_3}{2nc_2 - c_1} < b_3,$$

from which together with (6.6) it follows that

$$y(b_3) > 0.$$

By (6.4) we see $y(0) = 0$ and $\lim_{x\to\infty} y(x) = -\infty$ and also by means of (6.5) we have $y'|_{[0,b_2)} < 0$, this yields that there exists two positive roots a_2 and a_3 of the equation $y(x) = 0$. By (6.3) and (6.4) we obtain

$$0 \leq f_0 = \sup f \leq a_2 \quad \text{or} \quad f_0 = \sup f \geq a_3.$$

It completes the proof. □

Remark 6.1. The values a_2 and a_3 are two positive roots of the cubic equation

$$x^3 - 2(2nc_2 - c_1)x + 2c_3 = 0 \tag{6.7}$$

and hence we can require concretely the values. They are depended on the constant c_1, c_2, c_3 and n.

Owing to Theorem 5.1, Theorem 6.1 and Remark 6.1, the following property is easily verified.

Corollary 6.2. *Let M' be an $(n+1)$-dimensional locally symmetric space which satisfies the condition $(*)$ and let M be a complete minimal hypersurface. If it satisfies*

$$\sup h_2 < 2nc_2 - c_1, \tag{6.8}$$

then M is totally geodesic.

Proof. By the assumption (6.8) the Ricci curvature on M is bounded from below, because we see

$$S_{ij} \geq (n-1)c_2 - \lambda_j^2\delta_{ij} \geq (n-1)c_2 - h_2$$

On the other hand, since M' is locally symmetric, we see $c_3 = 0$, from which it follows that the algebraic equation (6.7) is reduced to

$$x^3 - 2(2nc_2 - c_1)x = 0.$$

So the supremum f_0 of the function $f = h_2$, i.e., the squared norm of the second fundamental form, satisfies

$$f_0^3 - 2(2nc_2 - c_1)f_0 = 0.$$

The assumption (6.8) implies $f_0 = 0$, which yields that M is totally geodesic.

It completes the proof. □

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Hyang Sook Kim[1]
Department of Mathematics
School of Computer Aided Science
Inje University
Kimhae 621-749, Korea
E-mail : mathkim@ijnc.inje.ac.kr

Yong-Soo Pyo
Division of Mathematical Sciences
Pukyong National University
Pusan 608-737, Korea
E-mail : yspyo@dolphin.pknu.ac.kr

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