

Some Examples of Almost Kähler 4-Manifolds

Takuji Sato

*Dedicated to Prof. Dr. Constantin UDRIȘTE
on the occasion of his sixtieth birthday*

Abstract

In the framework of P. Nurowski and M. Przanowski [3], we construct 4-dimensional examples of Ricci flat almost Kähler manifolds, almost Kähler manifolds of pointwise constant holomorphic sectional curvature, and weakly *-Einstein almost Kähler manifolds.

Mathematics Subject Classification: 53C25, 53C55

Key words: almost Kähler manifolds, pointwise constant holomorphic sectional curvature, weakly *-Einstein manifolds

1 Introduction

An almost Hermitian manifold $M = (M, J, g)$ is called an almost Kähler manifold if its Kähler form Ω is closed. One of the interest problems on almost Kähler manifolds is so called Goldberg's conjecture, which asserts the almost complex structure of a compact Einstein almost Kähler manifold is integrable (and the manifold is necessarily Kähler) [2]. In connection with this conjecture, P. Nurowski and M. Przanowski [3] recently constructed a non-compact example of a strictly almost Kähler, Ricci-flat manifold. This shows that the assumption about compactness of the Einstein manifold is essential for the Goldberg conjecture. This example is also a space of pointwise *positive* constant holomorphic sectional curvature and a weakly *-Einstein manifold (see also [4]).

By considering a real expression for the Nurowski-Przanowski example, the author [5] has constructed a new example of almost Kähler manifold of pointwise *negative* constant holomorphic sectional curvature. In the present paper, we shall mainly deal with 4-dimensional almost Kähler manifolds $M(f, u, v, \phi)$ which will be defined in §2, and show that there exists a family of Ricci flat almost Kähler manifolds which include the Nurowski-Przanowski example. In [5], we investigated almost Kähler manifolds $M(f, u, v, \phi)$ with $u = v = 0$ and $\phi = 0$. For the case where u and v are any constants and ϕ is arbitrary, we can obtain the examples of almost Kähler manifolds with pointwise constant holomorphic sectional curvature which are generalizations of [5]. These examples are also weakly *-Einstein, but not Einstein. We also show that there are other examples of weakly *-Einstein almost Kähler manifolds. Our examples

are 4-dimensional and non-compact. We do not know whether or not there exist arbitrary dimensional, compact almost Kähler manifolds which are of (pointwise) constant holomorphic sectional curvature, or which are (weakly) *-Einstein.

In §2, we recall a characterization for a 4-dimensional almost Kähler manifold of pointwise constant holomorphic sectional curvature by using the expressions A_{ij} introduced by J. T. Cho and K. Sekigawa [1]. We give a real version of the Nurowski-Przanowski's construction, and define an almost Kähler manifold $M(f, u, v, \phi)$. §3 is devoted to the construction of Ricci-flat examples. By calculating Ricci tensor ρ_{ij} , we find functions u, v and f for ρ_{ij} vanishing. In §4, we show that the Ricci flat almost Kähler manifolds in §3 are also of pointwise constant holomorphic sectional curvature, by using the characterization of [1]. Moreover, when u, v are constants α, β , we derive conditions for $M(f, \alpha, \beta, \phi)$ to be of pointwise constant holomorphic sectional curvature. Taking account of these conditions, we shall obtain examples (Theorem 4.4). In the last §5, we give other examples of weakly *-Einstein almost Kähler manifolds. Especially, the last example (Theorem 5.3) shows that it depends on the value of ϕ that $M(f, \alpha, \beta, \phi)$ is weakly *-Einstein.

2 Preliminaries

Let $M = (M, J, g)$ be a four-dimensional almost Hermitian manifold with an almost Hermitian structure (J, g) . We denote by Ω and N the Kähler form and the Nijenhuis tensor of M defined respectively by $\Omega(X, Y) = g(X, JY)$ and $N(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY]$ for $X, Y \in \mathcal{X}(M)$, where $\mathcal{X}(M)$ is the Lie algebra of all smooth vector fields on M . The Nijenhuis tensor N has the properties

$$N(JX, Y) = N(X, JY) = -JN(X, Y), \quad X, Y \in \mathcal{X}(M).$$

Further we denote by $\nabla, R, \rho, \tau, \rho^*$ and τ^* the Riemannian connection, the Riemannian curvature tensor, the Ricci tensor, the scalar curvature, the Ricci *-tensor and the *-scalar curvature of M , respectively. The Ricci *-tensor ρ^* satisfies

$$\rho^*(JX, JY) = \rho^*(Y, X), \quad X, Y \in \mathcal{X}(M).$$

An almost Hermitian manifold M is called a *weakly *-Einstein manifold* if it satisfies $\rho^* = \lambda^*g$ for some function λ^* on M . In particular, if λ^* is constant on M , then M is called a **-Einstein manifold*.

The holomorphic sectional curvature $H = H(x) = -R(x, Jx, x, Jx)$ ($x \in T_p(M), \|x\| = 1$) can be regarded as a differentiable function on the unit tangent bundle $U(M)$ of M . If the function H is constant along each fiber, then M is called a *space of pointwise constant holomorphic sectional curvature*. Especially, if H is constant on the whole of $U(M)$, then M is called a *space of constant holomorphic sectional curvature*.

Now we assume that $M = (M, J, g)$ is a four-dimensional almost Kähler manifold. Then we have

$$(2.1) \quad 2g((\nabla_X J)Y, Z) = g(JX, N(Y, Z)),$$

$$(2.2) \quad \tau - \tau^* = -\frac{1}{2}\|\nabla J\|^2 = -\frac{1}{8}\|N\|^2.$$

In the sequel, we adopt the following notational convention: for an orthonormal basis $\{e_i\}$ of a tangent space T_pM , we put

$$\begin{aligned} J_{ij} &= g(Je_i, e_j), & \nabla_i J_{jk} &= g((\nabla_{e_i} J)e_j, e_k), \\ N_{ijk} &= g(e_i, N(e_j, e_k)), & R_{ijkl} &= R(e_i, e_j, e_k, e_l), \\ \nabla_{\bar{i}} J_{jk} &= g((\nabla_{J e_i} J)e_j, e_k), & N_{\bar{i}jk} &= g(Je_i, N(e_j, e_k)), \\ N_{\bar{i}jk} &= g(Je_i, N(Je_j, e_k)), & R_{ij\bar{k}\bar{l}} &= R(e_i, e_j, Je_k, Je_l), \end{aligned} \quad \text{etc.}$$

Then it is easy to see that

$$\begin{aligned} \nabla_i J_{jk} + \nabla_{\bar{i}} J_{\bar{j}\bar{k}} &= 0, \\ \nabla_{\bar{i}} J_{jk} &= \nabla_i J_{\bar{j}\bar{k}} = \nabla_i J_{j\bar{k}}, \\ N_{ijk} &= -2\nabla_{\bar{i}} J_{jk}, & 2\nabla_i J_{jk} &= N_{\bar{i}jk}. \end{aligned}$$

We set

$$(2.3) \quad A_{ij} = g(e_i, (\nabla_{e_j} N)(e_1, e_3)) = \nabla_j N_{i13},$$

for a unitary basis $\{e_i\} = \{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ of $T_pM, p \in M$. We note that

$$A_{ij} - A_{ji} = -2(R_{ij13} - R_{ij24}).$$

By using these A_{ij} , J. T. Cho and K. Sekigawa obtained the following characterization of almost Kähler manifolds of pointwise constant holomorphic sectional curvature:

Proposition 2.1 ([1]). *Let M be a 4-dimensional almost Kähler manifold of pointwise constant holomorphic sectional curvature $c = c(p)(p \in M)$. Then*

$$\begin{aligned} R_{1212} &= R_{3434} = -c(p), \\ R_{1234} &= -\frac{c(p)}{2} - \frac{1}{16}(\tau^* - \tau), \\ R_{1324} &= -\frac{c(p)}{4} + \frac{1}{32}(\tau^* - \tau) + \frac{1}{8}(A_{13} - A_{31} - A_{24} + A_{42}), \\ R_{1423} &= \frac{c(p)}{4} + \frac{3}{32}(\tau^* - \tau) + \frac{1}{8}(A_{13} - A_{31} - A_{24} + A_{42}), \end{aligned}$$

$$\begin{aligned}
R_{1313} &= -\frac{c(p)}{4} + \frac{1}{32}(\tau^* - \tau) - \frac{3}{8}(A_{13} - A_{31}) - \frac{1}{8}(A_{24} - A_{42}), \\
R_{1414} &= -\frac{c(p)}{4} + \frac{5}{32}(\tau^* - \tau) - \frac{1}{8}(A_{13} + A_{42}) - \frac{3}{8}(A_{31} + A_{24}), \\
R_{2323} &= -\frac{c(p)}{4} + \frac{5}{32}(\tau^* - \tau) + \frac{1}{8}(A_{31} + A_{24}) + \frac{3}{8}(A_{13} + A_{42}), \\
R_{2424} &= -\frac{c(p)}{4} + \frac{1}{32}(\tau^* - \tau) + \frac{3}{8}(A_{24} - A_{42}) + \frac{1}{8}(A_{13} - A_{31}), \\
R_{1334} &= -R_{2434} = -\frac{1}{4}(A_{34} - A_{43}), \\
R_{1213} &= -R_{1224} = -\frac{1}{4}(A_{12} - A_{21}), \\
R_{1434} &= R_{2334} = -\frac{1}{4}(A_{33} + A_{44}), \\
R_{1214} &= R_{1223} = -\frac{1}{4}(A_{11} + A_{22}), \\
R_{1323} &= \frac{1}{8}(A_{14} + A_{41} + A_{32} - 3A_{23}), \\
R_{2324} &= \frac{1}{8}(A_{14} + A_{41} + A_{23} - 3A_{32}), \\
R_{1314} &= -\frac{1}{8}(A_{23} + A_{32} + A_{14} - 3A_{41}), \\
R_{1424} &= -\frac{1}{8}(A_{23} + A_{32} + A_{41} - 3A_{14}),
\end{aligned}$$

for any unitary basis $\{e_i\}$ of $T_p M$ at each point $p \in M$.

Let M be an open set of \mathbf{R}^4 , and let (x_1, x_2, x_3, x_4) be the Euclidean coordinates on M . We put

$$z_1 = x_1 + \sqrt{-1}x_2, \quad z_2 = x_3 + \sqrt{-1}x_4.$$

Let f be a non-zero real function and h be a complex function on M . Then P. Nurowski and M. Przanowski proved the following

Lemma 2.2 ([3]). *Let $(z_1, \bar{z}_1, z_2, \bar{z}_2)$ be coordinates on M . Then for each value of the real constant $\phi \in [0, 2\pi)$, the metric*

$$g = 2f^2(dz_1 + hdz_2)(d\bar{z}_1 + \bar{h}d\bar{z}_2) + \frac{2}{f^2}dz_2d\bar{z}_2$$

and the almost complex structure

$$J_{e^{\sqrt{-1}\phi}}^+ = 2\text{Re} \left[\sqrt{-1} e^{\sqrt{-1}\phi} \left\{ f^2(dz_1 + hdz_2) \otimes \left(\frac{\partial}{\partial \bar{z}_2} - \bar{h} \frac{\partial}{\partial \bar{z}_1} \right) - \frac{1}{f^2} dz_2 \otimes \frac{\partial}{\partial \bar{z}_1} \right\} \right]$$

define an almost Kähler structure on M .

The Riemannian metric $g = (g_{ij})$ and the almost complex structure $J_{e^{\sqrt{-1}\phi}}^+ = (J_i^j)$ in the above Lemma 2.2 are given with respect to the real coordinates (x_1, x_2, x_3, x_4) by

$$\begin{aligned}
(2.4) \quad & g_{11} = g_{22} = 2f^2, \\
& g_{12} = g_{21} = g_{34} = g_{43} = 0, \\
& g_{13} = g_{31} = g_{24} = g_{42} = f^2u, \\
& -g_{14} = -g_{41} = g_{23} = g_{32} = f^2v, \\
& g_{33} = g_{44} = 2f^2(u^2 + v^2 + \frac{1}{f^4}),
\end{aligned}$$

and

$$\begin{aligned}
(2.5) \quad & J_1^1 = -J_2^2 = -v \cos \phi + u \sin \phi, \\
& J_1^2 = J_2^1 = u \cos \phi + v \sin \phi, \\
& J_1^3 = -J_2^4 = -\sin \phi, \\
& J_1^4 = J_2^3 = -\cos \phi, \\
& J_3^1 = -J_4^2 = (u^2 + v^2 + \frac{1}{f^4}) \sin \phi, \\
& J_3^2 = J_4^1 = (u^2 + v^2 + \frac{1}{f^4}) \cos \phi, \\
& -J_3^3 = J_4^4 = v \cos \phi + u \sin \phi, \\
& J_3^4 = J_4^3 = -u \cos \phi + v \sin \phi,
\end{aligned}$$

where u and v are the real and imaginary part of the complex function h , respectively. It is easy to see that the Kähler form Ω is given by

$$\Omega = 2 \sin \phi dx_1 \wedge dx_3 + 2 \cos \phi dx_1 \wedge dx_4 + 2 \cos \phi dx_2 \wedge dx_3 - 2 \sin \phi dx_2 \wedge dx_4,$$

and $(J_{e^{\sqrt{-1}\phi}}^+, g)$ is an almost Kähler structure. In the present paper, we shall mainly deal with this almost Kähler manifold $(M, J_{e^{\sqrt{-1}\phi}}^+, g)$. Since the almost Kähler structure $(J_{e^{\sqrt{-1}\phi}}^+, g)$ is determined by functions f, u, v and a real constant ϕ , we denote this almost Kähler manifold by $M(f, u, v, \phi)$.

We define a unitary frame field $\{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ on $M(f, u, v, \phi)$ by

$$\begin{aligned}
(2.6) \quad & e_1 = \frac{1}{\sqrt{2}f} \frac{\partial}{\partial x_1}, \\
& e_2 = \frac{f}{\sqrt{2}} \left\{ (-v \cos \phi + u \sin \phi) \frac{\partial}{\partial x_1} + (u \cos \phi + v \sin \phi) \frac{\partial}{\partial x_2} \right. \\
& \quad \left. - \sin \phi \frac{\partial}{\partial x_3} - \cos \phi \frac{\partial}{\partial x_4} \right\}, \\
& e_3 = \frac{1}{\sqrt{2}f} \frac{\partial}{\partial x_2}, \\
& e_4 = \frac{f}{\sqrt{2}} \left\{ (u \cos \phi + v \sin \phi) \frac{\partial}{\partial x_1} + (v \cos \phi - u \sin \phi) \frac{\partial}{\partial x_2} \right. \\
& \quad \left. - \cos \phi \frac{\partial}{\partial x_3} + \sin \phi \frac{\partial}{\partial x_4} \right\}.
\end{aligned}$$

With respect to this unitary frame $\{e_i\}_{i=1,2,3,4}$ we set

$$\nabla_{e_i} e_j = \sum \Gamma_{ijk} e_k.$$

Then we have

$$\begin{aligned}
(2.7) \quad & \Gamma_{ijj} = 0, \\
& \Gamma_{112} = -\Gamma_{121} = \frac{1}{\sqrt{2}} \{ \cos \phi (\partial_4 f + v \partial_1 f - u \partial_2 f + f \partial_1 v) \\
& \quad \quad \quad + \sin \phi (\partial_3 f - u \partial_1 f - v \partial_2 f - f \partial_1 u) \}, \\
& \Gamma_{113} = -\Gamma_{131} = -\frac{\partial_2 f}{\sqrt{2} f^2}, \\
& \Gamma_{114} = -\Gamma_{141} = \frac{1}{\sqrt{2}} \{ \cos \phi (\partial_3 f - u \partial_1 f - v \partial_2 f - f \partial_1 u) \\
& \quad \quad \quad - \sin \phi (\partial_4 f + v \partial_1 f - u \partial_2 f + f \partial_1 v) \}, \\
& \Gamma_{123} = -\Gamma_{132} = \frac{f}{2\sqrt{2}} \{ \cos \phi (\partial_1 u - \partial_2 v) + \sin \phi (\partial_2 u + \partial_1 v) \}, \\
& \Gamma_{124} = -\Gamma_{142} = \frac{f^3}{2\sqrt{2}} \{ (\partial_3 v - u \partial_1 v - v \partial_2 v) + (\partial_4 u + v \partial_1 u - u \partial_2 u) \}, \\
& \Gamma_{134} = -\Gamma_{143} = -\frac{f}{2\sqrt{2}} \{ \cos \phi (\partial_2 u + \partial_1 v) - \sin \phi (\partial_1 u - \partial_2 v) \}, \\
& \Gamma_{212} = -\Gamma_{221} = -\frac{\partial_1 f}{\sqrt{2} f^2}, \\
& \Gamma_{213} = -\Gamma_{231} = -\frac{f}{2\sqrt{2}} \{ \cos \phi (\partial_1 u + \partial_2 v) - \sin \phi (\partial_2 u - \partial_1 v) \}, \\
& \Gamma_{214} = -\Gamma_{241} = \frac{f^3}{2\sqrt{2}} \{ (\partial_3 v - u \partial_1 v - v \partial_2 v) + (\partial_4 u + v \partial_1 u - u \partial_2 u) \}, \\
& \Gamma_{223} = -\Gamma_{232} = \frac{\partial_2 f}{\sqrt{2} f^2}, \\
& \Gamma_{224} = -\Gamma_{242} = -\frac{1}{\sqrt{2}} \{ \cos \phi (\partial_3 f - u \partial_1 f - v \partial_2 f) \\
& \quad \quad \quad - \sin \phi (\partial_4 f + v \partial_1 f - u \partial_2 f) \},
\end{aligned}$$

$$\Gamma_{234} = -\Gamma_{243} = -\frac{f^3}{2\sqrt{2}}\{(\partial_3 u - u\partial_1 u - v\partial_2 u) - (\partial_4 v + v\partial_1 v - u\partial_2 v)\},$$

$$\Gamma_{312} = -\Gamma_{321} = -\frac{f}{2\sqrt{2}}\{\cos\phi(\partial_1 u - \partial_2 v) + \sin\phi(\partial_2 u + \partial_1 v)\},$$

$$\Gamma_{313} = -\Gamma_{331} = \frac{\partial_1 f}{\sqrt{2}f^2},$$

$$\Gamma_{314} = -\Gamma_{341} = -\frac{f}{2\sqrt{2}}\{\cos\phi(\partial_2 u + \partial_1 v) - \sin\phi(\partial_1 u - \partial_2 v)\},$$

$$\begin{aligned}\Gamma_{323} = -\Gamma_{332} = &-\frac{1}{\sqrt{2}}\{\cos\phi(\partial_4 f + v\partial_1 f - u\partial_2 f - f\partial_2 u) \\ &+ \sin\phi(\partial_3 f - u\partial_1 f - v\partial_2 f - f\partial_2 v)\},\end{aligned}$$

$$\Gamma_{324} = -\Gamma_{342} = -\frac{f^3}{2\sqrt{2}}\{(\partial_3 u - u\partial_1 u - v\partial_2 u) - (\partial_4 v + v\partial_1 v - u\partial_2 v)\},$$

$$\begin{aligned}\Gamma_{334} = -\Gamma_{343} = &\frac{1}{\sqrt{2}}\{\cos\phi(\partial_3 f - u\partial_1 f - v\partial_2 f - f\partial_2 v) \\ &- \sin\phi(\partial_4 f + v\partial_1 f - u\partial_2 f - f\partial_2 u)\},\end{aligned}$$

$$\Gamma_{412} = -\Gamma_{421} = -\frac{f^3}{2\sqrt{2}}\{(\partial_3 v - u\partial_1 v - v\partial_2 v) + (\partial_4 u + v\partial_1 u - u\partial_2 u)\},$$

$$\Gamma_{413} = -\Gamma_{431} = \frac{f}{2\sqrt{2}}\{\cos\phi(\partial_2 u - \partial_1 v) + \sin\phi(\partial_1 u + \partial_2 v)\},$$

$$\Gamma_{414} = -\Gamma_{441} = -\frac{\partial_1 f}{\sqrt{2}f^2},$$

$$\Gamma_{423} = -\Gamma_{432} = -\frac{f^3}{2\sqrt{2}}\{(\partial_3 u - u\partial_1 u - v\partial_2 u) - (\partial_4 v + v\partial_1 v - u\partial_2 v)\},$$

$$\begin{aligned}\Gamma_{424} = -\Gamma_{442} = &\frac{1}{\sqrt{2}}\{\cos\phi(\partial_4 f + v\partial_1 f - u\partial_2 f) \\ &+ \sin\phi(\partial_3 f - u\partial_1 f - v\partial_2 f)\},\end{aligned}$$

$$\Gamma_{434} = -\Gamma_{443} = -\frac{\partial_2 f}{\sqrt{2}f^2},$$

where we denote

$$\partial_i f = \frac{\partial f}{\partial x_i}.$$

By (2.7), we find

$$\begin{aligned}
(2.8) \quad \nabla_1 J_{13} &= -\nabla_1 J_{31} = -\nabla_1 J_{24} = \nabla_1 J_{42} = -\nabla_2 J_{14} = \nabla_2 J_{41} = -\nabla_2 J_{23} = \nabla_2 J_{32} \\
&= \frac{1}{2\sqrt{2}} \{2 \cos \phi (\partial_3 f - u \partial_1 f - v \partial_2 f) - 2 \sin \phi (\partial_4 f + v \partial_1 f - u \partial_1 f) \\
&\quad - f \cos \phi (\partial_1 u + \partial_2 v) + f \sin \phi (\partial_2 u - \partial_1 v)\}, \\
\nabla_1 J_{14} &= -\nabla_1 J_{41} = \nabla_1 J_{23} = -\nabla_1 J_{32} = \nabla_2 J_{13} = -\nabla_2 J_{31} = -\nabla_2 J_{24} = \nabla_2 J_{42} \\
&= \frac{1}{2\sqrt{2}f^2} \{2\partial_2 f - f^5 u (\partial_2 u + \partial_1 v) + f^5 v (\partial_1 u - \partial_2 v) + f^5 (\partial_4 u + \partial_3 v)\}, \\
\nabla_3 J_{13} &= -\nabla_3 J_{31} = -\nabla_3 J_{24} = \nabla_3 J_{42} = -\nabla_4 J_{14} = \nabla_4 J_{41} = -\nabla_4 J_{23} = \nabla_4 J_{32} \\
&= -\frac{1}{2\sqrt{2}} \{2 \sin \phi (\partial_3 f - u \partial_1 f - v \partial_2 f) + 2 \cos \phi (\partial_4 f + v \partial_1 f - u \partial_1 f) \\
&\quad - f \sin \phi (\partial_1 u + \partial_2 v) - f \cos \phi (\partial_2 u - \partial_1 v)\}, \\
\nabla_3 J_{14} &= -\nabla_3 J_{41} = \nabla_3 J_{23} = -\nabla_3 J_{32} = \nabla_4 J_{13} = -\nabla_4 J_{31} = -\nabla_4 J_{24} = \nabla_4 J_{42} \\
&= \frac{1}{2\sqrt{2}f^2} \{-2\partial_1 f + f^5 u (\partial_1 u - \partial_2 v) + f^5 v (\partial_2 u + \partial_1 v) - f^5 (\partial_3 u - \partial_4 v)\}, \\
\nabla_i J_{jk} &= 0 \quad (\text{otherwise}).
\end{aligned}$$

By (2.1) and (2.8), we then have

$$\begin{aligned}
(2.9) \quad N_{113} &= -N_{131} = -N_{214} = N_{241} = -N_{223} = N_{232} = -N_{124} = N_{142} \\
&= -\frac{1}{\sqrt{2}f^2} \{2\partial_2 f - f^5 u (\partial_2 u + \partial_1 v) + f^5 v (\partial_1 u - \partial_2 v) + f^5 (\partial_4 u + \partial_3 v)\}, \\
N_{213} &= -N_{231} = N_{114} = -N_{141} = N_{123} = -N_{132} = -N_{224} = N_{242} \\
&= \frac{1}{\sqrt{2}} \{2 \cos \phi (\partial_3 f - u \partial_1 f - v \partial_2 f) - 2 \sin \phi (\partial_4 f + v \partial_1 f - u \partial_1 f) \\
&\quad - f \cos \phi (\partial_1 u + \partial_2 v) + f \sin \phi (\partial_2 u - \partial_1 v)\}, \\
N_{313} &= -N_{331} = -N_{414} = N_{441} = -N_{423} = N_{432} = -N_{324} = N_{342} \\
&= -\frac{1}{\sqrt{2}f^2} \{-2\partial_1 f + f^5 u (\partial_1 u - \partial_2 v) + f^5 v (\partial_2 u + \partial_1 v) - f^5 (\partial_3 u - \partial_4 v)\}, \\
N_{413} &= -N_{431} = N_{314} = -N_{341} = N_{323} = -N_{332} = -N_{424} = N_{442} \\
&= -\frac{1}{\sqrt{2}} \{2 \sin \phi (\partial_3 f - u \partial_1 f - v \partial_2 f) + 2 \cos \phi (\partial_4 f + v \partial_1 f - u \partial_1 f) \\
&\quad - f \sin \phi (\partial_1 u + \partial_2 v) - f \cos \phi (\partial_2 u - \partial_1 v)\}, \\
N_{ijk} &= 0 \quad (\text{otherwise}).
\end{aligned}$$

Now, let $u = \alpha, v = \beta$, where α, β are constants. In this case, we introduce a new coordinates system $(\xi_1, \xi_2, \xi_3, \xi_4)$ on $M(f, \alpha, \beta, \phi)$ by

$$(2.10) \quad \xi_1 = x_1 + \alpha x_3 - \beta x_4, \quad \xi_2 = x_2 + \beta x_3 + \alpha x_4, \quad \xi_3 = x_3, \quad \xi_4 = x_4.$$

With respect to this coordinates system $(\xi_1, \xi_2, \xi_3, \xi_4)$, we adopt the notation $\tilde{\partial}_i f = \frac{\partial f}{\partial \xi_i}$. Then, (2.9) can be written as

$$\begin{aligned}
\Gamma_{112} &= -\Gamma_{121} = -\Gamma_{323} = \Gamma_{332} = \Gamma_{424} = -\Gamma_{442} \\
&= \frac{1}{\sqrt{2}} \{ \cos \phi \tilde{\partial}_4 f + \sin \phi \tilde{\partial}_3 f \}, \\
\Gamma_{113} &= -\Gamma_{131} = -\Gamma_{223} = \Gamma_{232} = \Gamma_{434} = -\Gamma_{443} = -\frac{\tilde{\partial}_2 f}{\sqrt{2} f^2}, \\
(2.11) \quad \Gamma_{114} &= -\Gamma_{141} = -\Gamma_{224} = \Gamma_{242} = \Gamma_{334} = -\Gamma_{343} \\
&= \frac{1}{\sqrt{2}} \{ \cos \phi \tilde{\partial}_3 f - \sin \phi \tilde{\partial}_4 f \}, \\
\Gamma_{212} &= -\Gamma_{221} = -\Gamma_{313} = \Gamma_{331} = \Gamma_{414} = -\Gamma_{441} = -\frac{\tilde{\partial}_1 f}{\sqrt{2} f^2}, \\
\Gamma_{ijk} &= 0 \quad (\text{otherwise}).
\end{aligned}$$

3 Ricci flat examples

In this section, we shall find out a family of Ricci flat metrics in the framework of Nurowski and Przanowski [3]. Let $M(f, u, v, \phi)$ be the almost Kähler 4-manifold defined in §2. Since the Nurowski-Przanowski's Ricci flat example is given by

$$u = -2x_3, \quad v = 2x_4, \quad f = \frac{1}{\sqrt{2}} \{ 2x_1 - 2(x_3^2 + x_4^2) \}^{-\frac{1}{4}},$$

we suppose that $u = u(x_3, x_4)$, $v = v(x_3, x_4)$. Then, by a straightforward computation, we obtain, with respect to the unitary basis $\{e_i\}$ in (2.6),

$$\begin{aligned}
\rho_{11} &= \frac{1}{4 f^4} f^{10} (\partial_4 u + \partial_3 v)^2 - 10 (\partial_1 f)^2 + 6 (\partial_2 f)^2 \\
(3.1) \quad &- 2 f^4 (\partial_4 f + v \partial_1 f - u \partial_2 f)^2 + (\partial_3 f - u \partial_1 f - v \partial_2 f)^2 \\
&+ 2 f (\partial_1^2 f - \partial_2^2 f) \\
&- 2 f^5 \{ (\partial_4 + v \partial_1 - u \partial_2)^2 f + (\partial_3 - u \partial_1 - v \partial_2)^2 f \},
\end{aligned}$$

$$\begin{aligned}
\rho_{22} &= -\frac{1}{4f^4} \left[6 \{ (\partial_1 f)^2 + (\partial_2 f)^2 \} - 2f \{ \partial_1^2 f + \partial_2^2 f \} \right. \\
&\quad + f^{10} \{ (\partial_3 u - \partial_4 v)^2 + (\partial_4 u + \partial_3 v)^2 \} \\
&\quad + 2f^4 \{ (1 + 2 \cos 2\phi) (\partial_4 f + v \partial_1 f - u \partial_2 f)^2 \\
&\quad + (1 - 2 \cos 2\phi) (\partial_3 f - u \partial_1 f - v \partial_2 f)^2 \\
&\quad + 4 \sin 2\phi (\partial_3 f - u \partial_1 f - v \partial_2 f) (\partial_4 f + v \partial_1 f - u \partial_2 f) \} \\
&\quad + 2f^5 \{ \cos 2\phi ((\partial_4 + v \partial_1 - u \partial_2)^2 f - (\partial_3 - u \partial_1 - v \partial_2)^2 f) \\
&\quad + \sin 2\phi ((\partial_3 - u \partial_1 - v \partial_2) (\partial_4 + v \partial_1 - u \partial_2) f \\
&\quad \left. + (\partial_4 + v \partial_1 - u \partial_2) (\partial_3 - u \partial_1 - v \partial_2) f \} \right], \tag{3.2}
\end{aligned}$$

$$\begin{aligned}
\rho_{33} &= \frac{1}{4f^4} \left[f^{10} (\partial_3 u - \partial_4 v)^2 + 6 (\partial_1 f)^2 - 10 (\partial_2 f)^2 \right. \\
&\quad - 2f^4 \{ (\partial_4 f + v \partial_1 f - u \partial_2 f)^2 + (\partial_3 f - u \partial_1 f - v \partial_2 f)^2 \} \\
&\quad - 2f (\partial_1^2 f - \partial_2^2 f) \\
&\quad \left. - 2f^5 \{ (\partial_4 + v \partial_1 - u \partial_2)^2 f + (\partial_3 - u \partial_1 - v \partial_2)^2 f \} \right], \tag{3.3}
\end{aligned}$$

$$\begin{aligned}
\rho_{44} &= \frac{1}{4f^4} \left[-6 \{ (\partial_1 f)^2 + (\partial_2 f)^2 \} + 2f \{ \partial_1^2 f + \partial_2^2 f \} \right. \\
&\quad - f^{10} \{ (\partial_3 u - \partial_4 v)^2 + (\partial_4 u + \partial_3 v)^2 \} \\
&\quad - 2f^4 \{ (1 - 2 \cos 2\phi) (\partial_4 f + v \partial_1 f - u \partial_2 f)^2 \\
&\quad + (1 + 2 \cos 2\phi) (\partial_3 f - u \partial_1 f - v \partial_2 f)^2 \\
&\quad - 4 \sin 2\phi (\partial_3 f - u \partial_1 f - v \partial_2 f) (\partial_4 f + v \partial_1 f - u \partial_2 f) \} \\
&\quad + 2f^5 \{ \cos 2\phi ((\partial_4 + v \partial_1 - u \partial_2)^2 f - (\partial_3 - u \partial_1 - v \partial_2)^2 f) \\
&\quad + \sin 2\phi ((\partial_3 - u \partial_1 - v \partial_2) (\partial_4 + v \partial_1 - u \partial_2) f \\
&\quad \left. + (\partial_4 + v \partial_1 - u \partial_2) (\partial_3 - u \partial_1 - v \partial_2) f \} \right], \tag{3.4}
\end{aligned}$$

$$\begin{aligned}
\rho_{12} &= \frac{1}{4f^2} \left[f^6 \{ \sin \phi \partial_4 (\partial_4 u + \partial_3 v) - \cos \phi \partial_3 (\partial_4 u + \partial_3 v) \} \right. \\
&\quad + 4 \partial_1 f \{ \cos \phi (\partial_4 f + v \partial_1 f - u \partial_2 f) + \sin \phi (\partial_3 f - u \partial_1 f - v \partial_2 f) \} \\
&\quad \left. - 6 f^5 (\partial_4 u + \partial_3 v) \{ \cos \phi (\partial_3 f - u \partial_1 f - v \partial_2 f) - \sin \phi (\partial_4 f + v \partial_1 f - u \partial_2 f) \} \right], \tag{3.5}
\end{aligned}$$

$$\rho_{13} = -\frac{1}{4f^4} \left[f^{10} (\partial_3 u - \partial_4 v) (\partial_4 u + \partial_3 v) + 16 \partial_1 f \partial_2 f - 4f \partial_1 \partial_2 f \right], \tag{3.6}$$

$$\begin{aligned}
(3.7) \quad \rho_{14} &= \frac{1}{4f^2} \left[f^6 \{ \sin \phi \partial_3 (\partial_4 u + \partial_3 v) + \cos \phi \partial_4 (\partial_4 u + \partial_3 v) \} \right. \\
&+ 4 \partial_1 f \{ \cos \phi (\partial_3 f - u \partial_1 f - v \partial_2 f) - \sin \phi (\partial_4 f + v \partial_1 f - u \partial_2 f) \} \\
&+ \left. 6 f^5 (\partial_4 u + \partial_3 v) \{ \cos \phi (\partial_4 f + v \partial_1 f - u \partial_2 f) + \sin \phi (\partial_3 f - u \partial_1 f - v \partial_2 f) \} \right],
\end{aligned}$$

$$\begin{aligned}
(3.8) \quad \rho_{23} &= \frac{1}{4f^2} \left[f^6 \{ \cos \phi \partial_3 (\partial_3 u - \partial_4 v) - \sin \phi \partial_4 (\partial_3 u - \partial_4 v) \} \right. \\
&+ 4 \partial_2 f \{ \cos \phi (\partial_4 f + v \partial_1 f - u \partial_2 f) + \sin \phi (\partial_3 f - u \partial_1 f - v \partial_2 f) \} \\
&+ \left. 6 f^5 (\partial_3 u - \partial_4 v) \{ \cos \phi (\partial_3 f - u \partial_1 f - v \partial_2 f) - \sin \phi (\partial_4 f + v \partial_1 f - u \partial_2 f) \} \right],
\end{aligned}$$

$$\begin{aligned}
(3.9) \quad \rho_{24} &= \frac{1}{2} f \partial_2 f (\partial_3 u - \partial_4 v) - \frac{1}{2} f \partial_1 f (\partial_4 u + \partial_3 v) \\
&+ \sin 2\phi \{ (\partial_4 f + v \partial_1 f - u \partial_2 f)^2 - (\partial_3 f - u \partial_1 f - v \partial_2 f)^2 \} \\
&- 2 \cos 2\phi (\partial_3 f - u \partial_1 f - v \partial_2 f) (\partial_4 f + v \partial_1 f - u \partial_2 f) \\
&+ f \cos \phi \sin \phi \partial_4 (\partial_4 f + v \partial_1 f - u \partial_2 f) - f \cos^2 \phi \partial_4 (\partial_3 f - u \partial_1 f - v \partial_2 f) \\
&+ f \sin^2 \phi \partial_3 (\partial_4 f + v \partial_1 f - u \partial_2 f) - f \cos \phi \sin \phi \partial_3 (\partial_3 f - u \partial_1 f - v \partial_2 f) \\
&+ f (u \cos \phi + v \sin \phi) \{ \cos \phi \partial_2 (\partial_3 f - u \partial_1 f - v \partial_2 f) \\
&- \sin \phi \partial_2 (\partial_4 f + v \partial_1 f - u \partial_2 f) \} \\
&+ f (u \sin \phi - v \cos \phi) \{ \cos \phi \partial_1 (\partial_3 f - u \partial_1 f - v \partial_2 f) \\
&- \sin \phi \partial_1 (\partial_4 f + v \partial_1 f - u \partial_2 f) \},
\end{aligned}$$

$$\begin{aligned}
(3.10) \quad \rho_{34} &= -\frac{1}{4f^2} \left[f^6 \{ \sin \phi \partial_3 (\partial_3 u - \partial_4 v) + \cos \phi \partial_4 (\partial_3 u - \partial_4 v) \} \right. \\
&+ 4 \partial_2 f \{ \sin \phi (\partial_4 f + v \partial_1 f - u \partial_2 f) - \cos \phi (\partial_3 f - u \partial_1 f - v \partial_2 f) \} \\
&+ \left. 6 f^5 (\partial_3 u - \partial_4 v) \{ \sin \phi (\partial_3 f - u \partial_1 f - v \partial_2 f) + \cos \phi (\partial_4 f + v \partial_1 f - u \partial_2 f) \} \right].
\end{aligned}$$

In order that our almost Kähler manifold $M(f, u, v, \phi)$ is Einstein, it is necessary that $\rho_{ij} = 0$ for $i \neq j$. From (3.6), if

$$\partial_3 u - \partial_4 v = 0 \quad \text{or} \quad \partial_4 u + \partial_3 v = 0$$

and

$$\partial_1 f = 0 \quad \text{or} \quad \partial_2 f = 0,$$

then $\rho_{13} = 0$. Taking account of the Nurowski-Przanowski example, we assume that

$$(3.11) \quad \partial_4 u + \partial_3 v = 0 \quad \text{and} \quad \partial_2 f = 0.$$

By (3.5) and (3.7), if

$$(3.12) \quad \partial_3 f = u \partial_1 f \quad \text{and} \quad \partial_4 f = -v \partial_1 f,$$

then $\rho_{12} = \rho_{14} = \rho_{24} = 0$. Further, from (3.8), if

$$(3.13) \quad \partial_3 u - \partial_4 v = \lambda (= \text{constant}),$$

then $\rho_{23} = \rho_{34} = 0$.

In this case, (3.1) \sim (3.4) reduce to

$$(3.14) \quad \rho_{11} = -\frac{1}{4f^4} \left[10 (\partial_1 f)^2 - 2f \partial_1^2 f \right],$$

$$(3.15) \quad \rho_{22} = -\frac{1}{4f^4} \left[6 (\partial_1 f)^2 - 2f \partial_1^2 f + \lambda^2 f^{10} \right],$$

$$(3.16) \quad \rho_{33} = \frac{1}{4f^4} \left[6 (\partial_1 f)^2 - 2f \partial_1^2 f + \lambda^2 f^{10} \right],$$

$$(3.17) \quad \rho_{44} = -\frac{1}{4f^4} \left[6 (\partial_1 f)^2 - 2f \partial_1^2 f + \lambda^2 f^{10} \right].$$

Since $\rho_{22} = -\rho_{33}$, it must be $\rho = 0$ for (M, J, g) is Einstein. Comparing (3.14) with (3.15), we find

$$4(\partial_1 f)^2 = \lambda^2 f^{10} \quad \text{and} \quad 2\partial_1 f = \pm \lambda f^5.$$

If $2\partial_1 f = -\lambda f^5$, then $N = 0$ by (2.9), and J is integrable. So, we suppose

$$(3.18) \quad 2\partial_1 f = \lambda f^5, \quad \lambda \neq 0.$$

From (3.18), we obtain that f is of in the following form:

$$(3.19) \quad f = \{-2\lambda x_1 - \varphi(x_3, x_4)\}^{-\frac{1}{4}},$$

where φ is an arbitrary function of x_3 and x_4 .

By (3.12), we have

$$(3.20) \quad \partial_3 \varphi = 2\lambda u \quad \text{and} \quad \partial_4 \varphi = -2\lambda v.$$

Taking account of (3.13) and (3.20), we deduce

$$(3.21) \quad \partial_3^2 \varphi + \partial_4^2 \varphi = 2\lambda^2.$$

Here, we suppose that φ is a polynomial of degree 2, for simplicity:

$$\varphi = ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k.$$

Then by (3.20) and (3.21), we have

$$\begin{aligned} u &= \frac{1}{\lambda}(ax_3 + bx_4 + d), \\ v &= -\frac{1}{\lambda}(bx_3 + cx_4 + e), \\ a + c &= \lambda^2 (> 0). \end{aligned}$$

Summing up the above arguments, we obtain the following

Theorem 3.1. *Let*

$$\begin{aligned} u &= \pm \frac{1}{\sqrt{a+c}}(ax_3 + bx_4 + d), \\ v &= \mp \frac{1}{\sqrt{a+c}}(bx_3 + cx_4 + e), \\ f &= \left\{ \mp 2\sqrt{a+c}x_1 - (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k) \right\}^{-\frac{1}{4}}, \end{aligned}$$

where a, b, c, d, e, k are arbitrary constants such that $a + c > 0$. Then Riemannian metric g given by (2.4) is a Ricci flat, i.e., $M(f, u, v, \phi)$ is a Ricci flat strictly almost Kähler manifold.

Note that the Nurowski-Przanowski's Ricci flat example is obtained by putting $a = c = 8, \lambda = -4, b = d = e = k = 0$.

Next, we assume that

$$(3.22) \quad \partial_3 u - \partial_4 v = 0 \quad \text{and} \quad \partial_1 f = 0.$$

By (3.8) and (3.10), if

$$(3.23) \quad \partial_3 f = v \partial_2 f \quad \text{and} \quad \partial_4 f = u \partial_2 f,$$

then $\rho_{23} = \rho_{24} = \rho_{34} = 0$. Further, from (3.5), if

$$(3.24) \quad \partial_3 u + \partial_4 v = \mu (= \text{constant}),$$

then $\rho_{12} = \rho_{14} = 0$.

In this case, (3.1) \sim (3.4) reduce to

$$(3.25) \quad \rho_{11} = \frac{1}{4f^4} \left[6(\partial_2 f)^2 - 2f \partial_2^2 f + \mu^2 f^{10} \right],$$

$$(3.26) \quad \rho_{22} = -\frac{1}{4f^4} \left[6(\partial_2 f)^2 - 2f \partial_2^2 f + \mu^2 f^{10} \right],$$

$$(3.27) \quad \rho_{33} = -\frac{1}{4f^4} \left[10(\partial_2 f)^2 - 2f \partial_2^2 f \right],$$

$$(3.28) \quad \rho_{44} = -\frac{1}{4f^4} \left[6(\partial_2 f)^2 - 2f \partial_2^2 f + \mu^2 f^{10} \right].$$

By the same argument as above, we have

$$(3.29) \quad f = \{-2\mu x_2 - \varphi(x_3, x_4)\}^{-\frac{1}{4}},$$

and suppose that

$$\varphi = ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k.$$

Then, we find

$$\begin{aligned} u &= \frac{1}{\mu}(bx_3 + cx_4 + e), \\ v &= \frac{1}{\mu}(ax_3 + bx_4 + d), \\ a + c &= \mu^2 (> 0). \end{aligned}$$

Consequently, we obtain the following

Theorem 3.2. *Let*

$$\begin{aligned} u &= \pm \frac{1}{\sqrt{a+c}}(bx_3 + cx_4 + e), \\ v &= \pm \frac{1}{\sqrt{a+c}}(ax_3 + bx_4 + d), \\ f &= \{\mp 2\sqrt{a+c}x_2 - (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k)\}^{-\frac{1}{4}}, \end{aligned}$$

where a, b, c, d, e, k are arbitrary constants such that $a + c > 0$. Then Riemannian metric g given by (2.4) is a Ricci flat, i.e., $M(f, u, v, \phi)$ is a Ricci flat strictly almost Kähler manifold.

In the next section, we shall show that the almost Kähler manifolds $M(f, u, v, \phi)$ in Theorems 3.1 and 3.2 are also of pointwise constant holomorphic sectional curvature and weakly *-Einstein.

Remark 3.3. By considering the integrable case: $2\partial_1 f = -\lambda f^5$ or $2\partial_2 f = -\mu f^5$, we get the Ricci flat Kähler manifolds,

Let

$$\begin{aligned} u &= \pm \frac{1}{\sqrt{a+c}}(ax_3 + bx_4 + d), \\ v &= \mp \frac{1}{\sqrt{a+c}}(bx_3 + cx_4 + e), \\ f &= \{\pm 2\sqrt{a+c}x_1 + (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k)\}^{-\frac{1}{4}}, \end{aligned}$$

or

$$\begin{aligned} u &= \pm \frac{1}{\sqrt{a+c}}(bx_3 + cx_4 + e), \\ v &= \pm \frac{1}{\sqrt{a+c}}(ax_3 + bx_4 + d), \\ f &= \{\pm 2\sqrt{a+c}x_2 + (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k)\}^{-\frac{1}{4}}. \end{aligned}$$

Then $M(f, u, v, \phi)$ are Ricci flat Kähler manifolds.

4 Examples of almost Kähler 4-manifolds with pointwise constant holomorphic sectional curvature

In this section, we construct 4-dimensional almost Kähler manifolds of pointwise constant holomorphic sectional curvature.

First, we show that the almost Kähler manifolds $M(f, u, v, \phi)$ in Theorem 3.1 have pointwise constant holomorphic sectional curvature. Indeed, by direct computations, we obtain

$$\begin{aligned}
 (4.1) \quad & A_{13} = -A_{24} = \\
 & -\frac{1}{2}(a+c) \left\{ \mp 2\sqrt{a+c}x_1 - (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k) \right\}^{-\frac{3}{2}}, \\
 & A_{31} = -A_{42} \\
 & = \frac{3}{2}(a+c) \left\{ \mp 2\sqrt{a+c}x_1 - (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k) \right\}^{-\frac{3}{2}}, \\
 & A_{ij} = 0 \quad (\text{otherwise}),
 \end{aligned}$$

$$\begin{aligned}
 (4.2) \quad & R_{1212} = R_{1234} = R_{1414} = R_{1423} = R_{2323} = R_{3434} \\
 & = -\frac{1}{4}(a+c) \left\{ \mp 2\sqrt{a+c}x_1 - (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k) \right\}^{-\frac{3}{2}}, \\
 & R_{1313} = -R_{1324} = R_{2424} \\
 & = \frac{1}{2}(a+c) \left\{ \mp 2\sqrt{a+c}x_1 - (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k) \right\}^{-\frac{3}{2}}, \\
 & R_{ijkl} = 0 \quad (\text{otherwise } i < j, k < l),
 \end{aligned}$$

$$\begin{aligned}
 (4.3) \quad & \rho_{11}^* = \rho_{22}^* = \rho_{33}^* = \rho_{44}^* \\
 & = \frac{1}{2}(a+c) \left\{ \mp 2\sqrt{a+c}x_1 - (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k) \right\}^{-\frac{3}{2}}, \\
 & \rho_{ij}^* = 0 \quad (i \neq j),
 \end{aligned}$$

$$(4.4) \quad \tau^* = 2(a+c) \left\{ \mp 2\sqrt{a+c}x_1 - (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k) \right\}^{-\frac{3}{2}}.$$

By virtue of Proposition 2.1, (4.1), (4.2), (4.3), and (4.4), we have the following **Theorem 4.1**. Let $M(f, u, v, \phi)$ be the Ricci flat strictly almost Kähler manifold in Theorem 3.1, i.e., functions u, v and f are given by

$$\begin{aligned}
 u &= \pm \frac{1}{\sqrt{a+c}}(ax_3 + bx_4 + d), \\
 v &= \mp \frac{1}{\sqrt{a+c}}(bx_3 + cx_4 + e), \\
 f &= \left\{ \mp 2\sqrt{a+c}x_1 - (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k) \right\}^{-\frac{1}{4}}.
 \end{aligned}$$

Then $M(f, u, v, \phi)$ is of pointwise constant holomorphic sectional curvature

$$c = \frac{1}{4}(a+c) \left\{ \mp 2\sqrt{a+c}x_1 - (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k) \right\}^{-\frac{3}{2}},$$

and weakly $*$ -Einstein.

Next, let $M(f, u, v, \phi)$ be the almost Kähler manifold in Theorem 3.2 Then, similarly we obtain

$$\begin{aligned} A_{13} &= -A_{24} \\ &= -\frac{3}{2}(a+c) \left\{ \mp 2\sqrt{a+c}x_2 - (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k) \right\}^{-\frac{3}{2}}, \\ (4.5) \quad A_{31} &= -A_{42} \\ &= \frac{1}{2}(a+c) \left\{ \mp 2\sqrt{a+c}x_2 - (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k) \right\}^{-\frac{3}{2}}, \\ A_{ij} &= 0 \quad (\text{otherwise}), \end{aligned}$$

$$\begin{aligned} R_{1212} &= R_{1234} = R_{1414} = R_{1423} = R_{2323} = R_{3434} \\ &= -\frac{1}{4}(a+c) \left\{ \mp 2\sqrt{a+c}x_2 - (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k) \right\}^{-\frac{3}{2}}, \\ (4.6) \quad R_{1313} &= -R_{1324} = R_{2424} \\ &= \frac{1}{2}(a+c) \left\{ \mp 2\sqrt{a+c}x_2 - (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k) \right\}^{-\frac{3}{2}}, \\ R_{ijkl} &= 0 \quad (\text{otherwise } i < j, k < l), \end{aligned}$$

$$\begin{aligned} \rho_{11}^* &= \rho_{22}^* = \rho_{33}^* = \rho_{44}^* \\ (4.7) \quad &= \frac{1}{2}(a+c) \left\{ \mp 2\sqrt{a+c}x_2 - (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k) \right\}^{-\frac{3}{2}}, \\ \rho_{ij}^* &= 0 \quad (i \neq j), \end{aligned}$$

$$(4.8) \quad \tau^* = 2(a+c) \left\{ \mp 2\sqrt{a+c}x_2 - (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k) \right\}^{-\frac{3}{2}}.$$

Therefore we have the following

Theorem 4.2. *Let $M(f, u, v, \phi)$ be the Ricci flat strictly almost Kähler manifold in Theorem 3.2, i.e., u, v and f are given by*

$$\begin{aligned} u &= \pm \frac{1}{\sqrt{a+c}}(bx_3 + cx_4 + e), \\ v &= \pm \frac{1}{\sqrt{a+c}}(ax_3 + bx_4 + d), \\ f &= \left\{ \mp 2\sqrt{a+c}x_2 - (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k) \right\}^{-\frac{1}{4}}. \end{aligned}$$

Then $M(f, u, v, \phi)$ is of pointwise constant holomorphic sectional curvature

$$c = \frac{1}{4}(a+c) \left\{ \mp 2\sqrt{a+c}x_2 - (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k) \right\}^{-\frac{3}{2}},$$

and weakly *-Einstein.

Now, we shall construct other examples. In the previous paper [5], we assumed that $u = v = 0$ and $\phi = 0$ for the sake of simplicity. Here, we shall consider the case where $u = \alpha, v = \beta$ (α and β are constants) and arbitrary ϕ . Then, with respect to the unitary frame $\{e_i\}$ in (2.6) and the coordinates system (ξ_i) in (2.10), we have the following expressions:

$$\begin{aligned}
(4.9) \quad A_{11} &= \frac{3\tilde{\partial}_1 f \tilde{\partial}_2 f}{f^4} - \frac{\tilde{\partial}_1 \tilde{\partial}_2 f}{f^3} - \cos 2\phi \tilde{\partial}_3 f \tilde{\partial}_4 f - \frac{\sin 2\phi}{2} \{(\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2\}, \\
A_{12} &= \frac{\cos \phi}{f^2} \{2\tilde{\partial}_1 f \tilde{\partial}_3 f - 2\tilde{\partial}_2 f \tilde{\partial}_4 f + f \tilde{\partial}_2 \tilde{\partial}_4 f\} \\
&\quad - \frac{\sin \phi}{f^2} \{2\tilde{\partial}_2 f \tilde{\partial}_3 f + 2\tilde{\partial}_1 f \tilde{\partial}_4 f - f \tilde{\partial}_2 \tilde{\partial}_3 f\}, \\
A_{13} &= -\frac{1}{f^4} \{(\tilde{\partial}_1 f)^2 - 2(\tilde{\partial}_2 f)^2 + f \tilde{\partial}_2^2 f\} - \{\cos \phi \tilde{\partial}_3 f - \sin \phi \tilde{\partial}_4 f\}^2, \\
A_{14} &= -\frac{\cos \phi}{f^2} \{\tilde{\partial}_2 f \tilde{\partial}_3 f + \tilde{\partial}_1 f \tilde{\partial}_4 f - f \tilde{\partial}_2 \tilde{\partial}_3 f\} \\
&\quad - \frac{\sin \phi}{f^2} \{\tilde{\partial}_1 f \tilde{\partial}_3 f - \tilde{\partial}_2 f \tilde{\partial}_4 f + f \tilde{\partial}_2 \tilde{\partial}_4 f\}, \\
A_{21} &= -\frac{\cos \phi}{f^2} \{2\tilde{\partial}_2 f \tilde{\partial}_4 f - f \tilde{\partial}_1 \tilde{\partial}_3 f\} - \frac{\sin \phi}{f^2} \{2\tilde{\partial}_2 f \tilde{\partial}_3 f + f \tilde{\partial}_1 \tilde{\partial}_4 f\}, \\
A_{22} &= \frac{\tilde{\partial}_1 f \tilde{\partial}_2 f}{f^4} - \cos 2\phi \tilde{\partial}_3 f \tilde{\partial}_4 f - \frac{\sin 2\phi}{2} \{(\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2\} \\
&\quad - f \cos 2\phi \tilde{\partial}_3 \tilde{\partial}_4 f - \frac{f \sin 2\phi}{2} \{\tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f\}, \\
A_{23} &= -\frac{\cos \phi}{f^2} \{\tilde{\partial}_2 f \tilde{\partial}_3 f - \tilde{\partial}_1 f \tilde{\partial}_4 f - f \tilde{\partial}_2 \tilde{\partial}_3 f\} \\
&\quad + \frac{\sin \phi}{f^2} \{\tilde{\partial}_1 f \tilde{\partial}_3 f + \tilde{\partial}_2 f \tilde{\partial}_4 f - f \tilde{\partial}_2 \tilde{\partial}_4 f\}, \\
A_{24} &= \frac{(\tilde{\partial}_2 f)^2}{f^4} + \{\cos \phi \tilde{\partial}_4 f + \sin \phi \tilde{\partial}_3 f\}^2 - \frac{f}{2} \{\tilde{\partial}_3^2 f + \tilde{\partial}_4^2 f\} \\
&\quad - \frac{f \cos 2\phi}{2} \{\tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f\} + f \sin 2\phi \tilde{\partial}_3 \tilde{\partial}_4 f, \\
A_{31} &= -\frac{1}{f^4} \{2(\tilde{\partial}_1 f)^2 - (\tilde{\partial}_2 f)^2 - f \tilde{\partial}_1^2 f\} + \{\cos \phi \tilde{\partial}_4 f + \sin \phi \tilde{\partial}_3 f\}^2,
\end{aligned}$$

$$\begin{aligned}
A_{32} &= \frac{\cos \phi}{f^2} \{ \tilde{\partial}_2 f \tilde{\partial}_3 f + \tilde{\partial}_1 f \tilde{\partial}_4 f - f \tilde{\partial}_1 \tilde{\partial}_4 f \} \\
&+ \frac{\sin \phi}{f^2} \{ \tilde{\partial}_1 f \tilde{\partial}_3 f - \tilde{\partial}_2 f \tilde{\partial}_4 f - f \tilde{\partial}_1 \tilde{\partial}_3 f \}, \\
A_{33} &= -\frac{3\tilde{\partial}_1 f \tilde{\partial}_2 f}{f^4} + \frac{\tilde{\partial}_1 \tilde{\partial}_2 f}{f^3} + \cos 2\phi \tilde{\partial}_3 f \tilde{\partial}_4 f + \frac{\sin 2\phi}{2} \{ (\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2 \}, \\
A_{34} &= \frac{\cos \phi}{f^2} \{ 2\tilde{\partial}_1 f \tilde{\partial}_3 f - 2\tilde{\partial}_2 f \tilde{\partial}_4 f - f \tilde{\partial}_1 \tilde{\partial}_3 f \} \\
&- \frac{\sin \phi}{f^2} \{ 2\tilde{\partial}_2 f \tilde{\partial}_3 f + 2\tilde{\partial}_1 f \tilde{\partial}_4 f - f \tilde{\partial}_1 \tilde{\partial}_4 f \}, \\
A_{41} &= -\frac{\cos \phi}{f^2} \{ \tilde{\partial}_2 f \tilde{\partial}_3 f - \tilde{\partial}_1 f \tilde{\partial}_4 f + f \tilde{\partial}_1 \tilde{\partial}_4 f \} \\
&+ \frac{\sin \phi}{f^2} \{ \tilde{\partial}_1 f \tilde{\partial}_3 f + \tilde{\partial}_2 f \tilde{\partial}_4 f - f \tilde{\partial}_1 \tilde{\partial}_3 f \}, \\
A_{42} &= -\frac{(\tilde{\partial}_1 f)^2}{f^4} - \{ \cos \phi \tilde{\partial}_3 f - \sin \phi \tilde{\partial}_4 f \}^2 + \frac{f}{2} \{ \tilde{\partial}_3^2 f + \tilde{\partial}_4^2 f \} \\
&- \frac{f \cos 2\phi}{2} \{ \tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f \} + f \sin 2\phi \tilde{\partial}_3 \tilde{\partial}_4 f, \\
A_{43} &= \frac{\cos \phi}{f^2} \{ 2\tilde{\partial}_1 f \tilde{\partial}_3 f - f \tilde{\partial}_2 \tilde{\partial}_4 f \} - \frac{\sin \phi}{f^2} \{ 2\tilde{\partial}_1 f \tilde{\partial}_4 f + f \tilde{\partial}_2 \tilde{\partial}_3 f \}, \\
A_{44} &= -\frac{\tilde{\partial}_1 f \tilde{\partial}_2 f}{f^4} + \cos 2\phi \tilde{\partial}_3 f \tilde{\partial}_4 f + \frac{\sin 2\phi}{2} \{ (\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2 \} \\
&+ f \cos 2\phi \tilde{\partial}_3 \tilde{\partial}_4 f + \frac{f \sin 2\phi}{2} \{ \tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f \}.
\end{aligned}$$

Furthermore, we find from (2.11)

$$\begin{aligned}
R_{1212} &= \frac{1}{2f^4} \{ 3(\tilde{\partial}_1 f)^2 - (\tilde{\partial}_2 f)^2 - f \tilde{\partial}_1^2 f \} - \frac{\cos 2\phi}{2} \{ (\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2 \} \\
&+ \sin 2\phi \tilde{\partial}_3 f \tilde{\partial}_4 f + \frac{f}{4} \{ \tilde{\partial}_3^2 f + \tilde{\partial}_4^2 f \} - \frac{f \cos 2\phi}{4} \{ \tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f \} \\
&+ \frac{f \sin 2\phi}{2} \tilde{\partial}_3 \tilde{\partial}_4 f, \\
(4.10) \quad R_{3434} &= \frac{1}{2f^4} \{ -(\tilde{\partial}_1 f)^2 + 3(\tilde{\partial}_2 f)^2 - f \tilde{\partial}_2^2 f \} + \frac{\cos 2\phi}{2} \{ (\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2 \} \\
&- \sin 2\phi \tilde{\partial}_3 f \tilde{\partial}_4 f + \frac{f}{4} \{ \tilde{\partial}_3^2 f + \tilde{\partial}_4^2 f \} + \frac{f \cos 2\phi}{4} \{ \tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f \} \\
&- \frac{f \sin 2\phi}{2} \tilde{\partial}_3 \tilde{\partial}_4 f,
\end{aligned}$$

$$\begin{aligned}
R_{1234} &= 0, \\
R_{1324} &= 0, \\
R_{1423} &= 0, \\
R_{1313} &= \frac{1}{2f^4} \{ -(\tilde{\partial}_1 f)^2 - (\tilde{\partial}_2 f)^2 + f\tilde{\partial}_1^2 f + f\tilde{\partial}_2^2 f \} + \frac{1}{2} \{ (\tilde{\partial}_3 f)^2 + (\tilde{\partial}_4 f)^2 \}, \\
R_{1414} &= \frac{1}{2f^4} \{ 3(\tilde{\partial}_1 f)^2 - (\tilde{\partial}_2 f)^2 - f\tilde{\partial}_1^2 f \} + \frac{\cos 2\phi}{2} \{ (\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2 \} \\
&\quad - \sin 2\phi \tilde{\partial}_3 f \tilde{\partial}_4 f + \frac{f}{4} \{ \tilde{\partial}_3^2 f + \tilde{\partial}_4^2 f \} + \frac{f \cos 2\phi}{4} \{ \tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f \} \\
&\quad - \frac{f \sin 2\phi}{2} \tilde{\partial}_3 \tilde{\partial}_4 f, \\
R_{2323} &= \frac{1}{2f^4} \{ -(\tilde{\partial}_1 f)^2 + 3(\tilde{\partial}_2 f)^2 - f\tilde{\partial}_2^2 f \} - \frac{\cos 2\phi}{2} \{ (\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2 \} \\
&\quad + \sin 2\phi \tilde{\partial}_3 f \tilde{\partial}_4 f + \frac{f}{4} \{ \tilde{\partial}_3^2 f + \tilde{\partial}_4^2 f \} - \frac{f \cos 2\phi}{4} \{ \tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f \} \\
&\quad + \frac{f \sin 2\phi}{2} \tilde{\partial}_3 \tilde{\partial}_4 f, \\
R_{2424} &= \frac{1}{2f^4} \{ (\tilde{\partial}_1 f)^2 + (\tilde{\partial}_2 f)^2 \} + \frac{1}{2} \{ (\tilde{\partial}_3 f)^2 + (\tilde{\partial}_4 f)^2 \} - \frac{f}{2} \{ \tilde{\partial}_3^2 f + \tilde{\partial}_4^2 f \}, \\
R_{1334} &= \frac{\cos \phi}{2f} \tilde{\partial}_1 \tilde{\partial}_3 f - \frac{\sin \phi}{2f} \tilde{\partial}_1 \tilde{\partial}_4 f, \\
R_{2434} &= -\frac{\cos \phi}{f^2} \tilde{\partial}_2 f \tilde{\partial}_4 f - \frac{\sin \phi}{f^2} \tilde{\partial}_2 f \tilde{\partial}_3 f + \frac{\cos \phi}{2f} \tilde{\partial}_2 \tilde{\partial}_4 f + \frac{\sin \phi}{2f} \tilde{\partial}_2 \tilde{\partial}_3 f, \\
R_{1213} &= -\frac{\cos \phi}{2f} \tilde{\partial}_2 \tilde{\partial}_4 f - \frac{\sin \phi}{2f} \tilde{\partial}_2 \tilde{\partial}_3 f, \\
R_{1224} &= \frac{\cos \phi}{f^2} \tilde{\partial}_1 f \tilde{\partial}_3 f - \frac{\sin \phi}{f^2} \tilde{\partial}_1 f \tilde{\partial}_4 f - \frac{\cos \phi}{2f} \tilde{\partial}_1 \tilde{\partial}_3 f + \frac{\sin \phi}{2f} \tilde{\partial}_1 \tilde{\partial}_4 f, \\
R_{1434} &= \frac{1}{2f^4} \{ 4\tilde{\partial}_1 f \tilde{\partial}_2 f - f\tilde{\partial}_1 \tilde{\partial}_2 f \}, \\
R_{2334} &= -\cos 2\phi \tilde{\partial}_3 f \tilde{\partial}_4 f - \frac{\sin 2\phi}{2} \{ (\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2 \} \\
&\quad - \frac{f \cos 2\phi}{2} \tilde{\partial}_3 \tilde{\partial}_4 f - \frac{f \sin 2\phi}{4} \{ \tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f \}, \\
R_{1214} &= \cos 2\phi \tilde{\partial}_3 f \tilde{\partial}_4 f + \frac{\sin 2\phi}{2} \{ (\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2 \} \\
&\quad + \frac{f \cos 2\phi}{2} \tilde{\partial}_3 \tilde{\partial}_4 f + \frac{f \sin 2\phi}{4} \{ \tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f \},
\end{aligned}$$

$$\begin{aligned}
R_{1223} &= -\frac{1}{2f^4}\{4\tilde{\partial}_1 f \tilde{\partial}_2 f - f \tilde{\partial}_1 \tilde{\partial}_2 f\}, \\
R_{1323} &= -\frac{\cos \phi}{2f} \tilde{\partial}_1 \tilde{\partial}_4 f - \frac{\sin \phi}{2f} \tilde{\partial}_1 \tilde{\partial}_3 f, \\
R_{2324} &= -\frac{\cos \phi}{f^2} \tilde{\partial}_2 f \tilde{\partial}_3 f + \frac{\sin \phi}{f^2} \tilde{\partial}_2 f \tilde{\partial}_4 f + \frac{\cos \phi}{2f} \tilde{\partial}_2 \tilde{\partial}_3 f - \frac{\sin \phi}{2f} \tilde{\partial}_2 \tilde{\partial}_4 f, \\
R_{1314} &= -\frac{\cos \phi}{2f} \tilde{\partial}_2 \tilde{\partial}_3 f + \frac{\sin \phi}{2f} \tilde{\partial}_2 \tilde{\partial}_4 f, \\
R_{1424} &= -\frac{\cos \phi}{f^2} \tilde{\partial}_1 f \tilde{\partial}_4 f - \frac{\sin \phi}{f^2} \tilde{\partial}_1 f \tilde{\partial}_3 f + \frac{\cos \phi}{2f} \tilde{\partial}_1 \tilde{\partial}_4 f + \frac{\sin \phi}{2f} \tilde{\partial}_1 \tilde{\partial}_3 f.
\end{aligned}$$

Then, by (4.1), we obtain easily

$$\begin{aligned}
(4.11) \quad \tau &= \frac{1}{f^3}\{\tilde{\partial}_1^2 f + \tilde{\partial}_2^2 f\} - \frac{4}{f^4}\{(\tilde{\partial}_1 f)^2 + (\tilde{\partial}_2 f)^2\} \\
&\quad - 2\{(\tilde{\partial}_3 f)^2 + (\tilde{\partial}_4 f)^2\} - f\{\tilde{\partial}_3^2 f + \tilde{\partial}_4^2 f\},
\end{aligned}$$

$$(4.12) \quad \tau^* = \frac{1}{f^3}\{\tilde{\partial}_1^2 f + \tilde{\partial}_2^2 f\} - \frac{2}{f^4}\{(\tilde{\partial}_1 f)^2 + (\tilde{\partial}_2 f)^2\} - f\{\tilde{\partial}_3^2 f + \tilde{\partial}_4^2 f\},$$

$$(4.13) \quad \tau^* - \tau = \frac{2}{f^4}\{(\tilde{\partial}_1 f)^2 + (\tilde{\partial}_2 f)^2\} + 2\{(\tilde{\partial}_3 f)^2 + (\tilde{\partial}_4 f)^2\}.$$

By (4.13), we see that $M(f, \alpha, \beta, \phi)$ is non-Kählerian if and only if f is not constant. By Proposition 2.1, (4.9), (4.1) and (4.13), we find the following

Lemma 4.3. *The almost Kähler manifold $M(f, \alpha, \beta, \phi)$ is of pointwise constant holomorphic sectional curvature $c = c(p)$ if and only if*

$$\begin{aligned}
(4.14) \quad &\frac{1}{2f^3}\{\tilde{\partial}_1^2 f - \tilde{\partial}_2^2 f\} + \frac{f \cos 2\phi}{2}\{\tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f\} - f \sin 2\phi \tilde{\partial}_3 \tilde{\partial}_4 f \\
&= \frac{2}{f^4}\{(\tilde{\partial}_1 f)^2 - (\tilde{\partial}_2 f)^2\} - \cos 2\phi\{(\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2\} + 2 \sin 2\phi \tilde{\partial}_3 f \tilde{\partial}_4 f,
\end{aligned}$$

$$(4.15) \quad c = -\frac{1}{8}(\tau^* - \tau) = -\frac{1}{4f^4}\{(\tilde{\partial}_1 f)^2 + (\tilde{\partial}_2 f)^2\} - \frac{1}{4}\{(\tilde{\partial}_3 f)^2 + (\tilde{\partial}_4 f)^2\},$$

$$\begin{aligned}
(4.16) \quad &\frac{1}{f^3}\{\tilde{\partial}_1^2 f + \tilde{\partial}_2^2 f\} - f\{\tilde{\partial}_3^2 f + \tilde{\partial}_4^2 f\} \\
&= \frac{1}{f^4}\{(\tilde{\partial}_1 f)^2 + (\tilde{\partial}_2 f)^2\} - \{(\tilde{\partial}_3 f)^2 + (\tilde{\partial}_4 f)^2\},
\end{aligned}$$

$$\begin{aligned}
(4.17) \quad &\cos \phi\{\tilde{\partial}_1 \tilde{\partial}_3 f + \tilde{\partial}_2 \tilde{\partial}_4 f\} - \sin \phi\{\tilde{\partial}_1 \tilde{\partial}_4 f - \tilde{\partial}_2 \tilde{\partial}_3 f\} \\
&= \frac{2 \cos \phi}{f} \tilde{\partial}_2 f \tilde{\partial}_4 f + \frac{2 \sin \phi}{f} \tilde{\partial}_2 f \tilde{\partial}_3 f,
\end{aligned}$$

$$\begin{aligned}
(4.18) \quad & \cos \phi \{ \tilde{\partial}_1 \tilde{\partial}_3 f + \tilde{\partial}_2 \tilde{\partial}_4 f \} - \sin \phi \{ \tilde{\partial}_1 \tilde{\partial}_4 f - \tilde{\partial}_2 \tilde{\partial}_3 f \} \\
& = \frac{2 \cos \phi}{f} \tilde{\partial}_1 f \tilde{\partial}_3 f - \frac{2 \sin \phi}{f} \tilde{\partial}_1 f \tilde{\partial}_4 f,
\end{aligned}$$

$$\begin{aligned}
(4.19) \quad & \frac{\tilde{\partial}_1 \tilde{\partial}_2 f}{f^3} - f \cos 2\phi \tilde{\partial}_3 \tilde{\partial}_4 f - \frac{f \sin 2\phi}{2} \{ \tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f \} \\
& = \frac{4 \tilde{\partial}_1 f \tilde{\partial}_2 f}{f^4} + 2 \cos 2\phi \tilde{\partial}_3 f \tilde{\partial}_4 f + \sin 2\phi \{ (\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2 \},
\end{aligned}$$

$$\begin{aligned}
(4.20) \quad & \cos \phi \{ \tilde{\partial}_1 \tilde{\partial}_4 f - \tilde{\partial}_2 \tilde{\partial}_3 f \} + \sin \phi \{ \tilde{\partial}_1 \tilde{\partial}_3 f + \tilde{\partial}_2 \tilde{\partial}_4 f \} \\
& = \frac{\cos \phi}{f} \{ \tilde{\partial}_1 f \tilde{\partial}_4 f - \tilde{\partial}_2 f \tilde{\partial}_3 f \} + \frac{\sin \phi}{f} \{ \tilde{\partial}_1 f \tilde{\partial}_3 f + \tilde{\partial}_2 f \tilde{\partial}_4 f \}.
\end{aligned}$$

To obtain solutions of the above equations, we first assume that $f = f(\xi_1, \xi_2)$ is a function of two variables ξ_1, ξ_2 . Then (4.14) \sim (4.20) reduce to

$$(4.21) \quad \tilde{\partial}_1^2 f - \tilde{\partial}_2^2 f = \frac{4}{f} \{ (\tilde{\partial}_1 f)^2 - (\tilde{\partial}_2 f)^2 \},$$

$$(4.22) \quad c = -\frac{1}{8}(\tau^* - \tau) = -\frac{1}{4f^4} \{ (\tilde{\partial}_1 f)^2 + (\tilde{\partial}_2 f)^2 \},$$

$$(4.23) \quad f(\tilde{\partial}_1^2 f + \tilde{\partial}_2^2 f) = (\tilde{\partial}_1 f)^2 + (\tilde{\partial}_2 f)^2,$$

$$(4.24) \quad f \tilde{\partial}_1 \tilde{\partial}_2 f = 4 \tilde{\partial}_1 f \tilde{\partial}_2 f.$$

By the same way as [5], we see that $f = K(\xi_1^2 + \xi_2^2)^{-\frac{1}{2}}$ satisfies (4.21), (4.23) and (4.24), where K is a constant.

Next, we assume that $f = f(\xi_3, \xi_4)$ is a function of ξ_3, ξ_4 . Then (4.14) \sim (4.15) reduce to

$$(4.25) \quad f(\tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f) = -2 \{ (\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2 \},$$

$$(4.26) \quad c = -\frac{1}{8}(\tau^* - \tau) = -\frac{1}{4} \{ (\tilde{\partial}_3 f)^2 + (\tilde{\partial}_4 f)^2 \},$$

$$(4.27) \quad f(\tilde{\partial}_3^2 f + \tilde{\partial}_4^2 f) = (\tilde{\partial}_3 f)^2 + (\tilde{\partial}_4 f)^2,$$

$$(4.28) \quad f \tilde{\partial}_3 \tilde{\partial}_4 f = -2 \tilde{\partial}_3 f \tilde{\partial}_4 f.$$

Similarly, we can easily see that $f = L(\xi_3^2 + \xi_4^2)^{\frac{1}{2}}$ satisfies (4.25), (4.27) and (4.28), where L is a constant.

Consequently, we obtain the following

Theorem 4.4. *Let $u = \alpha, v = \beta$ (α, β are constants) and*

$$f = K\{(x_1 + \alpha x_3 - \beta x_4)^2 + (x_2 + \beta x_3 + \alpha x_4)^2\}^{-\frac{1}{3}},$$

or

$$f = L(x_3^2 + x_4^2)^{\frac{1}{3}},$$

where K and L are non-zero constants. Then the almost Kähler manifold $M(f, u, v, \phi)$ is of pointwise constant holomorphic sectional curvature with

$$c = -\frac{1}{9K^2}\{(x_1 + \alpha x_3 - \beta x_4)^2 + (x_2 + \beta x_3 + \alpha x_4)^2\}^{-\frac{1}{3}}$$

or

$$c = -\frac{1}{9}L^2(x_3^2 + x_4^2)^{-\frac{1}{3}},$$

respectively.

We note that the pointwise constant c in the above theorem is negative contrary to the ones in Theorems 4.1 and 4.2.

5 Weakly *-Einstein examples

In this section, we provide weakly *-Einstein almost Kähler 4-manifolds. In §4, we have already shown that the Ricci flat almost Kähler manifolds $M(f, u, v, \phi)$ in Theorems 3.1 and 3.2 are weakly *-Einstein.

Here, we consider the almost Kähler manifolds $M(f, \alpha, \beta, \phi)$ for any constants α, β . In this case, by using the coordinates $(\xi_1, \xi_2, \xi_3, \xi_4)$ of (2.10), the components of the Ricci *-tensor with respect to the unitary frame $\{e_i\}$ in (2.6) is given by

$$\begin{aligned} \rho_{11}^* &= \rho_{22}^* = \frac{\tilde{\partial}_1^2 f}{2f^3} - \frac{f}{4}\{\tilde{\partial}_3^2 f + \tilde{\partial}_4^2 f\} + \frac{f \cos 2\phi}{4}\{\tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f\} - \frac{f \sin 2\phi}{2}\tilde{\partial}_3 \tilde{\partial}_4 f \\ &+ \frac{1}{2f^4}\{-3(\tilde{\partial}_1 f)^2 + (\tilde{\partial}_2 f)^2\} + \frac{\cos 2\phi}{2}\{(\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2\} - \sin 2\phi \tilde{\partial}_3 f \tilde{\partial}_4 f, \\ \rho_{33}^* &= \rho_{44}^* = \frac{\tilde{\partial}_2^2 f}{2f^3} - \frac{f}{4}\{\tilde{\partial}_3^2 f + \tilde{\partial}_4^2 f\} - \frac{f \cos 2\phi}{4}\{\tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f\} + \frac{f \sin 2\phi}{2}\tilde{\partial}_3 \tilde{\partial}_4 f \\ &+ \frac{1}{2f^4}\{(\tilde{\partial}_1 f)^2 - 3(\tilde{\partial}_2 f)^2\} - \frac{\cos 2\phi}{2}\{(\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2\} + \sin 2\phi \tilde{\partial}_3 f \tilde{\partial}_4 f, \\ \rho_{12}^* &= \rho_{21}^* = \rho_{34}^* = \rho_{43}^* = 0, \end{aligned}$$

$$\begin{aligned}
\rho_{13}^* &= \rho_{42}^* = \rho_{24}^* = \rho_{31}^* = \frac{\tilde{\partial}_1 \tilde{\partial}_2 f}{2f^3} - 2 \frac{\tilde{\partial}_1 f \tilde{\partial}_2 f}{f^4} - \frac{f \cos 2\phi}{2} \tilde{\partial}_3 \tilde{\partial}_4 f - \cos 2\phi \tilde{\partial}_3 f \tilde{\partial}_4 f \\
&\quad - \frac{f \sin 2\phi}{4} \{\tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f\} - \frac{\sin 2\phi}{2} \{(\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2\}, \\
\rho_{14}^* &= -\rho_{32}^* = \frac{\cos \phi}{2f} (\tilde{\partial}_1 \tilde{\partial}_3 f - \tilde{\partial}_2 \tilde{\partial}_4 f) - \frac{\sin \phi}{2f} (\tilde{\partial}_2 \tilde{\partial}_3 f + \tilde{\partial}_1 \tilde{\partial}_4 f), \\
\rho_{23}^* &= -\rho_{41}^* = \frac{\cos \phi}{2f} (\tilde{\partial}_1 \tilde{\partial}_3 f - \tilde{\partial}_2 \tilde{\partial}_4 f) - \frac{\cos \phi}{f^2} (\tilde{\partial}_1 f \tilde{\partial}_3 f - \tilde{\partial}_2 f \tilde{\partial}_4 f) \\
&\quad - \frac{\sin \phi}{2f} (\tilde{\partial}_2 \tilde{\partial}_3 f + \tilde{\partial}_1 \tilde{\partial}_4 f) + \frac{\sin \phi}{f^2} (\tilde{\partial}_2 f \tilde{\partial}_3 f + \tilde{\partial}_1 f \tilde{\partial}_4 f),
\end{aligned}$$

By (5.1), we have the following

Lemma 5.1. *The almost Kähler 4-manifold $M(f, \alpha, \beta, \phi)$ is weakly *-Einstein if and only if*

$$\begin{aligned}
(5.2) \quad & \frac{1}{2f^3} (\tilde{\partial}_1^2 f - \tilde{\partial}_2^2 f) + \frac{f \cos 2\phi}{2} (\tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f) - f \sin 2\phi \tilde{\partial}_3 \tilde{\partial}_4 f \\
&= \frac{2}{f^4} \{(\tilde{\partial}_1 f)^2 - (\tilde{\partial}_2 f)^2\} - \cos 2\phi \{(\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2\} + 2 \sin 2\phi \tilde{\partial}_3 f \tilde{\partial}_4 f,
\end{aligned}$$

$$\begin{aligned}
(5.3) \quad & \frac{\tilde{\partial}_1 \tilde{\partial}_2 f}{2f^3} - \frac{f \cos 2\phi}{2} \tilde{\partial}_3 \tilde{\partial}_4 f - \frac{f \sin 2\phi}{4} (\tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f) \\
&= 2 \frac{\tilde{\partial}_1 f \tilde{\partial}_2 f}{f^4} + \cos 2\phi \tilde{\partial}_3 f \tilde{\partial}_4 f + \frac{\sin 2\phi}{2} \{(\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2\},
\end{aligned}$$

$$(5.4) \quad \cos \phi (\tilde{\partial}_1 \tilde{\partial}_3 f - \tilde{\partial}_2 \tilde{\partial}_4 f) - \sin \phi (\tilde{\partial}_2 \tilde{\partial}_3 f + \tilde{\partial}_1 \tilde{\partial}_4 f) = 0,$$

$$\begin{aligned}
(5.5) \quad & \cos \phi (\tilde{\partial}_1 \tilde{\partial}_3 f - \tilde{\partial}_2 \tilde{\partial}_4 f) - \sin \phi (\tilde{\partial}_2 \tilde{\partial}_3 f + \tilde{\partial}_1 \tilde{\partial}_4 f) \\
&= \frac{2}{f} \{ \cos \phi (\tilde{\partial}_1 f \tilde{\partial}_3 f - \tilde{\partial}_2 f \tilde{\partial}_4 f) - \sin \phi (\tilde{\partial}_2 f \tilde{\partial}_3 f + \tilde{\partial}_1 f \tilde{\partial}_4 f) \}.
\end{aligned}$$

If f is a function of ξ_1, ξ_2 , then (5.2) \sim (5.5) reduce to

$$(5.6) \quad f(\tilde{\partial}_1^2 f - \tilde{\partial}_2^2 f) = 4\{(\tilde{\partial}_1 f)^2 - (\tilde{\partial}_2 f)^2\},$$

$$(5.7) \quad f \tilde{\partial}_1 \tilde{\partial}_2 f = 4 \tilde{\partial}_1 f \tilde{\partial}_2 f.$$

Since (5.6), (5.7) are same as (4.21), (4.24), we see that

$$(5.8) \quad f = K(\xi_1^2 + \xi_2^2)^{-\frac{1}{2}}$$

is a solution.

If f is a function of ξ_3, ξ_4 , then (5.2) \sim (5.5) reduce to

$$(5.9) \quad \begin{aligned} & \frac{f \cos 2\phi}{2} (\tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f) - f \sin 2\phi \tilde{\partial}_3 \tilde{\partial}_4 f \\ &= -\cos 2\phi \{(\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2\} + 2 \sin 2\phi \tilde{\partial}_3 f \tilde{\partial}_4 f, \end{aligned}$$

$$(5.10) \quad \begin{aligned} & -\frac{f \cos 2\phi}{2} \tilde{\partial}_3 \tilde{\partial}_4 f - \frac{f \sin 2\phi}{4} (\tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f) \\ &= \cos 2\phi \tilde{\partial}_3 f \tilde{\partial}_4 f + \frac{\sin 2\phi}{2} \{(\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2\}. \end{aligned}$$

Similarly, we see that

$$(5.11) \quad f = L(\xi_3^2 + \xi_4^2)^{\frac{1}{2}}$$

is a solution of (5.9), (5.10). Thus we have the following

Theorem 5.2. *Let*

$$f = K\{(x_1 + \alpha x_3 - \beta x_4)^2 + (x_2 + \beta x_3 + \alpha x_4)^2\}^{-\frac{1}{2}},$$

or

$$f = L(x_3^2 + x_4^2)^{\frac{1}{2}},$$

where K and L are non-zero constants. Then the almost Kähler manifold $M(f, \alpha, \beta, \phi)$ is weakly $*$ -Einstein. The $*$ -scalar curvature is given respectively by

$$\tau^* = -\frac{4}{9K^2} \{(x_1 + \alpha x_3 - \beta x_4)^2 + (x_2 + \beta x_3 + \alpha x_4)^2\}^{-\frac{1}{2}},$$

or

$$\tau^* = -\frac{4}{9} L^2 (x_3^2 + x_4^2)^{-\frac{1}{2}}.$$

Now, we give another example. It is easy to see that

$$(5.12) \quad f = (a_1 \xi_1 + a_2 \xi_2 + b_1)^{-\frac{1}{2}}$$

is also a solution of (5.6), (5.7), and

$$(5.13) \quad f = (a_3 \xi_3 + a_4 \xi_4 + b_2)^{\frac{1}{2}}$$

is a solution of (5.9), (5.10).

From (5.12) and (5.13),

$$(5.14) \quad f = (a_1 \xi_1 + a_2 \xi_2 + b_1)^{-\frac{1}{2}} (a_3 \xi_3 + a_4 \xi_4 + b_2)^{\frac{1}{2}}$$

satisfies (5.2), (5.3). Moreover, if

$$(5.15) \quad (a_1 a_3 - a_2 a_4) \cos \phi - (a_2 a_3 + a_1 a_4) \sin \phi = 0,$$

then f in (5.14) also satisfies (5.4) and (5.5). Note that the functions f in (5.12), (5.13) are given as special cases of (5.14) satisfying (5.15).

Consequently, we obtain the following

Theorem 5.3. *Let*

$$f = \left\{ \frac{a_3x_3 + a_4x_4 + b_2}{a_1(x_1 + \alpha x_3 - \beta x_4) + a_2(x_2 + \beta x_3 + \alpha x_4) + b_1} \right\}^{\frac{1}{3}},$$

where a_i ($i = 1, 2, 3, 4$) and b_j ($j = 1, 2$) are constants satisfying (5.15). Then the almost Kähler manifold $M(f, \alpha, \beta, \phi)$ is weakly *-Einstein, and

$$\begin{aligned} \tau^* &= \frac{2(a_1^2 + a_2^2)}{9} \{a_1(x_1 + \alpha x_3 - \beta x_4) + a_2(x_2 + \beta x_3 + \alpha x_4) + b_1\}^{-\frac{4}{3}} \\ &\times (a_3x_3 + a_4x_4 + b_2)^{-\frac{2}{3}} \\ &+ \frac{2(a_3^2 + a_4^2)}{9} \{a_1(x_1 + \alpha x_3 - \beta x_4) + a_2(x_2 + \beta x_3 + \alpha x_4) + b_1\}^{-\frac{2}{3}} \\ &\times (a_3x_3 + a_4x_4 + b_2)^{-\frac{4}{3}}. \end{aligned}$$

It should be remarked that the almost Kähler manifold $M(f, \alpha, \beta, \phi)$ in Theorem 5.2 is weakly*-Einstein for all ϕ . On the contrary $M(f, \alpha, \beta, \phi)$ in Theorem 5.3 is weakly*-Einstein for particular values of ϕ and a_i ($i = 1, 2, 3, 4$) satisfying (5.15).

Finally, we note that the Ricci tensors of these examples in Theorem 5.3 are J -anti-invariant, i.e., $\rho(JX, JY) = -\rho(X, Y)$, and hence $\tau = 0$.

References

- [1] J.T.Cho and K.Sekigawa, *Four-dimensional almost Kähler manifolds of pointwise constant holomorphic sectional curvature*, Arab. J. Math. Sci. 2(1996), 39-53.
- [2] S.I. Goldberg, *Integrability of almost Kähler manifolds*, Proc. Amer. Math. Soc. 21(1969), 96-100.
- [3] P.Nurowski and M.Przanowski, *A four-dimensional examples of a Ricci-flat metric admitting almost-Kähler non-Kähler structures*, Class. Quantum Grav. 16(1999), L9-L13.
- [4] T.Sato, *Some remarks on almost Kähler 4-manifolds of pointwise constant holomorphic sectional curvature*, Kodai Math. J. 22(1999), 286-303.
- [5] T.Sato, *An example of an almost Kähler manifold with pointwise constant holomorphic sectional curvature*, to appear in Tokyo J. Math.

Faculty of Engineering
Kanazawa University
Kanazawa, Japan
E-mail:tsato@t.kanazawa-u.ac.jp