Some Structural Considerations on the Theory of Gravitational Field

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Abstract

When the gravitational field is regarded as the time-sequence of space-slices (i.e., the evolution space), the gravitational field itself is treated by means of the differential geometry of total space of the vector bundle whose base manifold is the one-dimensional time-axis and fibre at each time is the three-dimensional space. From this vector bundle-like standpoint, new field equations and new conservation laws are proposed.

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1 Introduction

In this paper, the gravitational field is regarded as the ensemble of space-time events and then, as the time-sequence of space-slices. That is to say, the four-dimensional gravitational field itself is decomposed into the one-dimensional time-part and the three-dimensional space-part and the three-dimensional space evolves along the time-axis. This idea arises from the concept of evolution space in the theory of dynamical systems (cf. [1]). Therefore, from the vector bundle-like standpoint, the gravitational field can be adapted to the total space of the vector bundle whose base manifold is the one-dimensional time-axis ($x^0 = t$:time) and the fibre at each time is the three-dimensional space spanned by points $\{x^i\}$ $(i = 1, 2, 3)$.

Therefore, the gravitational field can be treated by means of the differential geometry of total space of the vector bundle [2], [3]. From this standpoint, the metrical and connection structures will be introduced and then, new field equations and new conservation laws will be proposed in the following.

2 On the vector bundle-like structures - I

Now, in the total space mentioned above, the so-called adapted frame is set as follows:

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$$
\begin{aligned}
&dX^A \equiv (dx^0, \delta x^i = dx^i + N_0^i dx^0),\\
&\frac{\partial}{\partial X^A} \equiv \left(\frac{\delta}{\delta x^0} = \frac{\partial}{\partial x^0} - N_0^i \frac{\partial}{\partial x^i},\ \frac{\partial}{\partial x^i}\right),
\end{aligned} \tag{2.1}
$$

where $X^A = (x^0, x^i)(A = (0,i);\ i = 1,2,3)$ and $x^0 = t$ (time). The quantity N_0^i denotes the nonlinear connection playing physically the role of velocity.

On the basis of (2.1), the connection structures is introduced by

$$
\begin{aligned}
&\nabla_{\frac{\partial}{\partial X^C}} \frac{\partial}{\partial X^B} = \Gamma^A_{BC} \frac{\partial}{\partial X^A};\\
&\Gamma^A_{BC} \equiv (L^0_{00}, L^i_{j0}, C^0_{0k}, C^i_{jk}),
\end{aligned} \tag{2.2}
$$

where

$$
\nabla_{\frac{\delta}{\delta x^0}} \frac{\delta}{\delta x^0} = L^0_{00} \frac{\delta}{\delta x^0}, \quad \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j} = C^i_{jk} \frac{\partial}{\partial x^i}, \text{ etc.}
$$

Namely, the following four kinds of covariant derivatives can be defined, for an arbitrary vector $V^A = (V^0, V^i)$

$$
\begin{cases}
V^0_{|0} = \dfrac{\delta V^0}{\delta x^0} + L^0_{00} V^0,\\
V^0|_k = \dfrac{\partial V^0}{\partial x^k} + C^0_{0k} V^0,\\
V^i_{|0} = \dfrac{\delta V^i}{\delta x^0} + L^i_{j0} V^j,\\
V^i|_k = \dfrac{\partial V^i}{\partial x^k} + C^i_{jk} V^j.
\end{cases} \tag{2.3}
$$

On the other hand, the metrical structure is introduced by

$$
G \equiv G_{AB} dX^A dX^B = g_{00} dx^0 \otimes dx^0 + g_{ij} \delta x^i \otimes \delta x^j, \tag{2.4}
$$

where g_{00} and g_{ij} are metric tensors depending on (x^0, x^i). The connection (2.2) can be made metrical by imposing the metrical conditions such as $g_{00|0} = 0$, $g_{00}|_k = 0$, $g_{ij|0} = 0$ and $g_{ij}|_k = 0$. In the metrical case, the canonical connection coefficients can be determined as follows [2], [3]

$$
\begin{cases}
L^0_{00} = \dfrac{1}{2} g^{00} \dfrac{\delta g_{00}}{\delta x^0},\\
L^i_{j0} = \dfrac{1}{2} g^{il} \dfrac{\delta g_{lj}}{\delta x^0},\\
C^0_{0k} = \dfrac{1}{2} g^{00} \dfrac{\partial g_{00}}{\partial x^k},\\
C^i_{jk} = \dfrac{1}{2} g^{il} \left(\dfrac{\partial g_{lj}}{\partial x^k} + \dfrac{\partial g_{kl}}{\partial x^j} - \dfrac{\partial g_{jk}}{\partial x^l} \right),
\end{cases} \tag{2.5}
$$

where g^{00} and g^{il} are the inverse of g_{00} and g_{li} respectively and the torsion tensor $S^i_{jk} (= C^i_{jk} - C^i_{kj})$ is assumed to vanish (see (3.5)). C^i_{jk} is nothing but the three-dimensional Christoffel symbol formed with g_{ij}. And also, in (2.5), if $g_{00} =$ constant, then $L^0_{00} = 0$ and $C^0_{0k} = 0$.

3 On the vector bundle-like structures - II

Now, the torsion and curvature tensors are defined by [2], [3], respectively,

$$T^A_{BC} = \Gamma^A_{BC} - \Gamma^A_{CB} + \omega^A_{BC}, \tag{3.1}$$

$$R^A_{BCD} = \frac{\partial}{\partial X^D}\Gamma^A_{BC} - \frac{\partial}{\partial X^C}\Gamma^A_{BD} + \Gamma^E_{BC}\Gamma^A_{ED} - \Gamma^E_{BD}\Gamma^A_{EC} + \Gamma^A_{DE}\omega^E_{CD}, \tag{3.2}$$

where ω^A_{BC} means the non-holonomic object defined by

$$\left[\frac{\partial}{\partial X^B}, \frac{\partial}{\partial X^C}\right] = \omega^A_{BC}\frac{\partial}{\partial X^A}. \tag{3.3}$$

In our case, only one component $\omega^i_{0j} = -\omega^i_{j0} = \dfrac{\partial N^i_0}{\partial x^j}$ appears (because $\omega^i_{00} = R^i_{00} = 0$, see below).

Five components of the torsion tensor

$$T^A_{BC} \equiv (T^0_{00} = 0,\ R^i_{00} = 0,\ C^0_{0i},\ P^i_{0j},\ S^i_{jk})$$

and six components of the curvature tensor

$$\mathcal{R}^{\mathcal{A}}_{\mathcal{BCD}} \equiv (\mathcal{R}'_{'''} = ',\ \mathcal{R}^{\rangle}_{|''} = ',\ \mathcal{P}^{\rangle}_{\|'|},\ \mathcal{P}'_{''\rangle},\ \mathcal{S}'_{'\rangle|},\ \mathcal{S}^{\rangle}_{|\|\updownarrow})$$

appear explicitly in the following Ricci-identities:

$$\left\{\begin{array}{l} V^0{}_{|0|0} - V^0{}_{|0|0} = R^0_{000}V^0 - T^0_{00}V^0{}_{|0} - R^i_{00}V^0|_i (= 0), \\ V^0{}_{|0}\big|_i - V^0|_{i|0} = P^0_{00i}V^0 - C^0_{0i}V^0_{|0} - P^j_{0i}V^0|_j, \\ V^0|_i|_j - V^0|_j|_i = S^0_{0ij}V^0 - S^k_{ij}V^0|_k, \end{array}\right.$$
$$\left\{\begin{array}{l} V^i{}_{|0|0} - V^i{}_{|0|0} = R^i_{j00}V^j - T^0_{00}V^i{}_{|0} - R^j_{00}V^i|_j (= 0), \\ V^i{}_{|0}\big|_j - V^i|_{j|0} = P^i_{k0j}V^k - C^0_{0j}V^i{}_{|0} - P^k_{0j}V^i|_k, \\ V^i|_j|_k - V^i|_k|_j = S^i_{ljk}V^l - S^l_{jk}V^i|_l. \end{array}\right. \tag{3.4}$$

The concrete definitions of those components are given as follows

$$\left\{\begin{array}{l} T^0_{00} = L^0_{00} - L^0_{00} = 0, \\ R^i_{00} = \dfrac{\delta N^i_0}{\delta x^0} - \dfrac{\delta N^i_0}{\delta x^0} = 0, \\ P^i_{0j} = \dfrac{\partial N^i_0}{\partial x^j} - L^i_{j0}, \\ S^i_{jk} = C^i_{jk} - C^i_{kj}. \end{array}\right. \tag{3.5}$$

$$
(3.6) \qquad \begin{cases}
R^0_{000} = \dfrac{\delta L^0_{00}}{\delta x^0} - \dfrac{\delta L^0_{00}}{\delta x^0} + L^0_{00}L^0_{00} - L^0_{00}L^0_{00} + C^0_{0i}R^i_{00} = 0, \\
R^i_{j00} = \dfrac{\delta L^i_{j0}}{\delta x^0} - \dfrac{\delta L^i_{j0}}{\delta x^0} + L^l_{j0}L^i_{l0} - L^l_{j0}L^i_{l0} + C^i_{jl}R^l_{00} = 0, \\
P^i_{j0k} = \dfrac{\partial L^i_{j0}}{\partial x^k} - C^i_{jk|0} + C^i_{jl}P^l_{0k}, \\
P^0_{00k} = \dfrac{\partial L^0_{00}}{\partial x^k} - C^0_{0k|0} + C^0_{0l}P^l_{0k}, \\
S^0_{0kh} = \dfrac{\partial C^0_{0k}}{\partial x^h} - \dfrac{\partial C^0_{0h}}{\delta x^k} + C^0_{0k}C^0_{0h} - C^0_{0h}C^0_{0k}, \\
S^i_{jkh} = \dfrac{\partial C^i_{jk}}{\delta x^h} - \dfrac{\partial C^i_{jh}}{\delta x^k} + C^l_{jk}C^i_{lh} - C^l_{jh}C^i_{lk}.
\end{cases}
$$

In the canonical case of (2.5), the tensor S^i_{jkh} is just the three-dimensional Riemann-Christoffel curvature tensor and also, if g_{00} = constant, then $P^0_{00k} = 0$ and $S^0_{0kh} = 0$ in (3.6).

From $\mathcal{R}^{\mathcal{A}}_{\mathcal{BCD}}$, the Ricci-tensor is given by

$$
(3.7) \quad \mathcal{R}_{\mathcal{BC}} \equiv \mathcal{R}^{\mathcal{A}}_{\mathcal{BCA}} \equiv (\mathcal{R}_{\prime\prime} \equiv \mathcal{R}'_{\prime\prime\prime} = \prime,\ \overset{\infty}{\mathcal{P}}_{|\prime} \equiv \mathcal{P}^{\rangle}_{|\prime\rangle},\ -\overset{\in}{\mathcal{P}}_{\prime\|} \equiv -\mathcal{P}'_{\prime\|\prime} = \mathcal{P}'_{\prime\prime\|},\ \mathcal{S}_{\rangle|} \equiv \mathcal{S}^{\|}_{\rangle|\|}).
$$

Namely, three non-vanishing components appear. And the total scalar is given by

$$
(3.8) \qquad \mathcal{R} \equiv \mathcal{R}_{\mathcal{AB}}\mathcal{G}^{\mathcal{AB}} = \mathcal{R}_{\prime\prime}\}^{\prime\prime} + \mathcal{S}_{\rangle|}\}^{\rangle|} = \mathcal{S}_{\rangle|}\}^{\rangle|} \equiv \mathcal{S}.
$$

(3.7) and (3.8) will be used in the next Section.

4 On the field equations and the conservation laws

As the field equation for the total space (i.e., the gravitational field), we shall put it in the form [2], [3], with use of the Ricci-tensor $\mathcal{R}_{\mathcal{AB}}$ (3.7) and the total scalar $\mathcal{R}$ (3.8),

$$
(4.1) \qquad \mathcal{R}_{\mathcal{AB}} - \frac{\infty}{\in}\mathcal{R}\mathcal{G}_{\mathcal{AB}} = \tau_{\mathcal{AB}},
$$

where τ_{AB} represents the energy-momentum tensor with four components $\tau_{AB} \equiv (\tau_{00}, \tau_{j0}, \tau_{0i}, \tau_{ij})$. Then, by use of the components of (3.7) and (3.8), we can obtain the following four kinds of field equations

$$
(4.2) \qquad \begin{cases}
-\dfrac{1}{2}Sg_{00} = \tau_{00}, \\
\overset{1}{P}_{j0} = \tau_{j0}, \\
\overset{2}{P}_{0i} = -\tau_{0i}, \\
S_{ij} - \dfrac{1}{2}Sg_{ij} = \tau_{ij}.
\end{cases}
$$

These are new equations, different from those obtained in [2], [3]. In the case of the canonical connection (2.5), the last equation is just the (three-dimensional) Einstein's field equation.

As to the conservative law, we can formulate it in the form [2], [3] (i.e., the divergence-zero of (4.1))

$$\frac{\nabla\partial}{\partial X^A}\left(\mathcal{R}^{\mathcal{A}}_{\mathcal{B}} - \frac{\infty}{\in}\mathcal{R}\delta^{\mathcal{A}}_{\mathcal{B}}\right) = 0, \tag{4.3}$$

where

$$\begin{aligned} \mathcal{R}^{\mathcal{A}}_{\mathcal{B}} &\equiv \mathcal{R}_{\mathcal{BC}}\mathcal{G}^{\mathcal{CA}} \equiv (\mathcal{R}'_{\prime} \equiv \mathcal{R}_{\prime\prime}\}'' = \prime,\ \overset{\infty}{\mathcal{P}}{}^{\rangle}_{\prime} \equiv \overset{\infty}{\mathcal{P}}_{|\prime}\ \}^{|\rangle}, \\ &- \overset{2}{P}{}^{0}_{j} \equiv -\overset{2}{P}_{0j}\ g^{00},\ S_j{}^i \equiv S_{jl}g^{li}). \end{aligned} \tag{4.4}$$

Therefore, we can obtain the following two kinds of conservation laws

$$\begin{cases} \overset{1}{P}{}^{i}_{0}\ |_i - \dfrac{1}{2}S_{|0} = 0, \\ (S^i_j - \dfrac{1}{2}S\delta^i_j)|_i - \overset{2}{P}{}^{0}_{j|0} = 0. \end{cases} \tag{4.5}$$

These are new conservation laws, different from those obtained in [2], [3]. In some special cases where $\overset{2}{P}{}^{0}_{j} = 0$ or $\overset{2}{P}{}^{0}_{j}\ {}_{|0} = 0$, we can obtain the pure conservation law

$$\left(S^i_j - \frac{1}{2}S\delta^i_j\right)|_i = 0,$$

which is the same as the Einstein's one. In those special cases, the conditions such as

$$(g_{00} = g_{00}(x^0) \& g_{ij} = g_{ij}(x^i)) \text{ or } (g_{00} = g_{00}(x^i) \& g_{ij} = g_{ij}(x^i)),$$

etc. must be taken into account.

5 Conclusion

Thus, we can treat the gravitational field by means of the differential geometry of total space of the vector bundle whose base manifold is the time-axis and fibre at each time is the space-slice. And we can propose new field equations (4.2) and new conservation laws (4.5). (Some physical aspects of these subjects are referred to [4]).

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